Spherical Black Holes in Quadratic Gravity

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Quadratic gravity: $S = \int \mathrm{d}^4x\,\sqrt{-g}\,\Big(\gamma\,(R-2\Lambda) + \beta\,R^2 - \alpha\,C_{abcd}\,C^{abcd}\Big)$

Fully general quadratic gravity in $D=4$:

$$\gamma\big(R_{ab} - \tfrac{1}{2}R\,g_{ab} + \Lambda\,g_{ab}\big) - 4\alpha\,B_{ab} + 2\beta\big(R_{ab} - \tfrac{1}{4}R\,g_{ab} + g_{ab}\,\Box - \nabla_b\nabla_a\big)R = 0$$

Employing assumption: $R = const.$

$$\gamma\big(R_{ab} - \tfrac{1}{2}R\,g_{ab} + \Lambda\,g_{ab}\big) - 4\alpha\,B_{ab} + 2\beta\big(R_{ab} - \tfrac{1}{4}R\,g_{ab}\big)R = 0$$

The trace: using $g^{ab}B_{ab} = 0$

$$R = 4\Lambda$$

Substituting the trace:

$$(\gamma + 8\beta\Lambda)\,(R_{ab} - \Lambda\,g_{ab}) = 4\alpha\,B_{ab}$$

Two independent subclasses:

- $\gamma + 8\beta\Lambda = 0$: new solutions via $B_{ab} = \Omega^{-2}\,B^{\mathrm{seed}}_{ab}$ preserving $R = const$ with $B^{\mathrm{seed}}_{ab} = 0$
  for more details see [Pravda et al. 2017]
- $\gamma + 8\beta\Lambda \neq 0$: our aim of study: we can introduce $k \equiv \frac{\alpha}{\gamma + 8\beta\Lambda}$

What is the most suitable metric ansatz?

Standard spherically symmetric line element:

$$ds^2 = -h(\bar{r})\,dt^2 + \frac{d\bar{r}^2}{f(\bar{r})} + \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

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An alternative metric form:

$$ds^2 = \Omega^2(r)\left[\,d\theta^2 + \sin^2\theta\,d\phi^2 - 2\,du\,dr + \mathcal{H}(r)\,du^2\,\right]$$

- coordinates and metric functions related via

$$\bar{r} = \Omega(r) \qquad t = u - \int \mathcal{H}(r)^{-1} dr \qquad h(\bar{r}) = -\Omega^2\,\mathcal{H} \qquad f(\bar{r}) = -\left(\frac{\Omega'}{\Omega}\right)^2 \mathcal{H}$$

- $ds^2 = \Omega^2\,ds^2_{\text{Kundt}}$: conformal to a type D direct-product Kundt 'seed' [Pravda et al. 2017]

Field equations: classic metric ansatz

Spherically symmetric line element:

$$ds^2 = -h(\bar{r})\,dt^2 + \frac{d\bar{r}^2}{f(\bar{r})} + \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

Field equations:

$$R_{ab} - \Lambda\,g_{ab} = 4k\,B_{ab}$$

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Field equations:

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$\bar{r}\,\bar{r}$- and $t\,t$-component of the field equations:

$$\begin{aligned}
-4\Lambda h^2 r - 2hfh''r - hf'h'r + fh^2 r - 4f'h^2 = -\frac{4}{3}\frac{1}{r^3h^2}\Big(k\Big(r^3f^2h^2(h'r-2h)h''' - \frac{1}{2}h^2f^2h'^2r^4 - \frac{3}{2}\Big(fh^2r^2 - \frac{2}{3}hr(f'r+6f)h' \\
+ \frac{4}{3}h^2(f'r+2f)\Big)fhr^2h'' + \frac{1}{2}h^2fr^2(h'r-2h)^2f'' + \frac{7}{8}f^2h^4r^4 - \frac{3}{4}\Big(f'r+\frac{10}{3}f\Big)fhr^3h^3 - \frac{1}{8}r^2h^2(f'^2r^2 - 16rf'f + 4f^2)h^2 \\
+ \frac{1}{2}rh^3(f'^2r^2 - 2rf'f + 8f^2)h' - \frac{1}{2}h^4(f'^2r^2 + 4f^2 - 4)\Big)\Big)
\end{aligned}$$

$$\begin{aligned}
2hfh''r - fh^2r + h(f'r+4f)h' + 4\Lambda h^2 r = -\frac{1}{r^3h^2}\Big(8k\Big(-\frac{1}{3}h^3f^2h''''r^4 + \Big(h'fr - \Big(f'r+\frac{4}{3}f\Big)h\Big)fh^2r^3h''' - \frac{1}{6}r^3fh^3(h'r-2h)f''' \\
+ \frac{3}{4}h^2f^2h'^2r^4 - \frac{29}{12}\Big(\frac{8}{29}h^2ff''r + f^2h^2r - \frac{27}{29}fh\Big(f'r+\frac{26}{27}f\Big)h' + \frac{3}{29}f'h^2\Big(f'r+\frac{26}{3}f\Big)\Big)hr^3h'' + \frac{1}{2}\Big(fh^2r^2 - \frac{1}{6}hr(f'r \\
+ 6f)h' + \frac{1}{3}h^2(f'r+2f)\Big)h^2r^2f'' + \frac{49}{48}f^2h^4r^4 - \frac{29}{24}f\Big(f'r+\frac{22}{29}f\Big)hr^3h^3 + \frac{3}{16}\Big(f'^2r^2 + \frac{16}{3}rf'f - \frac{20}{9}f^2\Big)h^2r^2h^2 \\
- \frac{1}{12}h^3f'r^2(f'r-10f)h' - \frac{1}{12}h^4(f'^2r^2 + 8rf'f - 4f^2 + 4)\Big)\Big)
\end{aligned}$$

Field equations: conformal to Kundt metric ansatz

An alternative form of spherically symmetric line element:

$$\mathrm{d}s^2 = \Omega^2(r)\left[\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 - 2\,\mathrm{d}u\,\mathrm{d}r + \mathcal{H}(r)\,\mathrm{d}u^2\,\right]$$

Field equations:

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rr-, ru- and $\phi\phi$-component of the field equations:

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Field equations:

$$R_{ab} - \Lambda\, g_{ab} = 4k\, B_{ab}$$

rr-, *ru*- and $\phi\phi$-component of the field equations:

$$\Omega''\Omega - 2\,\Omega'^2 = \frac{1}{3}\,k\,H''''$$

$$2\,\Lambda\,\Omega^4 - H''\Omega^2 - 2\,\Omega''H\,\Omega - 2\,\Omega'^2 H - 4\,\Omega' H'\Omega = -\frac{1}{3}\,k\left(H''^2 - 2\,H'H''' - 4\,H\,H'''' - 4\right)$$

$$-\Lambda\,\Omega^4 + \Omega'' H\,\Omega + \Omega'^2 H + \Omega' H'\Omega + \Omega^2 = -\frac{1}{6}\,k\left(H''^2 - 2\,H'H''' - 2\,H\,H'''' - 4\right)$$

The trace:

$$-4\,\Lambda\,\Omega^3 + \Omega\,H'' + 6\,H\,\Omega'' + 6\,H'\Omega' + 2\,\Omega = 0$$

Moreover, we employ the **Bianchi identities**

Simplifying the system of field equations

Three non-trivial equations for **two unknown functions** $\Omega(r)$ and $\mathcal{H}(r)$

Define: the **auxiliary tensor**

$$J_{ab} \equiv R_{ab} - \tfrac{1}{2}R\, g_{ab} + \Lambda g_{ab} - 4k\, B_{ab}$$

- the field equations are $J_{ab} = 0$
- the **Bianchi identities** $R_{ab}{}^{;b} = \frac{1}{2}\,R_{;a}$ and the Bach tensor property $B_{ab}{}^{;b} = 0$:

$$J_{ab}{}^{;b} \equiv 0$$

i.e.

$$J_{rb}{}^{;b} = -\Omega^{-3}\Omega'\big(J_{ij}\, g^{ij} + \mathcal{H}J_{rr}\big) - \Omega^{-2}\big(\mathcal{H}J_{rr,r} + J_{ru,r} + \tfrac{3}{2}\mathcal{H}'J_{rr}\big) \equiv 0$$

$$J_{ub}{}^{;b} = -2\Omega^{-3}\Omega'\big(J_{uu} + \mathcal{H}J_{ru}\big) - \Omega^{-2}\big(J_{uu} + \mathcal{H}J_{ru}\big)_{,r} \equiv 0$$

$$J_{ib}{}^{;b} = \Omega^{-2}J_{ik||l}\, g^{kl} \equiv 0$$

Direct calculation: $J_{uu} = -\mathcal{H}J_{ru}$ and $J_{kl} = \mathcal{J}(r)\, g_{kl}$

The first identity: $\Omega(r)$ and $\mathcal{H}(r)$ satisfy $J_{rr} = 0 = J_{ru} \;\Rightarrow\; J_{ij}\, g^{ij} \equiv 0 \;\Rightarrow\; \mathcal{J}(r) = 0$

Field equations: autonomous system of ODEs

Evaluation of the field equations yields two ODEs for $\Omega(r)$ and $\mathcal{H}(r)$:

$$\Omega\Omega'' - 2\Omega'^2 = \tfrac{1}{3} k\, \mathcal{B}_1 \mathcal{H}^{-1}$$

$$\Omega\Omega'\mathcal{H}' + 3\Omega'^2\mathcal{H} + \Omega^2 - \Lambda\Omega^4 = \tfrac{1}{3} k\, \mathcal{B}_2$$

where

$$\mathcal{B}_1 \equiv \mathcal{H}\mathcal{H}'''' \qquad \mathcal{B}_2 \equiv \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2$$

- equations do not explicitly depend on r – **autonomous system**
- solutions can be expressed as **power series** in r expanded around **any** point r_0

$$\Omega(r) = \Delta^n \sum_{i=0}^{\infty} a_i\, \Delta^i \qquad\qquad \mathcal{H}(r) = \Delta^p \sum_{i=0}^{\infty} c_i\, \Delta^i$$

with $\Delta \equiv r - r_0, \quad n, p \in \mathbb{R} \quad$ and $\quad a_0,\, c_0 \neq 0$

- asymptotic expansions as $r \to \infty$ in **negative powers** of r

$$\Omega(r) = r^N \sum_{i=0}^{\infty} A_i\, r^{-i} \qquad\qquad \mathcal{H}(r) = r^P \sum_{i=0}^{\infty} C_i\, r^{-i}$$

with $N, P \in \mathbb{R} \quad$ and $\quad A_0,\, C_0 \neq 0$

Example: dominant powers of $\triangle$

The first field equation:

$$\sum_{l=2n-2}^{\infty} \Delta^l \sum_{i=0}^{l-2n+2} a_i\, a_{l-i-2n+2}\,(l-i-n+2)(l-3i-3n+1)$$

$$= \tfrac{1}{3}k \sum_{l=p-4}^{\infty} \Delta^l\, c_{l-p+4}\,(l+4)(l+3)(l+2)(l+1)$$

- coefficients with the same powers of Δ^l on both sides relates c_j in terms of (products of) a_j
- the lowest order terms ($l = 2n-2$ and $l = p-4$) give

$$-a_0^2\, n(n+1)\,\Delta^{2n-2} = \tfrac{1}{3}k\, c_0\, p(p-1)(p-2)(p-3)\,\Delta^{p-4}$$

  i.e. **three distinct possibilities** w.r.t. $2n-2 \lesseqqgtr p-4$

- employing the second field equation (and/or trace) restricts $[n,\, p]$ and Λ:

n	0	0	1	-1	-1	0	0	< 0
p	1	0	0	2	0	2	≥ 2	$2n+2$
Λ	any	any	any	0	$\neq 0$	$\neq 0$	$\frac{3}{8k}$	$\frac{11n^2+6n+1}{1-4n^2}\,\frac{3}{8k}$

Generic solutions to QG and the Einstein–Weyl theory

We focus on the $\Lambda = 0$ case (for simplicity): $\Omega(r)$ and $\mathcal{H}(r)$

- the power series expanded around any constant value r_0

Class $[n, p]$	Family (s, t)	Interpretation
$[-1, 2]$	$(0, 0)^{\infty}$	Schwarzschild BH
$[0, 1]$	$(-1, 1)_{\bar{r}_0}$	Schwarzschild–Bach BH (near the horizon)
$[0, 0]$	$(0, 0)_{\bar{r}_0}$	generic solution (including the Sch–B BH and wormholes)
$[1, 0]$	$(2, 2)_0$	Bachian singularity (near the singularity)

- the power series expanded as $r \to \infty$

Class $[N, P]^{\infty}$	Family (s, t)	Interpretation
$[-1, 3]^{\infty}$	$(1, -1)_0$	Schwarzschild–Bach BH (near the singularity)
$[-1, 2]^{\infty}$	$(0, 0)_0$	Bachian vacuum (near the origin)

How to obtain this physical interpretation?

Example: $[n=0,\, p=1]$ class – gauge fixing

Coefficients of the solution:

$$a_1 = -\tfrac{a_0}{3c_0}\,(1+c_1) \qquad c_2 = \tfrac{1}{6kc_0}\left[a_0^2(2-c_1)+2k(c_1^2-1)\right] \qquad \text{and for } l \geq 2$$

$$a_l = \frac{1}{l^2 c_0}\left[-\tfrac{1}{3}\,a_{l-1} - \sum_{i=1}^{l} c_i\, a_{l-i}\left(l(l-i)+\tfrac{1}{6}i(i+1)\right)\right]$$

$$c_{l+1} = \frac{3}{k(l+2)(l+1)l(l-1)}\sum_{i=0}^{l-1} a_i\, a_{l-i}(l-i)(l-1-3i)$$

... **free parameters** $a_0,\ c_0,\ c_1$

On the horizon: $\mathcal{B}_1(r_h)=0 \qquad \mathcal{B}_2(r_h) = -\frac{1}{k}a_0^2\,(c_1-2)$ define: $b \equiv \frac{1}{3}(c_1-2)$

- for $b=0$ the Bach tensor vanishes **everywhere**
- **Schwarzschild** spacetime

$$\Omega(r) = -r^{-1} = \bar{r} \qquad \text{and} \qquad \mathcal{H}(r) = \left(r-r_h\right)\frac{r^2}{r_h}$$

- corresponding to the **gauge** fixing: $a_0 = -{r_h}^{-1}$ and $c_0 = r_h$

We have the **Schwarzschild background!**

Example: $[n=0,\ p=1]$ class – BH with the Bach extension

With 'background' gauge choice: we set $b \neq 0$

$$\Omega(r) = -\frac{1}{r} - \frac{b}{r_h} \sum_{i=1}^{\infty} \alpha_i \left(1 - \frac{r}{r_h}\right)^i$$

$$\mathcal{H}(r) = (r - r_h)\left[\frac{r^2}{r_h} + 3b\, r_h \sum_{i=1}^{\infty} \gamma_i \left(\frac{r}{r_h} - 1\right)^i\right]$$

where $\alpha_0 \equiv 0\,, \quad \alpha_1 \equiv 1\,, \quad \gamma_1 = 1\,, \quad \gamma_2 = \frac{1}{3}\left(4 - \frac{1}{2kr_h^2} + 3b\right)$

For $l \geq 2$: coefficients $\alpha_l,\ \gamma_{l+1}$ **recursively** given by

$$\alpha_l = \frac{1}{l^2}\Big[-\alpha_{l-2}(l-1)^2 + \alpha_{l-1}\big[\tfrac{1}{3} + 2\big(l(l-1) + \tfrac{1}{3}\big)\big] - 3\sum_{i=1}^{l} (-1)^i\, \gamma_i\, (1 + b\,\alpha_{l-i})\big(l(l-i) + \tfrac{1}{6}i(i+1)\big)\Big]$$

$$\gamma_{l+1} = \frac{(-1)^l}{kr_h^2\,(l+2)(l+1)l(l-1)} \sum_{i=0}^{l-1} [\alpha_i + \alpha_{l-i}(1 + b\,\alpha_i)]\,(l-i)(l-1-3i)$$

- **two-parameter family of spherically symmetric black holes** (with static regions)
- cosmological constant Λ can be added

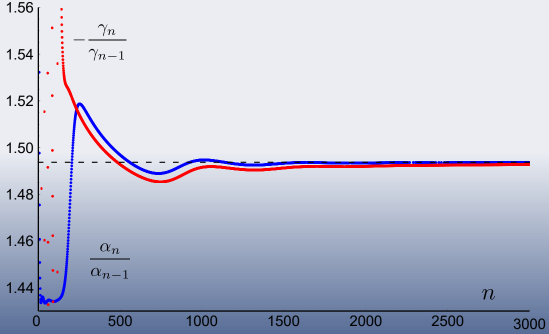
Example: $[n=0,\ p=1]$ class – invariants and convergence

The **scalar invariants** on the horizon: in terms of **two physical parameters**

$$C_{abcd}\, C^{abcd}(r_h) = 12\,(1+b)^2\, r_h^4 \qquad B_{ab}\, B^{ab}(r_h) = \frac{r_h^4}{4k^2}\, b^2$$

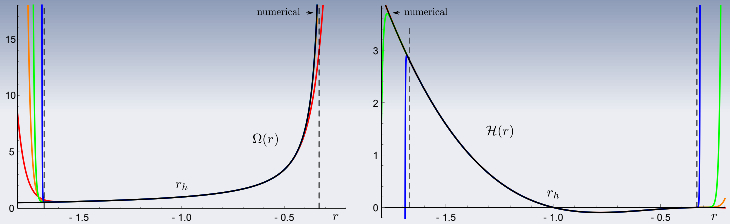
Convergence of the series: d'Alembert ratio test

- with n growing: the ratio between two subsequent terms approaches a specific constant
- the series thus **asymptotically behave as geometric series**



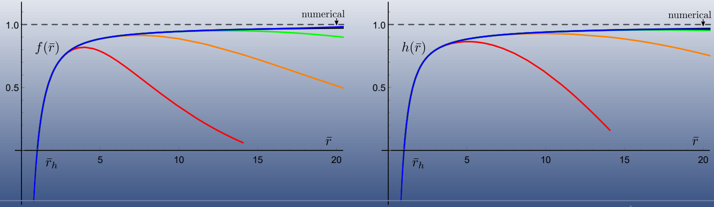
Expansion around horizon: Schwa-Bach-(A)dS black holes ($b \neq 0$)

Typical behaviour of the metric functions $\mathcal{H}(r)$ and $\Omega(r)$:



(first 20 (red), 50 (orange), 100 (green), 500 (blue) terms in the expansions; agreement with the numerical simulation up to the dashed lines)

The metric functions $f(\bar{r})$ and $h(\bar{r})$ of the standard spherically symmetric line element:



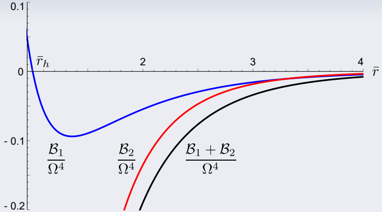
Tidal effects – the Jacobi equation: $\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\,\tau^2} = R^\mu{}_{\alpha\beta\nu}\, u^\alpha u^\beta Z^\nu$

The Bach components $\mathcal{B}_1, \mathcal{B}_2$ observable via a **specific relative motion of free test particles**:

- invariant description: an orthonormal frame associated with **initially static observer**
- projection of the equation of geodesic deviation onto the frame

$$\ddot{Z}^{(1)} = \frac{\Lambda}{3} Z^{(1)} + \frac{1}{6}\frac{\mathcal{H}'' + 2}{\Omega^2} Z^{(1)} - \frac{k}{3}\frac{\mathcal{B}_1 + \mathcal{B}_2}{\Omega^4} Z^{(1)}$$

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3} Z^{(i)} - \frac{1}{12}\frac{\mathcal{H}'' + 2}{\Omega^2} Z^{(i)} - \frac{k}{6}\frac{\mathcal{B}_1}{\Omega^4} Z^{(i)}$$



- classic parts: **isotropic influence** of Λ and the **Newtonian tidal effect** of the **Weyl tensor**
- **two additional effects** encoded in the non-trivial **Bach tensor** components
  - $\mathcal{B}_1$ affects particles in the **transverse** directions $\partial_\theta, \partial_\phi$
  - $\mathcal{B}_2$ induces their **radial** acceleration along $\partial_{\bar{r}}$
  - $\mathcal{B}_1(r_h) = 0$, **on any horizon** there is **only the radial** effect caused by $\mathcal{B}_2(r_h)$
- $\mathcal{B}_1, \mathcal{B}_2$ **cannot mimic** the classic tidal effects (i.e., cannot be “incorporated” into the Weyl part)

What have we done?

Spherically symmetric solutions to **quadratic gravity** with any cosmological constant:

- the alternative more **convenient metric** was employed
- the field equations as an **autonomous system** of two (simple) ODEs were formulated
- their **explicit solutions** (including BHs) in the form of power series were found
- the **mathematical analysis** and **physical properties** were discussed

This talk is based on our papers:

- *Explicit black hole solutions in higher-derivative gravity*
  Jiří Podolský, Robert Švarc, Vojtěch Pravda, Alena Pravdová
  arXiv:1806.08209 (Phys. Rev. D **98** (2018) 021502(R))
- *Exact black holes in quadratic gravity with any cosmological constant*
  Robert Švarc, Jiří Podolský, Vojtěch Pravda, Alena Pravdová
  arXiv:1806.09516 (Phys. Rev. Lett. **121** (2018) 231104)
- *Black holes and other exact spherical solutions in Quadratic Gravity*
  Jiří Podolský, Robert Švarc, Vojtěch Pravda, Alena Pravdová
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Thank you for your attention!