A class of conformal curves in spherically symmetric spacetimes

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Outline

1 Introduction

2 Conformal geodesics and other conformal curves

3 Main results

Motivation behind this research:

In brief:

- Develop a better understanding of the **conformal structure** of **spherically symmetric** spacetimes:
  - The Reissner-Nordström spacetime;
  - The Schwarzschild-de Sitter spacetime;
  - Schwarzschild-anti de Sitter spacetime.
- Understand the role of the **above spacetimes** as a solution of the **conformal Einstein(-Maxwell) field equations**.

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- Understand the role of the **above spacetimes** as a solution of the **conformal Einstein(-Maxwell) field equations**.

Strategy:

- Probe the spherically symmetric spacetimes by means of **conformally privileged** curves —the so-called **conformal geodesics** and generalisations thereof (**conformal curves**).

Why spherically symmetric spacetimes?

Leitmotiv:

- Clarify Folklore and investigate various insights concerning the global structure of black hole spacetimes.

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  - How far can one go with analogy?

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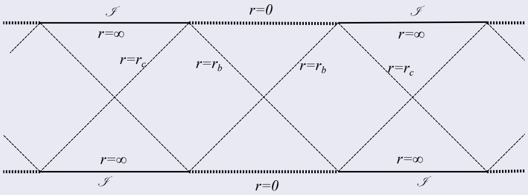
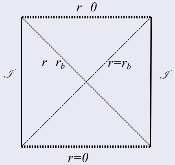
    - How far can one go with analogy?
- The extremal Reissner-Nordström spacetime provides the simplest example of **degenerate horizons**.
- There is also an **extremal** Schwarzschild-de Sitter spacetime.

Conformal diagrams for the Schwarzschild-de Sitter/-anti de Sitter

Schwarzschild-de Sitter:

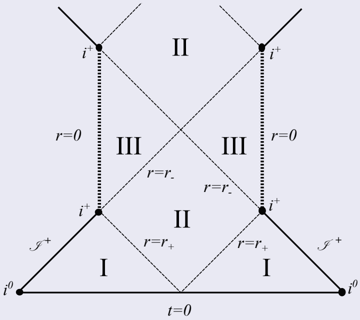


Schwarzschild-anti de Sitter:

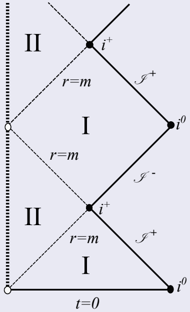


Conformal diagrams for the Reissner-Nordström spacetime

Non-extremal case:



Extremal case:



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Conformal geodesics and conformal curves

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- The class of conformal curves to be used to probe the spherically symmetric spacetimes are the so-called **conformal geodesics** in the vacuum case, and a suitable generalisation thereof (**conformal curves**) in the electrovacuum case.

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Conformal geodesics:

- Introduced to the context of GR by Friedrich & Schmidt ('87), Friedrich ('95), Friedrich ('03).
- Provide a way of constructing gauges based on conformal structures.
- Given $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ these are defined by a curve $\boldsymbol{x}(\tau)$ and a 1-form $\boldsymbol{b}(\tau)$ satisfying the equations

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle \boldsymbol{b}, \dot{\boldsymbol{x}}\rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{b}^{\sharp},$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}}\rangle \boldsymbol{b} - \tfrac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{b}, \boldsymbol{b})\dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot),$$

where $\tilde{L}_{\mu\nu} \equiv \frac{1}{2}\tilde{R}_{\mu\nu} - \frac{1}{12}\tilde{R}\tilde{g}_{\mu\nu}$ is the Schouten tensor, $\tau \in I \subset \mathbb{R}$.

Properties of conformal geodesics I

Conformal invariance:

- Given the conformal rescaling $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$, the pair

$$(\boldsymbol{x}(\tau), \boldsymbol{b} - \boldsymbol{\Upsilon}), \qquad \boldsymbol{\Upsilon} \equiv \Theta^{-1} \mathbf{d}\Theta$$

  satisfies the conformal geodesic equations with respect to the metric $\boldsymbol{g}$.
- This property can be extended to the case of **Weyl connections**.

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A priori conformal factor:

- If one requires Θ to satisfy

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1,$$

  then for a vacuum spacetime ($\tilde{\boldsymbol{L}} = 0$) one has

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2}\ddot{\Theta}_*(\tau - \tau_*)^2,$$

  where Θ_*, $\dot{\Theta}_*$, $\ddot{\Theta}_*$ are determined by some initial data at $\tau = \tau_*$.
- The parameter τ is the $\boldsymbol{g}$-proper time.

Properties of conformal geodesics II

But...

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Question:

- How could one recover this property?
  - This property offers (among other things) a **systematic approach** for the construction of conformal extensions of spacetimes.

A class of conformal curves

The equations (C Lübbe & JA Valiente K, 2011):

- Require the pair $(\boldsymbol{x}(\tau), \boldsymbol{b}(\tau))$ to satisfy

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle \boldsymbol{b}, \dot{\boldsymbol{x}}\rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{b}^{\sharp},$$
$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}}\rangle \boldsymbol{b} - \tfrac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{b}, \boldsymbol{b})\dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{H}}(\dot{\boldsymbol{x}}, \cdot),$$

where $\tilde{\boldsymbol{H}}$ is an arbitrary rank 2 tensor transforming via

$$\tilde{H}_{\mu\nu} - H_{\mu\nu} = \nabla_{\mu}\Upsilon_{\nu} + \Upsilon_{\mu}\Upsilon_{\nu} - \tfrac{1}{2}g^{\lambda\rho}\Upsilon_{\lambda}\Upsilon_{\rho}\, g_{\mu\nu}.$$

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A particular choice:

- The choice $\tilde{\boldsymbol{H}} = \lambda \tilde{\boldsymbol{g}}$ (independent of the matter content) allows to recover the property

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \tfrac{1}{2}\ddot{\Theta}_*(\tau - \tau_*)^2.$$

- With this choice the (physical) conformal curve equations are formally identical to the (physical) vacuum conformal geodesic equations.

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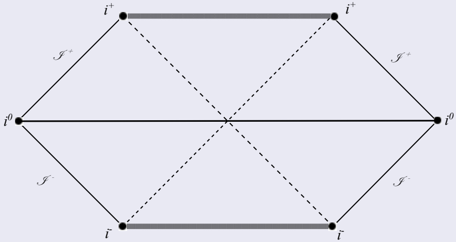
Conformal geodesics on the Schwarzschild spacetime

H. Friedrich, Comm. Math. Phys. **235**, 513 (2003):

- The Schwarzschild spacetime can be covered with a congruence of conformal geodesics which is **free of conjugate points**.
- The data for the congruence is prescribed on the time symmetric slice.

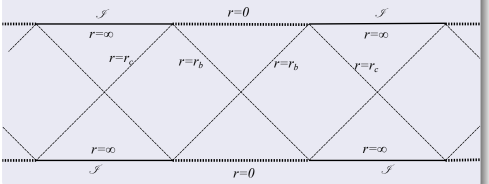


Conformal geodesics on the Schwarzschild-de Sitter/anti de Sitter spacetime

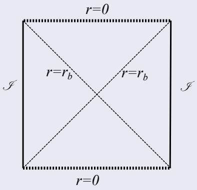
Main result:

- As in the case of the Schwarzschild spacetime, its generalisations with Cosmological constant can be covered with a congruence of conformal geodesics which is **free of conjugate points**.

Schwarzschild-de Sitter:



Schwarzschild-anti de Sitter:

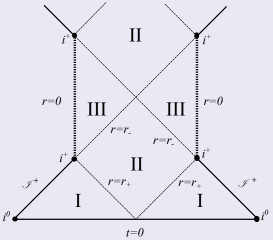


Conformal curves in the Reissner-Nordström spacetime

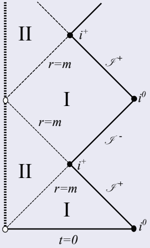
Main result:

- In this case the congruence is free of conjugate points only in the **domain of outer communication**.
- There are regions inside the black hole which cannot be accessed by the curves —in particular, the curves **avoid** the singularity.
- In the case of the extremal case, the congruence remains regular even at i^+.

Non-extremal case:



Extremal case:



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- JA Valiente Kroon. *Global evaluations of static black holes.* In preparation.
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