Solutions in the 2+1 null surface formulation

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The null surface formulation of general relativity (NSF) differs from the standard approach by featuring a function Z, describing families of null surfaces, as the prominent variable, rather than the metric tensor. It is possible to reproduce the metric, to within a conformal factor, by using Z (entering through its third derivative, which is denoted by Λ) and an auxiliary function Ω. The functions Λ and Ω depend upon the spacetime coordinates, which are usually introduced in a manner that is convenient for the null surfaces, and also upon an additional angular variable. A brief summary of the (2+1)-dimensional null surface formulation is presented, together with the NSF field equations for Λ and Ω. A few special solutions are found and the properties of one of them are explored in detail.

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1. Introduction

Frittelli, Kozameh and Newman [1-3] have introduced an alternative approach to general relativity called the null surface formulation (NSF). In this approach, it is not the metric g_{ab} that plays a primary role, but a function Z, which is used to specify families of null surfaces. If needed, a metric can be constructed up to a conformal factor from a knowledge of Z and an auxiliary function Ω. A (2+1)-dimensional version of the NSF has been developed by Forni, Iriondo, Kozameh and Parisi [4,5], Tanimoto [6] and Silva-Ortigoza [7]. Central to the NSF in 2+1 dimensions is a third-order ordinary differential equation,

$$u''' \;=\; \Lambda(u, u', u'', \varphi), \tag{1}$$

where the prime denotes differentiation with respect to the angular variable $\varphi \in S^1$. Solutions of Eq. (1) are written $u = Z(x^a;\varphi)$ with x^a $(a = 0, 1, 2)$ representing three constants of integration which are to be identified with coordinates in (2+1)-dimensional spacetime.

To see how Eq. (1) arises, consider the equation $u = Z(x^a;\varphi)$. For fixed (u,φ), this equation defines a surface $S_{(u,\varphi)}$. The principal requirement of the NSF is that $S_{(u,\varphi)}$ be a *null* surface with respect to some spacetime metric $g_{ab}(x^a)$ and so, for arbitrary values of the parameter φ, the gradient of Z satisfies

$$g^{ab}(x^a) Z_{,a}(x^a;\varphi)\, Z_{,b}(x^a;\varphi) \;=\; 0, \tag{2}$$

where $Z_{,a} \equiv \partial_a Z \equiv \partial Z/\partial x^a$. The NSF uses Eq. (2) and its derivatives with respect to φ to derive the so-called metricity conditions which ensure that this requirement of nullness can be satisfied. The first step in deriving the metricity conditions is to introduce the following coordinates (which are naturally adapted to the surfaces [4–6] and so are usually called *intrinsic* coordinates [2]),

$$\begin{aligned} u \;&\equiv\; \theta^0 \;:=\; Z(x^a;\varphi), \\ \omega \;&\equiv\; \theta^1 \;:=\; u' \;\equiv\; \eth u \;\equiv\; \eth Z(x^a;\varphi), \\ \rho \;&\equiv\; \theta^2 \;:=\; u'' \;\equiv\; \eth^2 u \;\equiv\; \eth^2 Z(x^a;\varphi), \end{aligned}$$

where $\eth := \partial/\partial\varphi$ denotes the derivative with respect to φ when x^a is held fixed. Since the intrinsic coordinate system is assumed to be well behaved, the equations above can in principle be inverted to give

$$x^a \;=\; x^a(u, \omega, \rho, \varphi). \tag{3}$$

Equation (3) and the third derivative, $\eth^3 Z(x^a;\varphi)$, are used to define the function

$$\Lambda(u, \omega, \rho, \varphi) := \eth^3 Z(x^a(u, \omega, \rho, \varphi);\varphi).$$

This is the origin of Eq. (1). The intrinsic coordinates u, ω and ρ are φ-dependent, and it can be shown that the action of the differential operator ∂ on a function $f(u,\omega,\rho,\varphi)$ is given by [4,6]

$$\partial \;=\; \partial' + \omega\,\partial_u + \rho\,\partial_\omega + \Lambda\,\partial_\rho, \tag{4}$$

where ∂' denotes the derivative with respect to φ when u, ω and ρ are held fixed.

It is convenient to define (coordinate) basis triads ${\theta^i}_a$ and their duals ${\theta_j}^a$ by

$${\theta^i}_a := {\theta^i}_{,a}\,, \qquad {\theta^i}_a{\theta_j}^a = \delta^i_j.$$

Using the ∂ operator, repeated differentiation of the null condition, Eq. (2), gives the components of the metric with respect to θ^i, i.e. in the u,ω,ρ coordinates:

$$g^{ij} \;=\; g^{ab}\partial^i Z_{,a}\partial^j Z_{,b} \;\equiv\; g^{ab}{\theta^i}_a{\theta^j}_b.$$

For example, Eq. (2) immediately implies

$$g^{00} \;\equiv\; g^{uu} \;=\; g^{ab}Z_{,a}Z_{,b} \;=\; 0,$$

and

$$g^{01} \;\equiv\; g^{u\omega} \;=\; g^{ab}Z_{,a}\,\partial Z_{,b} \;=\; 0.$$

An overall multiplicative factor can be conveniently extracted by defining

$$\Omega^2 \;:=\; g^{11} \;\equiv\; g^{\omega\omega} \;=\; g^{ab}\partial Z_{,a}\partial Z_{,b},$$

(or, alternatively, $\Omega^2 \;:=\; -\,g^{02} \equiv -g^{u\rho}$). The final result is [4,6]

$$\|g^{ij}\| \;=\; \Omega^2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & \frac{1}{3}\partial_\rho\Lambda \\ -1 & \frac{1}{3}\partial_\rho\Lambda & \frac{1}{3}\partial(\partial_\rho\Lambda) - \frac{1}{9}(\partial_\rho\Lambda)^2 - \partial_\omega\Lambda \end{pmatrix}. \tag{5}$$

Note that, in the present paper, we adopt the $(-++)$ signature convention for the metric g^{ab} and hence, by Eq. (5), for the metric g^{ij}. Both metrics will have negative determinants. Authors, such as Tanimoto [6], who adopt the $(+--)$ signature convention will have g^{ij} opposite in sign to that above and will have positive determinant.

The (inverse) metric of Eq. (5) can be itself inverted to give g_{ij}:

$$\|g_{ij}\| \;=\; \Omega^{-2} \begin{pmatrix} -\frac{1}{3}\partial(\partial_\rho\Lambda) + \frac{2}{9}(\partial_\rho\Lambda)^2 + \partial_\omega\Lambda & \frac{1}{3}\partial_\rho\Lambda & -1 \\ \frac{1}{3}\partial_\rho\Lambda & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{6}$$

or, equivalently and following from Eq. (6):

$$ds^2 = \Omega^{-2}\Big[\Big\{-\tfrac{1}{3}\partial(\partial_\rho\Lambda) + \tfrac{2}{9}(\partial_\rho\Lambda)^2 + \partial_\omega\Lambda\Big\}du^2 + \tfrac{2}{3}\partial_\rho\Lambda\, du d\omega \;-\; 2\,du d\rho \;+\; d\omega^2\Big]. \quad (7)$$

From Eq. (7), one can see that the vector $l^i := (\partial/\partial\rho)^i$ is null. [Of course, this is the vector whose components are $(0,\ 0,\ 1)$, when expressed in terms of intrinsic coordinates].

The metricity conditions derived from Eq. (2) are found to be [4,6]

$$3\,\partial\Omega \;=\; \Omega\,\partial_\rho\Lambda, \quad (8)$$

and

$$2[\partial(\partial_\rho\Lambda) - \partial_\omega\Lambda - \tfrac{2}{9}(\partial_\rho\Lambda)^2]\partial_\rho\Lambda - \partial^2(\partial_\rho\Lambda) + 3\,\partial(\partial_\omega\Lambda) - 6\,\partial_u\Lambda = 0. \quad (9)$$

Equations (8) and (9) ensure that a solution of Eq. (1) will define a null surface with respect to *some* spacetime metric $g_{ab}(x^a)$. Equation (9) is the main metricity condition. Equation (8) merely fixes the φ dependence of Ω. Despite the conformal invariance of the theory, Ω cannot be chosen arbitrarily since Ω^2 must equal g^{11}.

Forni et al [4] have shown that, if the two metricity conditions are satisfied, then the Einstein equations, $G_{ab} = \kappa T_{ab}$, will be satisfied if the following equation holds:

$$\partial^2_\rho\Omega \;=\; \kappa T_{\rho\rho}\Omega. \quad (10)$$

In the present paper, we shall follow Forni et al [4] by including the source term, T_{ab}, and not including the cosmological constant. The analysis of Tanimoto [6] includes the cosmological constant but does not have a source term, and so some of Tanimoto's comments (for example: that Ω is a polynomial of no more than first order in ρ) will not apply to the present paper. To find a solution in the NSF approach, one is required to solve the coupled set of equations, Eqs. (8), (9) and (10), for Λ and Ω and then to use the resulting Λ for Eq. (1), which must then be solved to obtain the desired null surface $u = Z(x^a;\varphi)$. The equations in the NSF appear difficult to solve when compared to those of standard general relativity, and the purpose of the present paper is to find a simple nontrivial solution by solving Eqs. (8) through (10) directly. (In a previous paper [8], the authors found a solution indirectly by considering the light cone cut interpretation of the NSF [9]).

2. Trivial solutions

If $\Lambda = \Lambda(\omega)$ then Eq. (8) implies that Ω is constant. Equation (10) indicates that $T_{\rho\rho} = 0$, i.e. empty space. Equation (9) implies that $\partial_\omega \Lambda$ is constant. There is no loss of generality in choosing $\Omega = 1$ and $\Lambda = -\omega$ to agree with the solution (corresponding to Minkowski spacetime) found by Tanimoto [6].

Alternatively, note that the form of the first bracketed term in Eq. (9) suggests the proposition $\Lambda = -(2/9)\omega + \rho$. It is easily verified that this satisfies Eq. (9) and that Eq. (8) can then be satisfied by choosing $\Omega = -(2/3)\omega + \rho$. Equation (10) then implies $T_{\rho\rho} = 0$, i.e. empty space, again.

3. Nontrivial solution

To move away from the previous trivial solutions, let Λ and Ω depend on ρ. For simplicity, assume that they depend *only* on ρ: $\Lambda = \Lambda(\rho)$ and $\Omega = \Omega(\rho)$. Equation (8), together with Eq. (4), implies

$$3\Lambda \partial_\rho \Omega \;=\; \Omega\, \partial_\rho \Lambda,$$

which leads to $\Omega = \Lambda^{1/3}$.

For further simplicity, assume that Λ takes the particular form $\Lambda = (a+\rho)^k$ where a and k are constants. Equation (9) leads to the quadratic, $(2/9)\,k^2 - k + 1 = 0$, which has solutions $k = 3$ and $k = 3/2$. For the choice $k = 3$, Eq. (10) leads to a trivial solution corresponding again to empty space [$\Lambda = (a+\rho)^3$, $\Omega = a + \rho$, and $T_{\rho\rho} = 0$]. Instead, we shall choose $k = 3/2$. This gives to the solution

$$\Lambda = (a+\rho)^{3/2}, \qquad \Omega = (a+\rho)^{1/2},$$

with the nonzero source term,

$$T_{\rho\rho} = -\frac{1}{4\kappa\,(a+\rho)^2}.$$

The properties of this solution will now be explored.

By Eq. (7), the solution $\Lambda = (a+\rho)^{3/2}$ and $\Omega = (a+\rho)^{1/2}$ corresponds to the metric

$$ds^2 \;=\; (a+\rho)^{-1} \Big[\, \tfrac{1}{4}(a+\rho)\, du^2 \;+\; (a+\rho)^{1/2}\, du d\omega \;-\; 2\, du d\rho \;+\; d\omega^2 \,\Big]. \qquad (11)$$

Further details concerning the metric of Eq. (11) and the resulting curvatures are given in the appendix. Note that in 2+1 dimensions there are three independent curvature scalars [10]: R, $R_{ab}R^{ab}$ and $\det\|R_{ab}\| / \det\|g_{ab}\|$. They are found to be

$$R = \frac{1}{32}, \qquad R_{ab}R^{ab} = \frac{3}{1024}, \qquad \frac{\det\|R_{ab}\|}{\det\|g_{ab}\|} = -\left(\frac{1}{32}\right)^3.$$

The null surface formulation of general relativity does not distinguish between conformally related spacetimes, and so a conformally flat spacetime would be an uninteresting example. Since the Weyl tensor is identically zero in 2+1 dimensions, it cannot be used to test for conformal flatness. Its role is played by the Cotton-York tensor [11-13],

$$C^a{}_b = \varepsilon^{acd}(R_{db} - \tfrac{1}{4}g_{db}R)_{;c}$$

which is identically zero if and only if the 2+1 spacetime is conformally flat. For the spacetime being considered, one finds that the Cotton-York tensor, $C^a{}_b$, has some nonzero components,

$$C^u{}_u = -\frac{1}{64}, \qquad C^\omega{}_\omega = C^\rho{}_\rho = \frac{1}{128},$$

$$C^u{}_\omega = -\frac{3}{64}(a+\rho)^{-1/2}, \qquad C^u{}_\rho = \frac{3}{32}(a+\rho)^{-1},$$

and so the spacetime is not conformally flat.

If the source of the gravitational field is assumed to be a fluid, then it is appropriate to introduce a timelike vector U^a: $U_aU^a = -1$. We introduce $U^a = (2, 0, (a+\rho)/2)$, so that $U_a = (0, (a+\rho)^{-1/2}, -2(a+\rho)^{-1})$. Thus

$$U^u = 2, \qquad U^\omega = 0, \qquad U^\rho = \frac{1}{2}(a+\rho),$$

and

$$U_u = 0, \qquad U_\omega = (a+\rho)^{-1/2}, \qquad U_\rho = -2(a+\rho)^{-1}.$$

The scalar expansion is defined by $\theta := U^a{}_{;a}$, and is found to be nonzero: $\theta = -1/4$. The nature of the fluid is unclear since the possibility of its being a perfect fluid,

$$T_{ab} = (\mu + p)U_aU_b + p\,g_{ab},$$

can quickly be eliminated, and further investigation shows that requiring the source to be an imperfect fluid would contradict the usual proportionality relationship between the anisotropic pressure (dynamic viscosity) tensor π_{ab} and the shear tensor σ_{ab}.

However, in 2+1 dimensions, the Einstein equations, $G_{ab} = \kappa\, T_{ab}$, are sometimes replaced by the Einstein-Cotton field equations of topologically massive gravity [13-15]. This generalization allows for gravitational excitations (which are absent in the 2+1 Einstein theory). Thus topologically massive gravity is often regarded as a more realistic 2+1 analog of standard (3+1)-dimensional general relativity. The most general form of topologically massive gravity includes the cosmological constant λ, and the field equations are

$$G_{ab} \; + \; \lambda\, g_{ab} \; + \; \frac{1}{m} C_{ab} \; = \; \kappa\, T_{ab}. \tag{12}$$

The constant m can take either sign. (In fact, in 2+1 dimensions, this is also true for κ). It is straightforward to show that the metric under consideration satisfies the field equations of topologically massive gravity [i.e. Eq. (12)] for a perfect fluid source with constant μ and p. Specifically:

$$m \; = \; -\,3/8, \qquad \mu \; = \; -\,p, \qquad p \; = \; \frac{1}{\kappa}\left(\lambda \; - \; \frac{1}{192}\right).$$

The most interesting case comes from choosing $\lambda = 1/192$. This gives a topologically massive gravity solution analogous to the regular de Sitter solution: a vacuum solution with nonzero cosmological constant and nonzero expansion θ.

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Appendix. Properties of solution with $\Lambda = (a+\rho)^{3/2}$ and $\Omega = (a+\rho)^{1/2}$

The metric corresponding to $\Lambda = (a+\rho)^{3/2}$ and $\Omega = (a+\rho)^{1/2}$ is

$$\begin{aligned} ds^2 \;\; &= \;\; (a+\rho)^{-1} \Big[\, \tfrac{1}{4}(a+\rho)\, du^2 \\ & \qquad + \; (a+\rho)^{1/2}\, dud\omega \; - \; 2\, dud\rho \; + \; d\omega^2 \, \Big]. \end{aligned}$$

Its determinant is $\det \|g_{ab}\| = -(a+\rho)^{-3}$. The Christoffel symbols are as follows.

$$\Gamma^u{}_{u\omega} \; = \; -\tfrac{1}{8}(a+\rho)^{-1/2}, \qquad \Gamma^u{}_{\omega\omega} \; = \; -\,\tfrac{1}{2}(a+\rho)^{-1}, \qquad \Gamma^\omega{}_{u\omega} \; = \; \tfrac{1}{16},$$

$$\Gamma^{\omega}{}_{u\rho} = -\tfrac{1}{8}(a+\rho)^{-1/2}, \qquad \Gamma^{\rho}{}_{u\rho} = -\tfrac{1}{16}, \qquad \Gamma^{\omega}{}_{\omega\omega} = \tfrac{1}{4}(a+\rho)^{-1/2},$$

$$\Gamma^{\omega}{}_{\omega\rho} = -\tfrac{1}{2}(a+\rho)^{-1}, \qquad \Gamma^{\rho}{}_{\omega\rho} = -\tfrac{1}{8}(a+\rho)^{-1/2}, \qquad \Gamma^{\rho}{}_{\rho\rho} = -(a+\rho)^{-1}.$$

The components of the Ricci tensor are

$$R_{uu} = -\frac{1}{128}, \qquad R_{u\omega} = -\frac{1}{64}(a+\rho)^{-1/2}, \qquad R_{u\rho} = \frac{1}{32}(a+\rho)^{-1},$$

$$R_{\omega\omega} = -\frac{1}{32}(a+\rho)^{-1}, \qquad R_{\omega\rho} = \frac{1}{8}(a+\rho)^{-3/2}, \qquad R_{\rho\rho} = -\frac{1}{4}(a+\rho)^{-2}.$$

The determinant of the Ricci tensor is $\det\|R_{ab}\| = [32(a+\rho)]^{-3}$. The scalar curvature is $R = 1/32$, whence the components of the Einstein tensor $G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$ are found to be

$$G_{uu} = -\frac{3}{256}, \qquad G_{u\omega} = -\frac{3}{128}(a+\rho)^{-1/2}, \qquad G_{u\rho} = \frac{3}{64}(a+\rho)^{-1},$$
$$G_{\omega\omega} = -\frac{3}{64}(a+\rho)^{-1}, \qquad G_{\omega\rho} = \frac{1}{8}(a+\rho)^{-3/2}, \qquad G_{\rho\rho} = -\frac{1}{4}(a+\rho)^{-2}.$$

In order to test for conformal flatness, the components of the Cotton-York tensor, C_{ab}, were computed. The fact that C_{ab} was found to be nonzero indicates that the spacetime is *not* conformally flat. Note that, regardless of the chosen metric, the Cotton-York tensor is traceless: $C^{a}{}_{a} \equiv C^{u}{}_{u} + C^{\omega}{}_{\omega} + C^{\rho}{}_{\rho} = 0$. The components of C_{ab} are

$$C_{uu} = -\frac{1}{256}, \qquad C_{u\omega} = -\frac{1}{128}(a+\rho)^{-1/2}, \qquad C_{u\rho} = \frac{1}{64}(a+\rho)^{-1},$$

$$C_{\omega\omega} = -\frac{1}{64}(a+\rho)^{-1}, \qquad C_{\omega\rho} = \frac{3}{64}(a+\rho)^{-3/2}, \qquad C_{\rho\rho} = -\frac{3}{32}(a+\rho)^{-2}.$$

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