Action-Angle formalism for geodesic motion in Kerr spacetime as a tool for EMRIs

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Outline

- What is the EMRI?
- Action-angle (AA) formalism.
- Canonical perturbation theory.
- Kerr geodesic Hamiltonian in the AA variables.

What is the EMRI?

- *Extreme mass ratio inspirals* (EMRI's) are one the most promising sources of gravitational wave that LISA will be observed.
- EMRI's consist of a primary supermassive black hole M, and a secondary much lighter compact stellar object μ.
- In an EMRI system, the mass ratio $q = \frac{\mu}{M}$ lies between 10^{-7} and 10^{-4}.

Modeling the EMRI's

To model an EMRI, we use the two time scale approximation.

- The slow time scale: evolution of the constants of motion: $\Longrightarrow$ Actions.
- The fast time scale: orbital phases of the secondary: $\Longrightarrow$ Angles.

Hamiltonian in action-angle variables

For a bounded motion when the system is integrable, there exist a canonical transformation

$$(\boldsymbol{q}, \boldsymbol{p}) \xrightarrow{\text{CT}} (\boldsymbol{\psi}, \boldsymbol{J}) \qquad \Longrightarrow \qquad H(\boldsymbol{q}, \boldsymbol{p}) \to H(\boldsymbol{J})$$

From Hamilton equations:

$$\dot{J}_i = -\frac{\partial H}{\partial \psi_i} = 0 \qquad \Longrightarrow \qquad J_i(\lambda) = Const.$$

$$\dot{\psi}_i = \frac{\partial H}{\partial J_i} = \Omega_i \qquad \Longrightarrow \qquad \psi_i(\lambda) = \Omega_i \lambda + \psi_i(0)$$

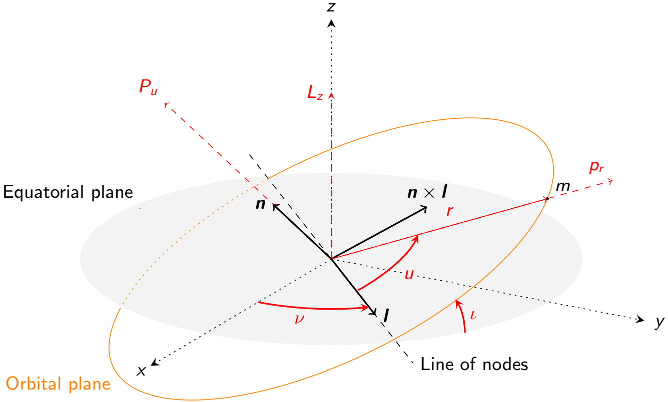
Geodesic motion in Kerr background in AA variables

- In the Boyer–Lindquist coordinates the Hamiltonian in Mino time is

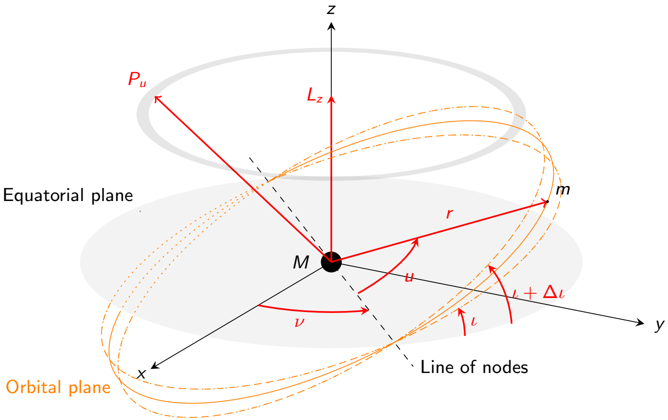
$$H_\lambda = \frac{1}{2}\left(\Delta p_r^2 - \frac{\left(\left(r^2+a^2\right)p_t + aL_z\right)^2}{\Delta} + r^2\right) + \frac{1}{2}\left(p_\theta^2 + a^2\cos^2\theta + \frac{\left(L_z + a\sin^2\theta p_t\right)^2}{\sin^2\theta}\right).$$

- Since we want to derive the Hamiltonian in the AA variables for off-equatorial orbits, it is useful to apply a canonical transformation from Boyer–Lindquist coordinates to the polar-nodal coordinates.

Polar-nodal coordinates



Kerr geodesics in polar-nodal coordinates



Canonical perturbation theory

- In the Birkhoff normal form theory we assume:

$$H^{(0)}(\psi_i, J_i) = Z_0(J_i) + \sum_{k=1}^{N} \epsilon^k H_k^{(0)}(\psi_i, J_i),$$

- We apply the Lie series to the Hamiltonian:

$$H^{(1)} = L_\chi H^{(0)} = Z_0(J_i) + \epsilon Z_1^{(0)}(J_i) + \epsilon^2 H_2^{(0)}(\psi_i, J_i) + \mathcal{O}(\epsilon^3),$$

where $L_\chi f = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathcal{L}_\chi^k f$ and $\mathcal{L}_\chi f = \{f, \chi\}$.

- In a similar fashion, we can apply n Lie series:

$$\begin{aligned} H^{(n)} &= L^n_{\chi_n} L^{n-1}_{\chi_{n-1}} ... L^2_{\chi_2} L_\chi H^{(0)} \\ &= Z_0(J_i) + \epsilon Z_1^{(n)}(J_i) + ... \epsilon^n Z_n^{(n)}(J_i) + R^{(n)}(\psi_i, J_i). \end{aligned}$$

Why Lie series?

- Lie series is are canonical transformations.
- We can express the old variables in terms of the new variables:

$$\psi_i^{(0)} = L^n_{\chi_n}\, L^{n-1}_{\chi_{n-1}} \ldots L^2_{\chi_2}\, L_\chi \psi_i^{(n)},$$
$$J_i^{(0)} = L^n_{\chi_n}\, L^{n-1}_{\chi_{n-1}} \ldots L^2_{\chi_2}\, L_\chi J_i^{(n)},$$

- New variables in terms of the old ones:

$$\psi_i^{(n)} = L_{-\chi}\, L^2_{-\chi_2} \ldots L^{n-1}_{-\chi_{n-1}}\, L^n_{-\chi_n} \psi_i^{(0)},$$
$$J_i^{(n)} = L_{-\chi}\, L^2_{-\chi_2} \ldots L^{n-1}_{-\chi_{n-1}}\, L^n_{-\chi_n} J_i^{(0)}.$$

- The resulting relations from the Lie series are in **closed form**! There are simple analytical relations without any integral, Hypergeometric function and etc.

Accuracy of the new system

- We applied 10 canonical transformation for the radial part and 7 canonical transformation for the angular part.
- We derived the new Hamiltonian in the action-angle variables:

$$H_{AA}(J_i) = H(J_t, J_r, J_u, J_z)$$

- From the Hamilton equations the radial and angular frequencies are obtained:

$$\Omega_r = \frac{\partial H_{AA}}{\partial J_r}, \qquad \Omega_u = \frac{\partial H_{AA}}{\partial J_u}.$$

- We derived the relative errors of the new actions and frequencies with their numerical values.

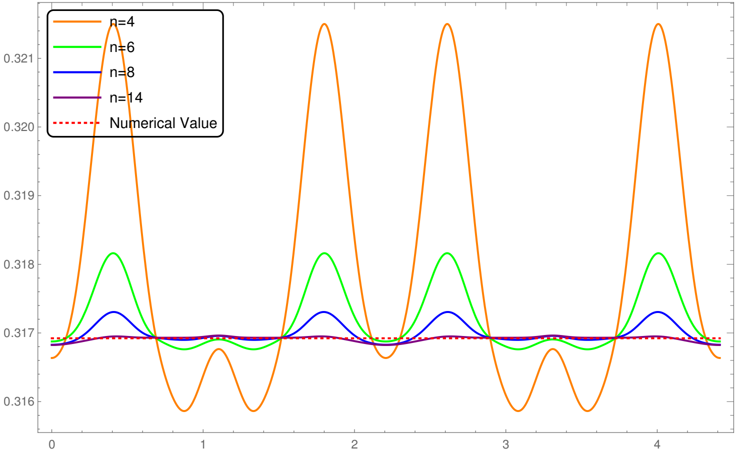


Figure: The radial action J_r in terms of the old variables for different numbers of the CT for a system with $a = 0.99$, $e = 0.4$, $p = 10$, and the initial inclination $\iota_0 = \pi/8$.

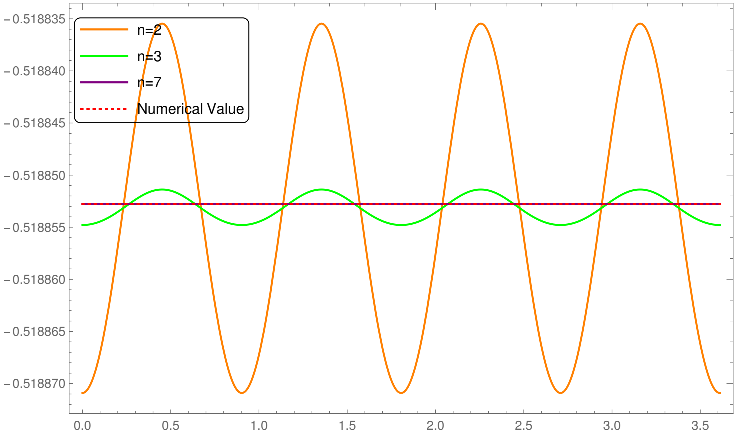


Figure: The angular action J_u in terms of the old variables for different numbers of the CT for a system with $a = 0.99$, $e = 0.4$, $p = 10$, and the initial inclination $\iota_0 = \pi/8$.

Accuracy of the new system

- The relative errors for a system $\{p, \iota_0\} = \{10, \pi/8\}$ for the Kerr parameter $0 < a \leq 0.99$ and the eccentricity $0 < e \leq 0.5$ are

$$\mathcal{O}(\Delta J_r)_{max} = \mid 1 - \frac{J_{r_{AA}}}{J_{r_n}} \mid \Longrightarrow \{10^{-10}, 10^{-4}\},$$

$$\mathcal{O}(\Delta \Omega_r)_{max} = \mid 1 - \frac{\Omega_{r_{AA}}}{\Omega_{r_n}} \mid \Longrightarrow \{10^{-12}, 10^{-4}\},$$

$$\mathcal{O}(\Delta J_u)_{max} = \mid 1 - \frac{J_{u_{AA}}}{J_{u_n}} \mid \Longrightarrow \{10^{-11}, 10^{-9}\},$$

$$\mathcal{O}(\Delta \Omega_u)_{max} = \mid 1 - \frac{\Omega_{u_{AA}}}{\Omega_{u_n}} \mid \Longrightarrow \{10^{-12}, 10^{-8}\}.$$

Thanks for your attention!