Exotic (dark) eigenspinors of the charge conjugation operator and cosmological applications

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Abstract. We report about some achievements and developments provided by the ELKO program, in particular the ones recently accomplished [1]. Exotic dark spinor fields has been investigated in the context of inequivalent spin structures on arbitrary curved spacetimes, which induces an additional term on the associated Dirac operator, related to a Čech cohomology class. It implies that the non-trivial topology associated to the spacetime can drastically engender — from the dynamics of ELKO dark spinor fields — constraints on the spacetime metric structure.

1. ELKO (Dark) Spinor Fields

An ELKO Ψ without loss of generality can be written as $\Psi(p) = \lambda(\mathbf{p})e^{\pm ip\cdot x}$ where $\lambda(\mathbf{p}) = \binom{i\Theta\phi^*(\mathbf{p})}{\phi(\mathbf{p})}$, and given the rotation generators $\mathfrak{J}$, the Wigner's spin-1/2 time reversal operator Θ satisfies $\Theta\mathfrak{J}\Theta^{-1} = -\mathfrak{J}^*$. Hereon the Weyl representation of γ^μ is used. ELKO spinor fields are eigenspinors of the charge conjugation operator C defined here by its action $C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})$, for $C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K$. The operator K $\mathbb{C}$-conjugates 2-component spinor fields appearing on the right. The plus sign stands for self-conjugate spinor fields, $\lambda^S(\mathbf{p})$, while the minus yields anti self-conjugate spinor fields, $\lambda^A(\mathbf{p})$. Explicitly, the complete form of ELKO can be found by solving the equation of helicity $(\sigma\cdot\widehat{\mathbf{p}})\phi^\pm = \pm\phi^\pm$ in the rest frame and subsequently make a boost, to recover the result for any $\mathbf{p}$.[2] Here $\widehat{\mathbf{p}} := \mathbf{p}/\|\mathbf{p}\|$. The boosted four spinor fields results in

$$\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1\mp\frac{p}{E+m}\right)\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}), \text{ for } \lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) = \begin{pmatrix} \pm i\Theta[\phi^{\pm}(\mathbf{0})]^{*} \\ \phi^{\pm}(\mathbf{0}) \end{pmatrix}. \tag{1}$$

As $\Theta[\phi^{\pm}(\mathbf{0})]^{*}$ and $\phi^{\pm}(\mathbf{0})$ have opposite helicities, ELKO cannot be an eigenspinor field of the helicity operator.

2. ELKO Dynamics: Exotic Spin Structure

To observe that dark spinor fields are a natural probe of the non-trivial topology one should firstly notice that such exotic spinor fields are parallel transported like standard spinor fields. Meanwhile, an outstanding property distinguishes both kinds of spinor fields: the covariant derivative acting on these exotic spinor fields changes by an additional one-form field that is manifestation of the non-trivial topology. The exotic structure endows the Dirac operator with an additional term $\frac{1}{2\pi i}\xi^{-1}(x)d\xi(x)$, which is real and closed, but not exact, and defines an integer cohomology class in the Čech sense. Using the relation between Čech and de Rham cohomologies, the integral of $\frac{1}{2\pi i}\xi^{-1}(x)d\xi(x)$ around any closed curve is an integer. The point is that for dark ELKO spinor fields such an exotic term cannot be absorbed by an external gauge field representing an element of $H^1(M,\mathbb{Z}_2)$, inasmuch as ELKO fields cannot carry gauge charges.

In addition to the ELKO spinor fields $\lambda(x)$ — that was indeed defined as sections in the bundle $P_{\mathrm{Spin}^e_{1,3}}(M)\times_\rho\mathbb{C}^4$, in Section 2 — one can get a second type of ELKO $\mathring{\lambda}(x)$, which can be described by sections in the inequivalent spin structure-induced spinor bundle $\mathring{P}_{\mathrm{Spin}^e_{1,3}}(M)\times_\rho\mathbb{C}^4$, with a variation of the covariant derivative, given by

$$i\gamma^{\mu}\mathring{\nabla}_{\mu} = i\gamma^{\mu}\nabla_{\mu} + \xi^{-1}(x)d\xi(x). \tag{2}$$

The exotic Dirac equation is given then by

$$(i\gamma^{\mu}\nabla_{\mu}+(\xi^{-1}(x)\,d\xi(x))-m\mathbb{I})\psi(x) = 0, \qquad \text{where } \psi \text{ denotes a Dirac spinor field.}$$

The exotic Dirac spinor fields are annihilated by $\left(i\gamma^{\mu}\nabla_{\mu} + (\xi^{-1}(x)\,d\xi(x)) \pm m\mathbb{I}\right)$ where the plus and minus signs stands respectively for particles and antiparticles. Hereon we denote $\xi^{-1}(x)\,d\xi(x)$ by $a(x)$ in order to shorten all formulæ notations.

ELKO spinor field can not be eigenspinors of the exotic Dirac operator $i\gamma^{\mu}\nabla_{\mu} + a(x)$. Namely, the mass terms carry opposite signs and consequently ELKO cannot be annihilated by $(i\gamma^{\mu}\nabla_{\mu} + a(x) \pm m\mathbb{I})$, because the term $\varepsilon^{\beta}_{\alpha}$ which implies that $\epsilon^{\mathrm{S}} = -1$ and $\epsilon^{\mathrm{A}} = +1$.

Furthermore, as comprehensively discussed we can express $\xi(x) = \exp(i\theta(x)) \in \mathrm{U}(1)$. The exotic spin structure term in this way reads

$$\xi^{-1}(x)d\xi(x) = \exp(-i\theta(x))(i\gamma^\mu\nabla_\mu\theta(x))\exp(i\theta(x)) = i\gamma^\mu\partial_\mu\theta(x). \tag{3}$$

The exotic Dirac operator $i\gamma^\mu\nabla_\mu + i\gamma^\mu\partial_\mu\theta - m\mathbb{I}$, annihilates each of the four exotic Dirac spinor fields does not annihilate ELKO.

Much has been extensively discussed about the subtle differences between Majorana and ELKO spinor fields. Both in the Lounesto spinor field classification are type-(5) spinor fields. It would be useful to discuss whether the exotic Dirac operator can be considered as a square root of the Klein–Gordon operator. This feature must remain true for the ELKO and its exotic partner:

$$((i\gamma^\mu\nabla_\mu + a(x))\delta_\alpha^\beta \pm m\mathbb{I}\varepsilon_\alpha^\beta)((i\gamma^\mu\nabla_\mu + a(x))\delta_\alpha^\beta \mp m\mathbb{I}\varepsilon_\alpha^\beta) = (g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\mathbb{I}\,\delta_\alpha^\beta \tag{4}$$

since the introduction of an exotic spin structure does not modify the Klein–Gordon propagator fulfillment by dark spinor fields.

The corresponding Klein-Gordon equation is given by

$$(\Box + m^2 + g^{\mu\nu}\nabla_\mu\nabla_\nu\theta + \partial^\mu\theta\nabla_\mu + \partial^\mu\theta\partial_\mu\theta)\mathring{\lambda}(x)^{S/A}_{\{\pm,\mp\}} = 0, \tag{5}$$

In order that the Klein-Gordon propagator for the exotic ELKO remains the same it follows that

$$(\Box\theta + \partial^\mu\theta\nabla_\mu + \partial^\mu\theta\partial_\mu\theta)\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x) = 0. \tag{6}$$

Since Eq.(6) holds for every exotic dark spinor field $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x)$, in particular let us analyze the solutions of Eq.(6) applied to, for instance, $\mathring{\lambda}^{S}_{\{-,+\}}(x)$. Using the expression

$$\nabla_\mu\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} = \partial_\mu\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} - \frac{1}{4}\Gamma_{\mu\rho\sigma}\gamma^\rho\gamma^\sigma\mathring{\lambda}^{S/A}_{\{\mp,\pm\}}, \tag{7}$$

for such case, after some calculation it follows that

$$\begin{aligned}
&(\Box\theta)\,\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} + (\partial_0\theta)\Big[\partial_0\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} - \frac{1}{4}\Big((\Gamma_{000} - \Gamma_{0jj})\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} + i\Gamma_{001}\mathring{\lambda}^{A/S}_{\{\pm,\mp\}} \\
&+\Gamma_{002}\mathring{\lambda}^{S/A}_{\{\pm,\mp\}} \mp \Gamma_{003}\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} \pm i\Gamma_{012}\mathring{\lambda}^{A/S}_{\{\mp,\pm\}} + i\Gamma_{013}\mathring{\lambda}^{A/S}_{\{\pm,\mp\}} \mp \Gamma_{023}\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}\Big)\Big] \\
&-g^{00}(\partial_0\theta)^2\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} = 0
\end{aligned} \tag{8}$$

where $\Gamma_{0jj} = \Gamma_{011} + \Gamma_{022} + \Gamma_{033}$. The equation above couples *again* all the four exotic spinor fields $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}$, in the case of spacetimes which the associated connection are non zero.

It is possible for cosmological applications to assume that the dark spinor fields depend only on the time variable t via a matter field $\kappa(t)$ compatible with homogeneity and isotropy and acts as the only dynamical cosmological variable, in such a way that $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x)$ can be explicitly written as

$$\mathring{\lambda}^{A/S}_{\{-,+\}}(x) = \kappa(t)\,\chi^{A/S}_{\{-,+\}}, \qquad \mathring{\lambda}^{A/S}_{\{+,-\}}(x) = \kappa(t)\,\zeta^{A/S}_{\{+,-\}}, \tag{9}$$

where $\zeta^{S/A}$ and $\chi^{S/A}$ are linearly independent constant spinor fields

$$\chi^{S}_{\{-,+\}} = \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \quad \chi^{A}_{\{-,+\}} = \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}, \quad \zeta^{S}_{\{+,-\}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}, \quad \zeta^{A}_{\{+,-\}} = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix} \tag{10}$$

The matter field $\kappa(t)$ satisfies $\frac{\dot{\kappa}}{\kappa} = -\frac{1}{3}\sqrt{\frac{1}{3M_{\rm Pl}^2}\Sigma_{tt}} + \mathcal{O}(\kappa^4)$, where $M_{\rm Pl}^{-2} = 8\pi G$ is the coupling constant, and total energy-momentum tensor is denoted by Σ_{tt}, involving also the Planck mass, and the Hubble constant. Therefore we can write $\kappa(t) = \exp(\tilde{a}t)$, where $\tilde{a}$ is the constant given in the equation above. Using now Eqs.(9) and (10) we have $(\partial_0\theta)\Gamma_{012} = 0$, what means that if θ is time dependent, it necessarily means that $\Gamma_{012} = 0$. Otherwise, in the case where θ does not depends on time, it implies that $\partial_0\theta = 0$, and then we obtain the Laplace equation $\nabla^2\theta = 0$. The dark spinor field dynamics can thus be used to probe the topological sector determined by θ. In addition, the above obtained second and third equations imply that

$$\Box\theta + (\partial_0\theta)\left(1 - \frac{1}{2}(\Gamma_{000} - \Gamma_{011} - \Gamma_{022} - \Gamma_{033} - \Gamma_{003})\right) - (\partial_0\theta)^2 = 0, \tag{11}$$

what means that if $\theta = \theta(t)$, so necessarily $4 - (\Gamma_{000} - \Gamma_{011} - \Gamma_{022} - \Gamma_{033}) = \partial_0\theta$.

Acknowledgments

R. da Rocha is grateful to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 472903/2008-0 and 304862/2009-6 for financial support.

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