Computing EMRI using canonical perturbation theory
(In progress)

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The simplest form of EMRI

- We assume a small particle with mass m is inspiralling into a central object with mass M due to the emission of gravitational waves. The ratio $\frac{m}{M} \ll 1$.
- For the radiation we employ quadrupole formalism:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{5}\sum_{i,j=1}^{3}\langle(\dddot{I}_{ij})^2\rangle, \qquad \frac{\mathrm{d}L_i}{\mathrm{d}t} = -\frac{2}{5}\sum_{j,k,l=1}^{3}\epsilon_{ijk}\langle(\dddot{I}_{jl}\ddot{I}_{kl})\rangle$$

$$I^{ij}(t) = m(x^i(t)x^j(t) - \frac{1}{3}\delta^{ij}x^k(t)x_k(t))$$

- $x^i(t)$ are modelled as circular orbits around a Newtonian source M $\vec{x}(t) = R(\cos(\Omega_\phi t), \sin(\Omega_\phi t), 0)$
- Adiabatic approximation: E and L_z evolve slowly according to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = f_E(E, L_z), \qquad \frac{\mathrm{d}L_z}{\mathrm{d}t} = f_{L_z}(E, L_z).$$

- We can then get $R(t)$ (passing from one circular orbit to another) as well as the gravitational waveforms

Our set-up

- Schwarzschid black hole with mass M instead of a Newtonian source.
- General orbits (bound geodesics) instead of circular orbits.
- Additional matter affecting the motion represented by an external quadrupole with potential $U(\rho, z) = -\frac{1}{4} Q \left(\rho^2 - 2\, z^2\right)$
- Quadrupole formalism for gravitational radiation (not accurate but easy to compute for qualitative analysis).
- Mostly analytical approach using canonical perturbation theory.

The steps

1. Approximately solve the geodesic equation using canonical perturbation theory $\Rightarrow$ obtaining explicit formulas $x^i_{geod}(t) \approx x^i_{geod}(t, J_i)$
2. Compute the gravitational wave fluxes for the independent integrals of motion J_i using the quadrupole formalism.
3. Adiabatically evolve the integrals of motion

$$\frac{\mathrm{d}J_i(t)}{\mathrm{d}t} = f_i(J_k(t)) \Rightarrow J_i(t)$$

4. Evolve quantities dependent on $J_i(t)$: $(\Omega_i(t),\ x^i_{EMRI}(t),..)$
5. Obtain the waveforms using $h^{TT}_{ij} = \frac{2}{r}\ddot{I}^{TT}_{ij}(t-r)$

The spacetime

- The metric can be written in the Weyl form

$$\mathrm{d}s^2 = -e^{2\nu(\rho,z)}\mathrm{d}t^2 + e^{2\lambda(\rho,z)-2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2 e^{-2\nu(\rho,z)}\mathrm{d}\phi^2.$$

- Metric functions of Schwarzschild + external quadrupole

$$\nu(\rho,z) = \frac{1}{2}\ln\left(\frac{d_1 + d_2 - 2M}{d_1 + d_2 + 2M}\right) - \frac{1}{4}Q\left(\rho^2 - 2z^2\right)$$

$$\lambda(\rho,z) = \frac{1}{2}\ln\left(\frac{(d_1 + d_2)^2 - 4M^2}{4d_1 d_2}\right) + \frac{1}{16}\rho^2 Q^2\left(\rho^2 - 8z^2\right) - \frac{1}{2}\left((z+M)\,d_1 + (M-z)\,d_2\right)Q$$

$$d_{1,2} = \sqrt{\rho^2 + (z \mp M)^2}$$

- The system is no longer integrable (integrals E and L_z) but can be approximated by an integrable one (Q is a small perturbation.) $\Rightarrow$ canonical perturbation theory

Action-angle coordinates

- Assumptions
  1. Integrable system with hamiltonian $H(q_i, p_i)$
  2. Motion in the phase space is bounded.

  $\Rightarrow$ canonical transformation to action-angle coordinates

  $$(q_i, p_i) \to (\psi_i, J_i), \;\; H(q_i, p_i) \to H(J_i)$$

  actions:

  $$J_i = \frac{1}{2\pi} \oint p_i \mathrm{d} q_i$$

  angles:

  $$\psi_j(t) = \Omega_j t + \psi_j(0), \qquad \Omega_i = \frac{\partial H(J_j)}{\partial J_i}$$

- Harmonic oscillator: $H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m \Omega^2 q^2$

  Transformation: $q = \sqrt{\frac{2J}{m\Omega}} \sin(\psi)$, $p = \sqrt{2J\Omega\, m} \cos(\psi)$

  $$\Rightarrow \;\; H(\psi, J) = \Omega J$$

- Kepler hamiltonian in action-angle coordinates:

  $$H(J_r, J_\theta, J_\phi) = \frac{G^2 M^2 m^3}{2(J_r + J_\theta + J_\phi)^2}$$

Lie series

- Class of canonical transformation defined by an arbitrary generating function $\omega(q_i, p_i)$
- z_i phase space coordinates $\Rightarrow \{z_i, z_j\} = \Omega_{ij}$

  Time evolution equation $\frac{\mathrm{d}z_i}{\mathrm{d}t} = \{z_i, H\}$ solution:
  $z_i(t) = z_i + \{z_i, H\}t + \frac{1}{2}\{\{z_i, H\}, H\}t^2 + \ldots = \exp(t\pounds_H)z_i$
  where $\pounds_g f = \{f, g\}$
- replacement $H \leftrightarrow \omega(q_i, p_i), \;\; t \leftrightarrow \varepsilon$:

$$Z_i = z_i(\varepsilon) = \exp(\varepsilon\pounds_\omega)z_i$$

- Inverse operator: $\Big(\exp(\varepsilon\pounds_\omega)\Big)^{-1} = \exp(-\varepsilon\pounds_\omega)$
- Identity $\{\exp(\pounds_\omega)f, \exp(\pounds_\omega)g\} = \exp(\pounds_\omega)\{f, g\}$

$$\Rightarrow \{z_i, z_j\} = \{Z_i, Z_j\}$$

Birkhoff normal form

- Assume we have $H^{(0)} = H_0(J_i) + \sum_{i=1} \varepsilon^i H_i^{(0)}(\psi_i, J_i)$
- First start with $\varepsilon H_1^{(0)}$: $H_1^{(0)} = Z_1(J_i) + h_1(\psi_i, J_i)$

  $$\exp(\varepsilon \pounds_{\omega_1}) H^{(0)} = H_0 + \varepsilon Z_1 + \varepsilon \{H_0, \omega_1\} + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$$

  $$\Rightarrow \ \{H_0, \omega_1\} + h_1 \stackrel{!}{=} 0 \text{ homological equation for } \omega_1$$

- $H^{(1)} = \exp(\varepsilon \pounds_{\omega_1}) H^{(0)}$ is independent of angles up to the first order in the perturbation parameter ε
- After r normalization steps we have $H^{(r)} = H_0(J_i) + \sum_{i=1}^{r} \varepsilon^i Z_i(J_i) + R^{(r)}(\psi_i, J_i)$,

  $R^{(r)} = \mathcal{O}(\varepsilon^{r+1})$ is a remainder.
- In total

$$H^{(n)} = \exp(\varepsilon^n \pounds_{\omega_n}) \exp(\varepsilon^{n-1} \pounds_{\omega_{n-1}}) \dots \exp(\varepsilon \pounds_{\omega_1}) H^{(0)} = U(\omega_i) H^{(0)}$$

$$\psi^{(0)} = U(\omega_i)\psi, \ \ J^{(0)} = U(\omega_i) J$$

The Schwarzschild hamiltonian

- In Schwarzschild coordinates hamiltonian $H_{Schw} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ is

$$H_{Schw} = \frac{1}{2}\left[-\frac{1}{1-\frac{2M}{r}} p_t^2 + \left(1 - \frac{2M}{r}\right) p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \right) \right],$$

- eliminating dependence on θ: $(\theta, p_\theta) \to (\vartheta, J_\vartheta)$

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \to (J_\theta + L_z)^2, \quad \theta = \pi - \arccos\left(\sqrt{1 - \frac{{L_z}^2}{(L_z + J_\theta)^2}} \sin(\psi_\theta) \right)$$

- Change of evolution parameter $\tau \to \lambda, \; \mathrm{d}\tau = r^2 \mathrm{d}\lambda \Rightarrow$ separation of the angular part:

$$H_{Schw} = \frac{1}{2} r^2 (g^{\mu\nu} p_\mu p_\nu + 1) = H_{rad} + \frac{1}{2}(J_\theta + L_z)^2$$

$$H_{Schw} = \frac{1}{2} r^2 \left[-\frac{1}{1-\frac{2M}{r}} p_t^2 + \left(1 - \frac{2M}{r}\right) p_r^2 + 1 \right] + \frac{1}{2}(J_\theta + L_z)^2$$

- Radial part: Expansion from a stable circular orbit: $r - r_c = \mathcal{O}(\varepsilon) = p_r, E - E_c = \delta E = \mathcal{O}(\varepsilon^2)$
- Radial part is then $H_{rad} = H_0(\delta E) + J_r\Omega + R(\delta E, \psi_r, J_r) =$ $= Z_0(J_r, \delta E) + R(\delta E, \psi_r, J_r), \;\; R = \mathcal{O}(\varepsilon^3)$
- Applying the Lie series transform:

$$\exp(\pounds_{\omega_2})\exp(\pounds_{\omega_1})H_{Schw} = \sum_{i=0}^{2} Z_i(J_r) + \frac{1}{2}(J_\theta + L_z)^2 + R^{(2)}(\psi_r, J_r) \approx H_{N0}(J_r, J_\theta) + \mathcal{O}(\varepsilon^5)$$

- After adding the perturbation:

$$H_{tot} = H_{Schw} + \frac{\partial H_{tot}}{\partial Q} Q + \mathcal{O}(Q^2)$$

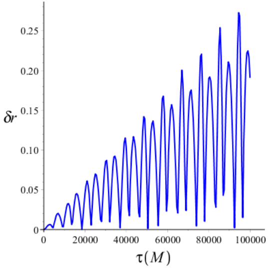
- We then get

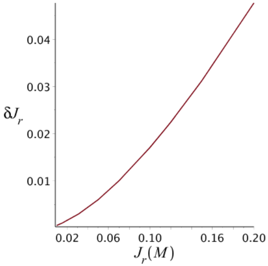
$$H_{Ntot} = H_{N0}(J_r, J_\theta) + QZ_{Q1}(J_r, J_\theta) + \mathcal{O}(Q^2)$$

- Neglecting the remainder $\Rightarrow$ integrable system $H_{Ntot}(E, J_r, J_\theta, L_z)$
- Coordinates $x^i(\lambda) = x^i(\psi_j(\lambda), E, J_r, J_\theta, L_z)$, $\qquad \psi_j(\lambda) = \Omega_j \lambda + \psi_j(0)$

Accuracy of the approximation

- $\frac{J_r}{M}$ and $\frac{QL^4}{M^2}$ need to be sufficiently small





Resonances

- Solving the homological equation

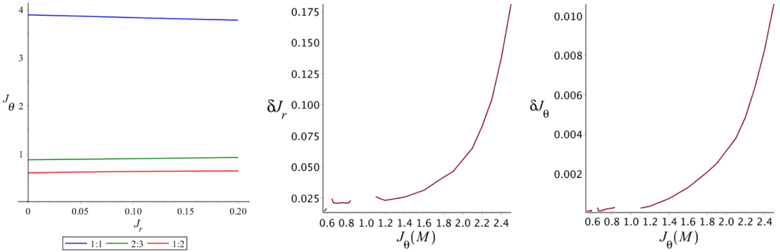
$$h_1 = \sum_{k,l} a_{kl} e^{i(k\psi_r + l\psi_\theta)} \rightarrow \chi = \sum_{k,l} a_{kl} \frac{1}{i(k\Omega_{r0} + l\Omega_{\theta 0})} e^{i(k\psi_r + l\psi_\theta)}$$

where $\Omega_{r0} = \frac{\partial H_{N0}}{\partial J_r}, \quad \Omega_{\theta 0} = \frac{\partial H_{N0}}{\partial J_\theta}$

- Resonance condition of the unperturbed system

$$k\Omega_{r0} + l\Omega_{\theta 0} = 0, \quad k, l \in \mathbb{Z}$$

- The approximation fails when close to a resonance



Adiabatic approximation

- Evolution of angles is much slower than evolution of actions (and energy):

$$\frac{\mathrm{d}}{\mathrm{d}t}x^i(t, E(t), J_i(t)) = \frac{\partial x^i}{\partial t} + \frac{\partial x^i}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial x^i}{\partial J_j}\frac{\mathrm{d}J_j}{\mathrm{d}t} \approx \frac{\partial x^i}{\partial t}.$$

- The averaging is trivial:

$$A = \sum_{k,l} c_{kl} e^{i(k\psi_r(t) + l\psi_\theta(t))} \quad \Rightarrow \quad \left\langle A \right\rangle = c_{00}(E, L_i).$$

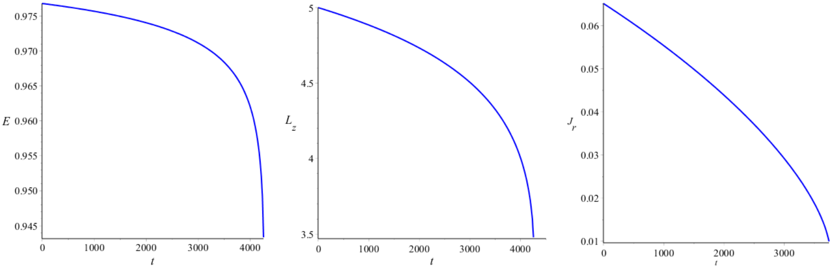
- The evolution equations for integrals of motion:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = f_E(E, J_\theta, L_z), \qquad \frac{\mathrm{d}J_\theta}{\mathrm{d}t} = f_{J_\theta}(E, J_\theta, L_z), \qquad \frac{\mathrm{d}L_z}{\mathrm{d}t} = f_{L_z}(E, J_\theta, L_z).$$

- The coordinates then evolve in time as $x^i_{EMRI}(t) = x^i(\psi_i(t), E(t), J_\theta(t), L_z(t))$ where $\psi_i(t) = \int_0^{\lambda(t)} \Omega_i(\lambda)\mathrm{d}\lambda + \psi_i(0), \qquad \Omega_i = (\Omega_i, E, J_\theta, L_z)$

Evolving the actions

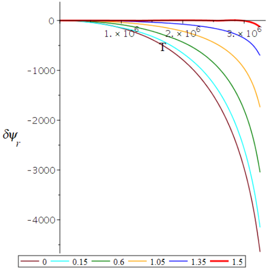
- Evolution of the integrals of motion during EMRI



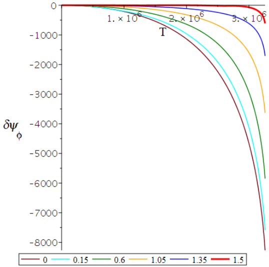
(a) Energy (b) Angular momentum (c) Radial action

Phase shifts

$$\delta\psi_i(t) = \psi_i(t) - \psi_i(t)|_{Q=0}$$

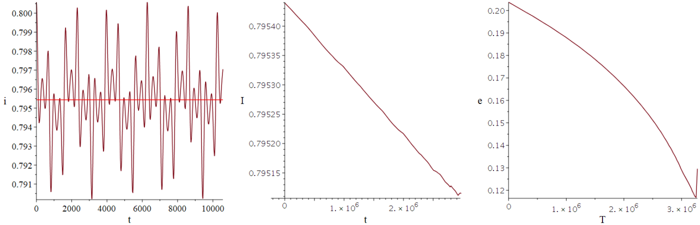


(a) Radial phase shift



(b) Azimuthal phase shift

$$i = \arccos\left(\frac{J_\phi}{J_\phi + J_\theta^{(0)}}\right), \qquad I = \langle i \rangle$$



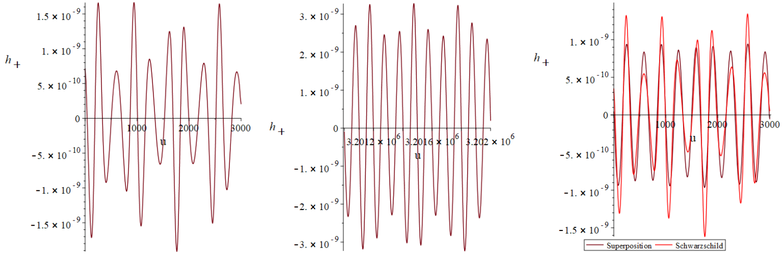
(a) geodesic evolution of Inclination i

(b) Adiabatic evolution of I

(c) Evolution of eccentricity

Extracting the waveforms

- Inserting $x^i(t)$ into the Einstein's quadrupole formula $h_{ij}^{TT} = \frac{2}{R}\ddot{I}_{ij}^{TT}(u)$ leads us to the gravitational waveforms.



(a) inspiral around $r = 21M$

(b) inspiral around $r = 9M$

(c) Comparison with Schwarzschild

Thank you for your attention