Mass, gauge conditions and spectral properties of the Sen-Witten operator

László Szabados

Wigner Research Centre for Physics,
Budapest

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1. THREE PROBLEMS

1.1 Mass

- Positive energy theorems
  $\mathrm{P}^{\underline{a}}_{ADM}$, $\mathrm{P}^{\underline{a}}_{BS}$ – future pointing and timelike,
  i.e. $\mathrm{m}^2 := \eta_{\underline{a}\,\underline{b}}\mathrm{P}^{\underline{a}}\mathrm{P}^{\underline{b}} > 0$, $\mathrm{m} > 0$
  Positive lower bound: $\mathrm{m} \geq \mathrm{M} > 0$?
  (For BH the Penrose and Dain inequalities are such.)

- Bäckdahl, Valiente-Kroon: In vacuum, asymptotically flat (AF)
  $\mathrm{m}_{ADM} \approx \|\mathcal{D}_{(AB}\lambda_{C)}\|^2_{L_2}$. For $\mathrm{m}_{BS} =?$ Or: in the presence of matter?

- In closed universes: NO energy-momentum by 2-surface integrals
  But: maybe in other way? E.g. mass as a positive measure of the strength of the gravitational field?

1.2 Gauge conditions

To reduce the huge gauge freedom of GR (e.g. in the energy positivity proofs, evolution problem, numerical calculations, ...)

$\mathcal{D}_{A'A}\lambda^A = 0$ — Witten's gauge,

$\mathcal{D}_{A'A}\psi^A + \alpha S_{A'A}\psi^A = 0$ — Parker's gauge,

$D_{AB}\phi^B = \beta\phi_A$ + no zero of ϕ^A — Nester's gauge.

These are known to admit solutions on asymptotically flat (AF) Σ.

Existence of their solutions in closed universes?

1.3 Spectral characterization of geometries

- E.g. by the eigenvalues and the structure of the spectrum of the Laplace, Dirac, ... operators? (Lichnerowicz, Friedrich, Bär, ...)
- Hijazi, Zhang: Sharp lower bound for the 1st eigenvalue of the SW operator:

  $$\alpha_1^2 \geq \frac{3}{4}\kappa \inf_{l^a} \frac{\int_\Sigma t^a T_{ab} l^b \mathrm{d}\Sigma}{\int_\Sigma t_c l^c \mathrm{d}\Sigma}$$

  Even greater lower bound, which is not trivial even in vacuum?

  Expression for the first eigenvalue itself?

2. RESULTS

The answer to the questions above in red.
In particular (Class. Quantum Grav. **29** (2012) 095001):

Notation:
Σ – spacelike hypersurface,
t^a – future pointing unit (timelike) normal,
$\mathcal{D}_{AB} := \sqrt{2} t_B{}^{A'} \mathcal{D}_{AA'} := \sqrt{2} t_{(B}{}^{A'} \nabla_{A)A'}$ – Sen connection,
$\|\psi^A\|^2_{L_2} := \int_\Sigma \sqrt{2} t_{AA'} \psi^A \bar{\psi}^{A'} \mathrm{d}\Sigma$ – the L_2 norm,

$$\mathtt{M} := \inf\Big\{ \frac{\sqrt{2}}{\kappa} \|\mathcal{D}_{(AB}\lambda_{C)}\|^2_{L_2} + \int_\Sigma t^a T_{ab} \lambda^B \bar{\lambda}^{B'} \mathrm{d}\Sigma \Big\};$$

where

- On AF/AH Σ: $\lambda^A - {}_\infty\lambda^A = o(r^{-\frac{1}{2}})$, ${}_\infty\lambda^A$ is constant/sol. of the asymptotic twistor eq., resp., normalization: ${}_\infty t_{AA'} {}_\infty\lambda^A {}_\infty\bar{\lambda}^{A'} = 1$.
- On closed Σ: $\|\lambda^A\|_{L_2} = 1$.

Then (using the Reula–Tod form of the Sen–Witten identity) :

Results

- On AF/AH Σ:
  - The ${}_\infty\lambda_{\underline{A}}\,{}_\infty\bar\lambda_{\underline{A}'}$-component of $\mathtt{P}^{\underline{a}}_{ADM}$, $\mathtt{P}^{\underline{a}}_{BS}$ in the Witten gauge can be rewritten as

$$\frac{\sqrt{2}}{\kappa}\|\mathcal{D}_{(AB}\lambda_{C)}\|^2_{L_2}+\int_\Sigma t^a T_{ab}\lambda^B\bar\lambda^{B'}\mathrm{d}\Sigma$$

    – generalizations of the result of Bäckdahl and Valiente-Kroon;
  - $\mathtt{m}_{ADM}, \mathtt{m}_{BS} \geq \mathtt{M} > 0$ – non-trivial positive lower bound.
- On closed Σ:
  - $\mathtt{M}=0$ iff (M, g_{ab}) is flat and $\Sigma \approx S^1\times S^1\times S^1$ – positive definite measure of the strength of gravity, given by the same formula as for $\mathtt{P}^{\underline{a}}\lambda_{\underline{A}}\bar\lambda_{\underline{A}'}$ – the dimension is *mass*;
  - Witten's gauge condition has a non-trivial solution iff $\mathtt{M}=0$;

Results

- For the first eigenvalue α_1 of $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B = \alpha^2\lambda^A$:

$$\alpha_1^2 = \frac{3}{2\sqrt{2}}\kappa\,\mathrm{M}$$

– mass of closed universes as the first eigenvalue of the SW operator;
– generalization of the result of Hijazi and Zhang;

Generalization/extension of Witten's gauge condition to closed universes: $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B = \alpha_1^2\lambda^A$.

Conjecture:
The first eigenspinors of $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}$ are nowhere vanishing on Σ.
Then there would be: Geometrically distinguished triad on Σ and lapse.