Geodesic chaos in perturbed black-hole fields

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Dynamics of time-like geodesics in the static and axially symmetric field of a black hole surrounded by a thin disc or ring is studied in several different ways: on Poincaré's sections, on phase-variable behaviours and their power spectra, and by two recurrence methods. The geodesic motion turns chaotic if the disc/ring is sufficiently massive and/or if the particle has sufficiently large energy. The occurrence of chaos due to the presence of ambient matter may be important for the evolution and appearance of astrophysical black-hole systems.

Introduction

Observational knowledge concerning black holes comes from their interaction with the surrounding matter. In theoretical models, this matter is usually treated as non-gravitating (test). With the possible exception of neutron tori, such an approximation is adequate as far as *intensity* of the field is concerned. However, the derivatives of the field may well be dominated by external matter, which would affect the latter's own stability and, consequently, the very configuration it assumes. The additional matter would also perturb the motion of (test) particles and light around. Here we study its influence on geodesic motion, neglecting all the other possible effects coming from EM fields, cosmological constant, higher multipoles of the test particles or their radiation. The geodesic dynamics, originally completely regular in the pure Schwarzschild field, grows chaotic when the external source (disc of ring) is gradually being "switched on". In the preceding work [5], this has been observed on Poincaré's surfaces of section and on phase-variable evolution and power spectra, in exact space-times describing the fields of a Schwarzschild black hole surrounded by several families of static and axially symmetric annular thin discs or rings.

It is especially interesting to study the chaotic dynamics in spaces which are themselves described by a non-linear theory like general relativity. At the same time, some of the classical indicators of chaos (e.g. the Lyapunov and related coefficients quantifying the rate of orbital-flow divergence) has to be used with caution there since they rely on the existence of a unique global time coordinate. Here we examine several other methods of recognising and classifying chaos in the relativistic black-hole–disc systems (two different recurrence methods, in particular) and compare their outcomes with what has been observed on Poincaré's diagrams and phase-variable power spectra.

The simplest among the RQA quantifiers is the ratio of the recurrence points ($\mathbf{R}_{i,j}(\varepsilon)=1$) within all points of the matrix, $RR(\varepsilon)=\frac{1}{N^2}\sum_{i,j=1}^{N}R_{i,j}(\varepsilon)$, called recurrence rate. Another, most important quantifier is the histogram of diagonal lines of a certain prescribed length l,

$$P(\varepsilon,l)=\sum_{i,j=1}^{N}(1-R_{i-1,j-1}(\varepsilon))(1-R_{i+l,j+l}(\varepsilon))\prod_{k=0}^{l-1}R_{i+k,j+k}(\varepsilon). \qquad (2)$$

From (2) several further quantities could be computed:
$DET=\sum_{l=l_{min}}^{N}lP(l)/\sum_{l=1}^{N}lP(l)$ gives the ratio between all recurrence points and those of them which lie at a diagonal line longer than $l_{\min}$.
$L_{\max}=\max(\{l_i\}_{i=1}^{N})$ stands for the length of the longest diagonal line, $DIV=1/L_{\max}$ is the inverse of this value.

Similarly, the histogram of vertical lines can be introduced,

$$P(\varepsilon,v)=\sum_{i,j=1}^{N}(1-R_{i,j}(\varepsilon))(1-R_{i,j+v}(\varepsilon))\prod_{k=0}^{v-1}R_{i,j+k}(\varepsilon), \qquad (3)$$

the measure of vertical structures $LAM=\sum_{v=v_{\min}}^{N}vP(v)/\sum_{v=1}^{N}vP(v)$. From the probabilities that some chosen diagonal/vertical line has length l/v, $p(l)=P(l)/N_l$ and $p(v)=P(v)/N_v$, where N_l and N_v are the total numbers of diagonal/vertical lines, one can compute the Shannon entropies

$$ENTR=-\sum_{l=l_{min}}^{N}p(l)\ln p(l);\quad VENTR=-\sum_{v=v_{min}}^{N}p(v)\ln p(v). \qquad (4)$$

Finally, we can compute the recurrence times given by differences between serial numbers of the consecutive recurrence points in one column $\vec{x}_{j_{k+1}}$, $\vec{x}_{j_k}$ multiplied by the respective proper-time step, $\{T_k=(j_{k+1}-j_k)\Delta\tau\}$. The mean of T_k is called the recurrence time of the first type, $T1$. In this set there are also the recurrence points (*sojourn points*) resulting from the tangential motion when ε is high enough to capture more successive points of the trajectory. These points need to be discarded from the set in to order to make the statistics only over the real recurrences, after which we get the recurrence times of the second type $T_k^{(2)}$ and the mean value $T2$.

Recurrence analysis

Recurrence analysis is a powerful tool for discovering the traces of chaos which are hidden in the time evolution of some dynamical quantity. Since 1987, when Eckmann et al. [1] invented the recurrence plots (RPs), this method has been used to analyse the behaviour of nonlinear dynamical systems in a very wide field of study, including physics, mathematics, financial markets and medicine. Very recently this method was also used for detecting chaos in relativistic system of charge particle moving around a magnetised rotating black hole [3].

The key feature for RP is the fact, that the patterns of recurrences of the motion are different for random, chaotic and regular trajectories. The recurrences are encoded in the recurrence matrix

$$\mathbf{R}_{i,j}(\varepsilon)=\Theta(\varepsilon-\parallel\vec{x}_i-\vec{x}_j\parallel),\qquad i,j=1,...,N, \qquad (1)$$

where $\vec{x}_i=\vec{x}(t_i)$ are (N) points of the phase trajectory, ε denotes a chosen threshold and Θ is the Heaviside step function. This matrix could be visualised in the RP and reveals long diagonal lines for regular motion and randomly scattered points for random processes. The diagonal lines correspond to the situation, when the trajectory evolves in a very similar way like it did some time ago, and are typical for the regular motion.

Number of quantifiers (called RQA) could be derived from this matrix which help us to find chaos in a very effective way without the necessity to actually look at the resulting recurrence plots (see [4]). Definitions of this RQA values are given in the box on the right. When we change some parameters of the motion (e.g. the energy of the particle or the relative mass of the external source), the dynamics could change its character abruptly from regular to chaotic and vice versa which causes changes in the recurrence patterns of the trajectory. This fact is reflected by the values of the quantifiers, so they are able to reveal changes in the dynamics.

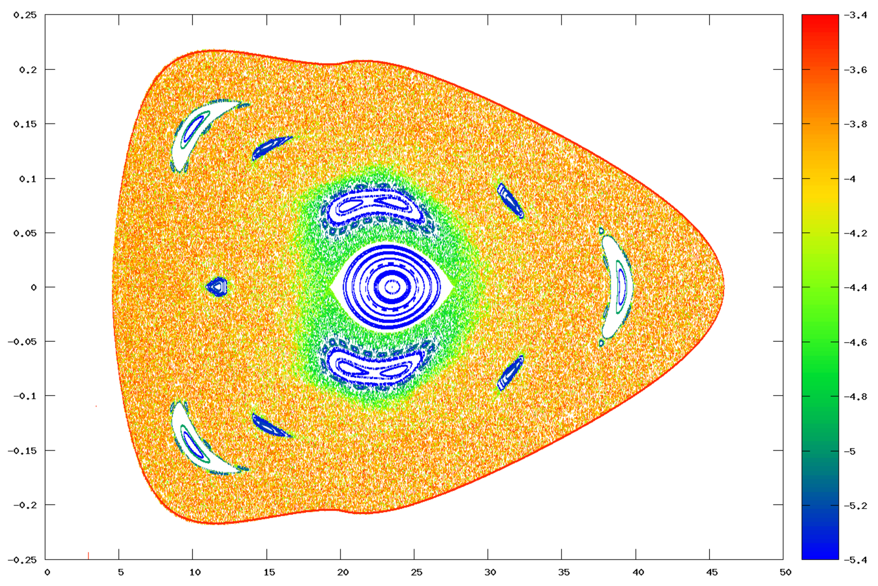
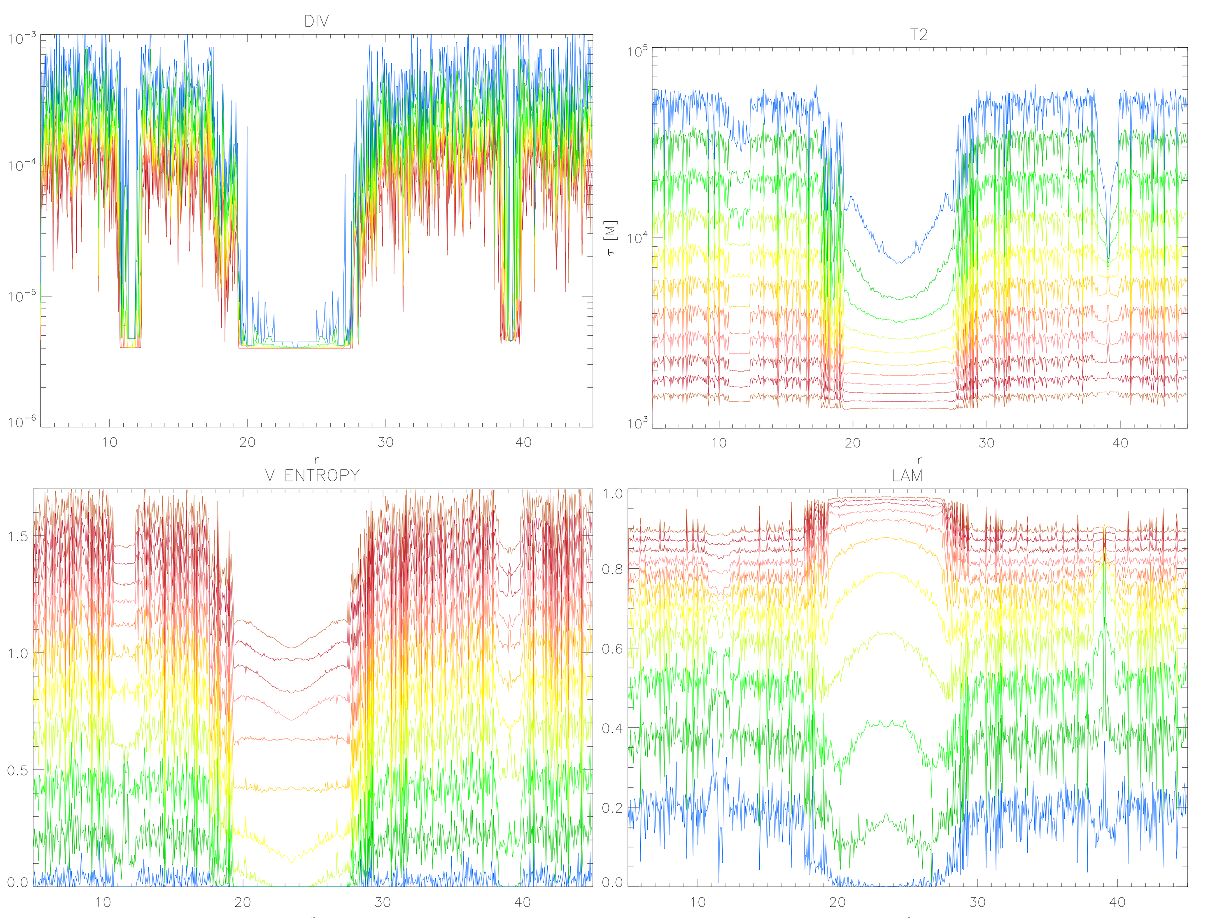


Fig. 1 Poincaré's surface of section colored with respected to the value of DIV of the particular trajectories, the decadic logarithmic scale is used.

Bottom is given the dependency of four RQA-quantites on the chosen treshold ε in the range $\varepsilon=0.5$ (blue) to $\varepsilon=2$ (red). The analysis is computed for orbits launched with zero radial velocity from $r=5M$ with the step $\Delta r=0.05M$. The results of the recurrence analysis depends on the proper choice of the parameters, but the general attributes could be seen in wide range of parameters.



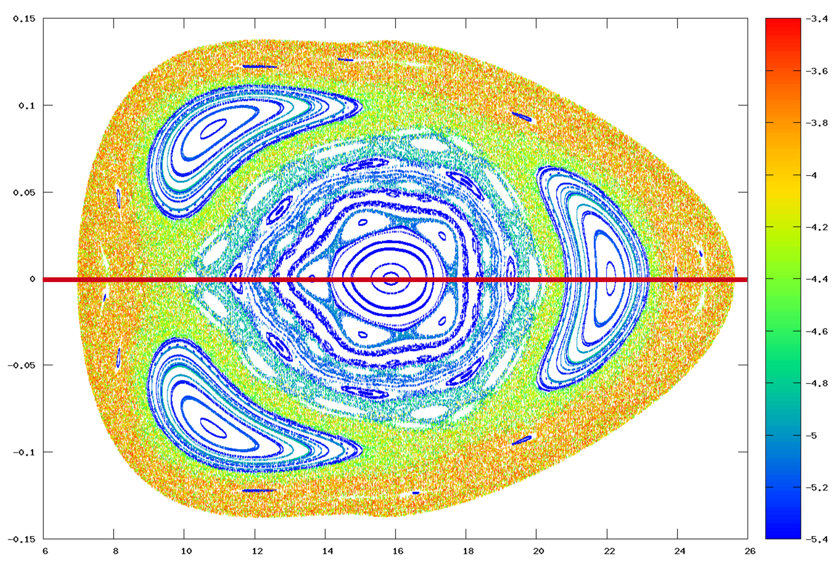
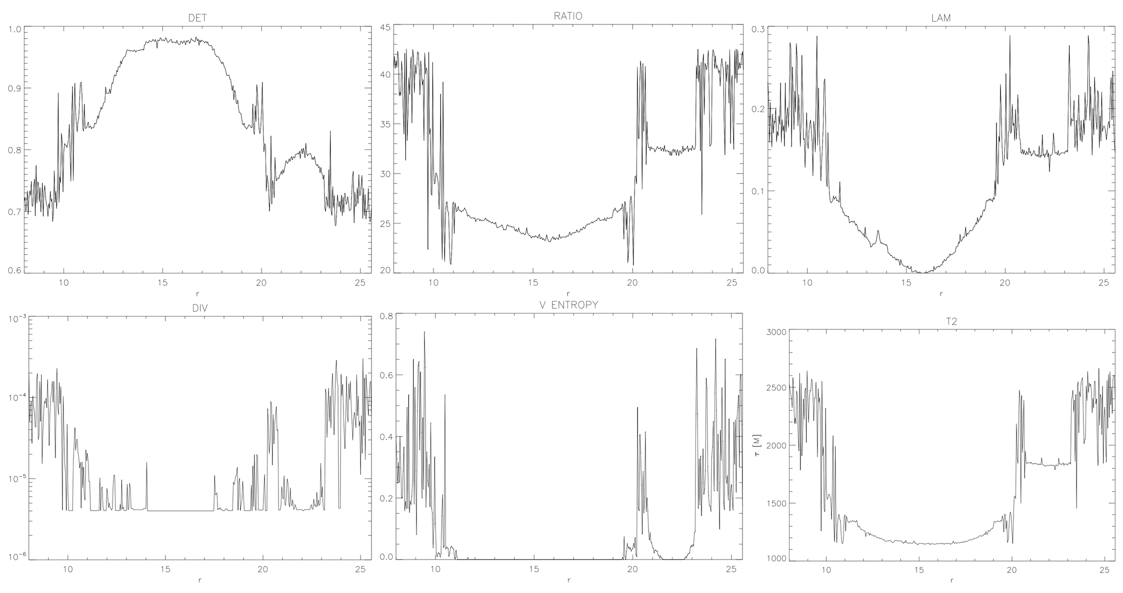


Fig. 2 Poincaré's surface of section colored with respected to the value of DIV of the particular trajectories. The plot is composed of hundreds of orbits.

On the right the course of six RQA-quantities DET, DIV, RATIO = DET/RR, V ENTR, LAM, and T2 is given for orbits launched with zero radial velocity with the radial step $\Delta r=0.04M$ (along the red line). The big regular regions could be seen in all pictures, the quantity DIV seems to be most sensitive to small chaotic layers inside regular islands.



Average of directional vectors

Another method of post-processing the measured time series of some dynamical variable is computing the averaged directional vectors in the reconstructed phase space. This procedure was brought in by Kaplan in [2] in 1992 to distinguish between deterministic and random data. It lies in reconstructing the phase space from the time series by its shifting by a constant time interval $\Delta\tau$ (in our case we use 3D embedding of the particle's z position, so the point in the phase space has the coordinates $z(\tau)$, $z(\tau-\Delta\tau)$ and $z(\tau-2\Delta\tau)$), then dividing the phase space into boxes and following the motion of the phase–space orbit through these boxes. Every time the trajectory crosses the (j-th) box, we record the unit vector in the direction of this passage (pointing from the point where the trajectory enters the box to the point where it leaves it), then we add all the vectors in one box and compute the resulting length $|V_j|$ divided by the number of passages through the box n_j. For some suitable value of the time lag $\Delta\tau$ the values of $|V_j/n_j|$ remain close to one with growing n for deterministic systems while it decreases with n as the average displacement per step for a random walk in a 3D space for random processes, given by

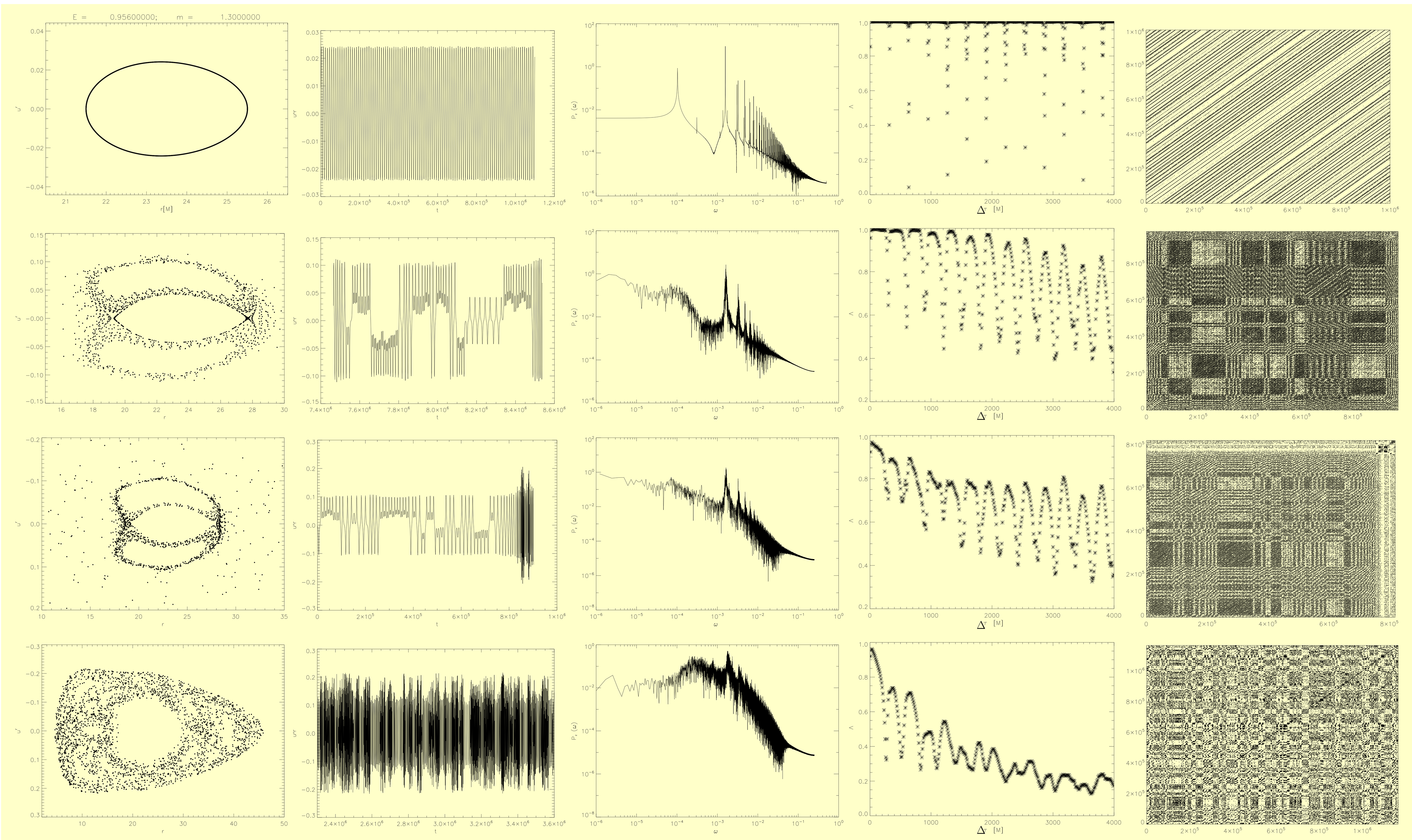
$$\bar{R}_{n_j}^3=\frac{4}{(6\pi)^{1/2}}n_j^{-\frac{1}{2}}. \qquad (5)$$

But instead of studying the dependence on n we compute the weighted average through all boxes

$$\bar{\Lambda}(\Delta\tau)=\left\langle\frac{(V_j/n_j)^2-(R_{n_j}^3)^2}{1-(R_{n_j}^3)^2}\right\rangle, \qquad (6)$$

and look at the dependence of this quantity on the chosen $\Delta\tau$. For regular trajectories it remains close to one for almost all values of $\Delta\tau$ except for some particular values connected with the orbital period of the motion. On the other hand for chaotic trajectories the deterministic connection between the points fades away quickly with the growing time lag because of the strong dependence on initial condition in the chaotic layer. Thus, the value $\bar{\Lambda}(\Delta\tau)$ decreases with growing $\Delta\tau$ more quickly for the very chaotic orbits and less quickly for the slightly chaotic orbits.

Fig. 3 Comparison of the mentioned quantities computed for four different orbits (each row for one). In the first row we have regular orbit belonging to the primary island. The other three orbits are parts of the orbit filling the big chaotic sea which differ in how much chaos they exhibit during the time we follow them. In the first column we plot the respective Poincaré's surfaces of section revealing the degree of chaocity, on the next plot there is the dependency of the radial velocity u^r on t at the moments of the passages through the equatorial plane. In the third column the spectra of the coordinate z are given, the next column displays $\bar{\Lambda}(\Delta\tau)$ and in the last column are the recurrence plots.



Concluding remarks

This work follow up with the previous paper [5], where we studied the geodesic dynamics in the black-hole–disc (or ring) field on Poincarés surfaces of section, on the behaviour of the "latitudinal action" and on phase-variable power spectra.

Here, we focus on particular trajectories in more detail and turn to other two powerful (recurrence) methods for comparison.

In particular, we illustrate the average-directional-vectors method [2] with changing time lag $\Delta\tau$ which deals with directions in which the orbit recurrently passes through phase-space cells, and the method developed in [4] and [7] which is based on statistics over the recurrences themselves. See [6] for further details; a more thorough paper with a number of illustrations has been submitted.

Acknowledgement

We thank M. Žáček and D. Heyrovský and the projects GAUK-428011, SVV-265301, GAČR-205/09/H033 (P.S.), GAČR-202/09/0772 and MSM0021610860 (O.S.).

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