Stability of Black Holes and Black Branes

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Stability

It is of considerable interest to determine the linear stablity of black holes in (D-dimensional) general relativity. We will consider vacuum general relativity without a cosmological constant here, but our methods apply to general theories of gravity. It is also of interest to determine the linear stability of the corresponding black branes in ($D + p$)-dimensions, i.e., spacetimes with metric of the form

$$d\tilde{s}^2_{D+p} = ds^2_D + \sum_{i=1}^{p} dz^2_i \,,$$

where ds^2_D is a black hole metric. One can analyze this

issue by writing out the linearized Einstein equation off of the black hole or black brane background spacetime. One can establish linear stability by finding a positive definite conserved norm for perturbations, whereas linear instability can be established by finding a solution with (gauge independent) unbounded growth in time.

However, even in the very simplest cases—such as the Schwarzschild black hole (Regge-Wheeler, Zerilli) and the Schwarzschild black string (Gregory-Laflamme)—it is quite nontrivial to carry out the decoupling of equations and the fixing of gauge needed to determine stability or instability directly from the equations of motion.

Thermodynamic Stability

Consider a homogeneous thermodynamic system, whose entropy, S, is a function of energy, E, and other extensive state parameters X_i, $S = S(E, X_i)$. The condition for thermodynamic instability is that the Hessian matrix

$$\mathbf{H}_S = \begin{pmatrix} \frac{\partial^2 S}{\partial E^2} & \frac{\partial^2 S}{\partial X_i \partial E} \\ \frac{\partial^2 S}{\partial E \partial X_i} & \frac{\partial^2 S}{\partial X_i \partial X_j} \end{pmatrix}.$$

admit a positive eigenvalue. If this happens, then one can increase total entropy by exchanging E and/or X_i between different parts of the system. For the case of E, this corresponds to having a negative heat capacity.

A positive eigenvalue of the Hessian is equivalent to

finding a perturbation of the system such that

$$\delta^2 E - T\delta^2 S - \sum_i Y_i \delta^2 X_i < 0$$

where $Y_i = (\partial E/\partial X_i)_S$.

Gubser-Mitra Conjecture for Black Branes

Black holes and black branes are thermodynamic systems, with

$$
\begin{aligned}
E &\leftrightarrow M \\
S &\leftrightarrow \frac{A}{4} \\
X_i &\leftrightarrow J_i, Q_i
\end{aligned}
$$

Thus, in the vacuum case $(Q_i = 0)$ a black hole or black brane is thermodynamically unstable if the Hessian matrix

$$
\mathbf{H}_A = \begin{pmatrix} \frac{\partial^2 A}{\partial M^2} & \frac{\partial^2 A}{\partial J_i \partial M} \\ \frac{\partial^2 A}{\partial M \partial J_i} & \frac{\partial^2 A}{\partial J_i \partial J_j} \end{pmatrix}.
$$

admits a positive eigenvalue. This is equivalent to finding a perturbation for which

$$\delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_A \Omega_A \delta^2 J_A < 0 \,.$$

One might expect that this condition for thermodynamic instability might imply dynamical instability. However, this is clearly false: The Schwarzschild black hole has negative heat capacity $(A = 16\pi M^2$, so $\partial^2 A/\partial M^2 > 0)$ but is well known to be stable. Nevertheless, the Schwarzschild black string is unstable. The Gubser-Mitra conjecture states that the above thermodynamic criterion for instability is a valid criterion dynamical instability for

black branes.

We will prove that the Gubser-Mitra conjecture follows as a consequence of a more fundamental stability criterion that we shall establish.

Local Penrose Inequality

Suppose one has a family of stationary, axisymmetric black holes parametrized by M and angular momenta $J_1, \ldots, J_N$. Consider an arbitrary axisymmetric perturbation. Let $\mathcal{A}$ denotes the area of the apparent horizon of the perturbed spacetime, $\bar{A}$ denotes the the event horizon area of the stationary black hole with the same mass and angular momentum as the perturbed spacetime. The event horizon of the perturbed black hole must lie outside the apparent horizon and (for small perturbations) must have larger area. This area can only increase with time, so the final area of the horizon will be larger than $\bar{A}$. Furthermore, the final mass of the black

hole cannot be larger than the initial total mass (by positivity of Bondi flux), and its final angular momenta must equal the initial angular momenta (by axisymmetry). It follows that if, to second order, we have

$$\delta^2 \mathcal{A} > \delta^2 \bar{A}$$

then there would be a contraction if the perturbed black hole solution were to settle down to a stationary black hole in the family. This implies that satisfaction of this inequality implies instablity—although it does not imply stability if $\delta^2 \mathcal{A} \leq \delta^2 \bar{A}$ always holds.

Our more fundamental stability criterion implies that satisfaction of $\delta^2 \mathcal{A} \leq \delta^2 \bar{A}$ is necessary and sufficient for black hole stability with respect to axisymmetric perturbations.

Our Results

We consider perturbations γ of a static or stationary-axisymmetric black hole or black brane with bifurcate Killing horizon and consider the *canonical energy* of the perturbation defined by

$$\mathcal{E} = \int_\Sigma \omega(g; \gamma, \pounds_t \gamma)$$

where Σ extends from the bifurcation surface to infinity.

We show that the necessary and sufficient condition for stability of a black hole (or black brane) with respect to axisymmetric perturbations is positivity of $\mathcal{E}$ on a Hilbert space, $\mathcal{V}$, of perturbations with vanishing perturbed mass, angular momentum, and linear momentum,

$$\delta M = \delta J_A = \delta P_i = 0.$$

We will also show that

$$\mathcal{E} = \delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_A \Omega_A \delta^2 J_A$$

In other words, dynamical stability is equivalent to thermodynamic stability *for perturbations with* $\delta M = \delta J_A = \delta P_i = 0.$ The "change of mass" perturbation of Schwarzschild—for which $\mathcal{E} < 0$—does not "count" for testing stability because, obviously, $\delta M \neq 0.$

However, if a black hole has a perturbation with $\mathcal{E} < 0$ with $\delta M \neq 0$ and/or $\delta J_A \neq 0$, we prove that there exists a sufficiently long wavelength perturbation of any

corresponding black brane for which $\mathcal{E} < 0$ but $\delta M = \delta J_A = 0$. This proves the Gubser-Mitra conjecture. Thus, for example, the calculation that $\partial^2 A / \partial M^2 = 32\pi > 0$ for Schwarzschild tells one nothing about the stability of Schwarzschild black hole, but it proves the instability of the Schwarzshild black string to sufficiently long wavelength perturbations.

Finally, we prove that if one can find a perturbation of a black hole for which $\delta^2 \mathcal{A} > \delta^2 \bar{A}$, if and only if one can find a perturbation for which $\delta M = \delta J_A = \delta P_i = 0$ and $\mathcal{E} < 0$. This proves that satisfaction of the local Penrose inequality is equivalent to dynamical stability.

Variational Formulas

Lagrangian for vacuum general relativity:

$$L_{a_1...a_D} = \frac{1}{16\pi} R \; \epsilon_{a_1...a_D} \,.$$

First variation:

$$\delta L = E \cdot \delta g + d\theta \,,$$

with

$$\theta_{a_1...a_{d-1}} = \frac{1}{16\pi} g^{ac} g^{bd} (\nabla_d \delta g_{bc} - \nabla_c \delta g_{bd}) \epsilon_{ca_1...a_{d-1}} \,.$$

Symplectic current ($(D-1)$-form):

$$\omega(g; \delta_1 g, \delta_2 g) = \delta_1 \theta(g; \delta_2 g) - \delta_2 \theta(g; \delta_1 g) \,.$$

Symplectic form:

$$W_\Sigma(g; \delta_1 g, \delta_2 g) \equiv \int_\Sigma \omega(g; \delta_1 g, \delta_2 g)$$

$$= -\frac{1}{32\pi} \int_\Sigma (\delta_1 h_{ab} \delta_2 p^{ab} - \delta_2 h_{ab} \delta_1 p^{ab}),$$

with

$$p^{ab} \equiv h^{1/2}(K^{ab} - h^{ab}K).$$

Noether current:

$$\mathcal{J}_X \equiv \theta(g, \pounds_X g) - X \cdot L$$

$$= X \cdot C + dQ_X.$$

Fundamental variational identity:

$$\omega(g;\delta g,\pounds_X g) \;=\; X\cdot[E(g)\cdot\delta g]+X\cdot\delta C$$
$$+d\left[\delta Q_X(g)-X\cdot\theta(g;\delta g)\right]$$

ADM conserved quantities:

$$\delta H_X = \int_\infty \left[\delta Q_X(g)-X\cdot\theta(g;\delta g)\right]$$

For a stationary black hole, choose X to be the horizon Killing field

$$K^a = t^a + \sum \Omega_A \phi_A^a$$

Integration of the fundamental identity yields the first

law of black hole mechanics:

$$0 = \delta M - \sum_A \Omega_A \delta J_A - \frac{\kappa}{8\pi} \delta A \,.$$

Horizon Gauge Conditions

Consider stationary black holes with surface gravity $\kappa > 0$, so the event horizon is of "bifurcate type," with bifurcation surface B. Consider an arbitrary perturbation $\gamma = \delta g$. Gauge condition that ensures that the location of the horizon does not change to first order:

$$\delta\vartheta|_B = 0\,.$$

Additional gauge condition that we impose:

$$\delta\epsilon|_B = \frac{\delta A}{A}\epsilon\,.$$

Canonical Energy

Define the *canonical energy* of a perturbation $\gamma = \delta g$ by

$$\mathcal{E} \equiv W_\Sigma \left(g; \gamma, \pounds_t \gamma \right)$$

The second variation of our fundamental identity then yields (for axisymmetric perturbations)

$$\mathcal{E} = \delta^2 M - \sum_A \Omega_A \delta^2 J_A - \frac{\kappa}{8\pi} \delta^2 A \, .$$

More generally, can view the canonical energy as a bilinear form $\mathcal{E}(\gamma_1, \gamma_2) = W_\Sigma(g; \gamma_1, \pounds_t \gamma_2)$ on perturbations. $\mathcal{E}$ can be shown to satisfy the following properties:

- $\mathcal{E}$ is conserved, i.e., it takes the same value if evaluated on another Cauchy surface Σ' extending from infinity to B.

- $\mathcal{E}$ is symmetric, $\mathcal{E}(\gamma_1, \gamma_2) = \mathcal{E}(\gamma_2, \gamma_1)$

- When restricted to perturbations for which $\delta A = 0$ and $\delta P_i = 0$ (where P_i is the ADM linear momentum), $\mathcal{E}$ is gauge invariant.

- When restricted to the subspace, $\mathcal{V}$, of perturbations for which $\delta M = \delta J_A = \delta P_i = 0$ (and hence, by the first law of black hole mechanics $\delta A = 0$), we have $\mathcal{E}(\gamma', \gamma) = 0$ for all $\gamma' \in \mathcal{V}$ if and only if γ is a perturbation towards another stationary and

axisymmetric black hole. (The proof of the "only if" part of this result was $\sim 40\%$ of the work of this paper!)

Thus, if we restrict to perturbations in the subspace, $\mathcal{V}'$, of perturbations in $\mathcal{V}$ modulo perturbations towards other stationary black holes, then $\mathcal{E}$ is a non-degenerate quadratic form. Consequently, on $\mathcal{V}'$, either (a) $\mathcal{E}$ is positive definite or (b) there is a $\psi \in \mathcal{V}'$ such that $\mathcal{E}(\psi) < 0$. If (a) holds, we have stability.

Flux Formulas

Let δN_{ab} denote the perturbed Bondi news tensor at null infinity, $\mathcal{I}^+$, and let $\delta \sigma_{ab}$ denote the perturbed shear on the horizon, $\mathcal{H}$. If the perturbed black hole were to "settle down" to another stationary black hole at late times, then $\delta N_{ab} \to 0$ and $\delta \sigma_{ab} \to 0$ at late times. We show that—for axisymmetric perturbations—the change in canonical energy would then be given by

$$\Delta \mathcal{E} = -\frac{1}{16\pi} \int_{\mathcal{I}} \delta \tilde{N}_{cd} \delta \tilde{N}^{cd} - \frac{1}{4\pi} \int_{\mathcal{H}} (K^a \nabla_a u) \, \delta \sigma_{cd} \delta \sigma^{cd} \leq 0 \,.$$

Thus, $\mathcal{E}$ can only decrease. Therefore if one has a perturbation $\psi \in \mathcal{V}'$ such that $\mathcal{E}(\psi) < 0$, then ψ cannot "settle down" to a stationary solution at late times

because $\mathcal{E} = 0$ for stationary perturbations with $\delta M = \delta J_A = \delta P_i = 0$. Indeed, $|\mathcal{E}(\psi)|$ should grow without bound, i.e., in case (b) we have instability.

Instability of Black Branes

Theorem: Suppose a family of black holes parametrized by (M, J_A) is thermodynamically unstable at (M_0, J_{0A}), i.e., there exists a perturbation within the black hole family for which $\mathcal{E} < 0$. Then, for any black brane corresponding to (M_0, J_{0A}) one can find a sufficiently long wavelength perturbation for which $\tilde{\mathcal{E}} < 0$ and $\delta \tilde{M} = \delta \tilde{J}_A = \delta \tilde{P}_i = \delta \tilde{A} = \delta \tilde{T}_i = 0$.

This result is proven by modifying the initial data for the perturbation to another black hole with $\mathcal{E} < 0$ by multiplying it by $\exp(ikz)$ and then re-adjusting it so that the modified data satisfies the constraints. The new data will automatically satisfy

$\delta\tilde{M} = \delta\tilde{J}_A = \delta\tilde{P}_i = \delta\tilde{A} = \delta\tilde{T}_i = 0$ because of the $\exp(ikz)$ factor. For sufficiently small k, it can be shown to satisfy $\tilde{\mathcal{E}} < 0$.

Equivalence to Local Penrose Inequality

Let $\bar{g}_{ab}(M, J_A)$ be a family of stationary, axisymmetric, and asymptotically flat black hole metrics on M. Let $g_{ab}(\lambda)$ be a one-parameter family of axisymmetric metrics such that $g_{ab}(0) = \bar{g}_{ab}(M_0, J_{0A})$. Let $M(\lambda), J_A(\lambda)$ denote the mass and angular momenta of $g_{ab}(\lambda)$ and let $\mathcal{A}(\lambda)$ denote the area of its apparent horizon. Let $\bar{g}_{ab}(\lambda) = \bar{g}_{ab}(M(\lambda), J_A(\lambda))$ denote the one-parameter family of stationary black holes with the same mass and angular momenta as $g_{ab}(\lambda)$.

Theorem: There exists a one-parameter family $g_{ab}(\lambda)$ for which

$$\mathcal{A}(\lambda) > \bar{\mathcal{A}}(\lambda)$$

to second order in λ if and only if there exists a perturbation γ'_{ab} of $\bar{g}_{ab}(M_0, (J_{0A})$ with $\delta M = \delta J_A = \delta P_i = 0$ such that $\mathcal{E}(\gamma') < 0$.

Proof: The first law of black hole mechanics implies $\mathcal{A}(\lambda) > \bar{\mathcal{A}}(\lambda)$ to first order in λ, so what counts are the second order variations. Since the families have the same mass and angular momenta, we have

$$
\begin{aligned}
\frac{\kappa}{8\pi} \left[\frac{d^2 A}{d\lambda^2}(0) - \frac{d^2 \bar{A}}{d\lambda^2}(0) \right] &= \mathcal{E}(\bar{\gamma}, \bar{\gamma}) - \mathcal{E}(\gamma, \gamma) \\
&= -\mathcal{E}(\gamma', \gamma') + 2\mathcal{E}(\gamma', \bar{\gamma}) \\
&= -\mathcal{E}(\gamma', \gamma')
\end{aligned}
$$

where $\gamma' = \bar{\gamma} - \gamma$.

Conclusion

Dynamical stability of a black hole is equivalent to its thermodynamic stability with respect to perturbations for which $\delta M = \delta J_A = \delta P_i = 0$.

Thus, the remarkable relationship between the laws of black hole physics and the laws of thermodynamics extends to dynamical stability.