Quadrupole formula in de Sitter: its application

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A brief history

- Well-defined notion of gravitational wave in full non-linear theory of general relativity for asymptotically flat space-time.

- Bondi and his collaborators provided a detailed analysis of gravitational waves in full non-linear theory along with a definition of energy carried away from an isolated system by gravitational radiation.

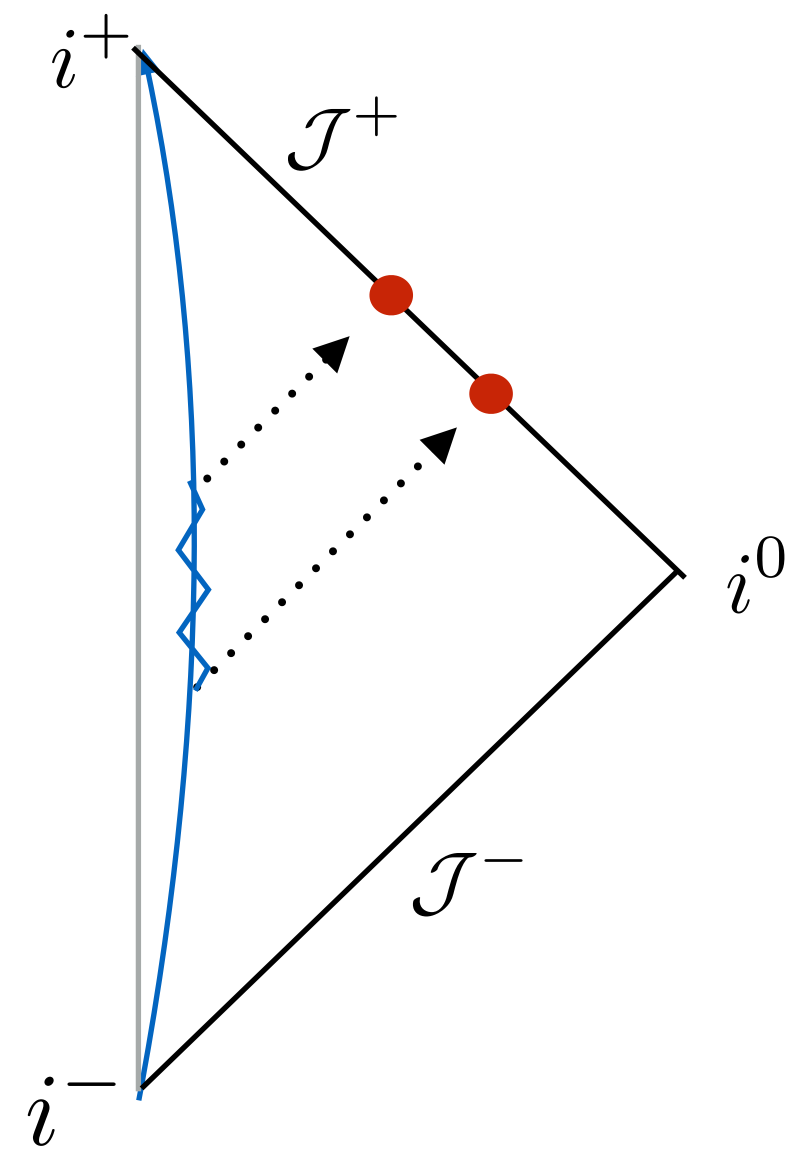
H. Bondi, M.G.J. Vander Burg, A. W. K. Metzner - 1962

- One of the remarkable milestones in gravitational radiation theory!

A brief history

- Bondi and his collaborators performed a systematic expansions of axis symmetric gravitational wave metric along outgoing null directions.
- Given an initial data on null hypersuface, solve Einstein's equations.
- Deduce the asymptotic fall-off condition for the gravitational fields.
- As a supplementary condition obtain mass-loss formula.
- 'News function' - absolute square integrated over the sphere at infinity, measures the rate of energy loss by a gravitating system.

$$\frac{dM}{du} = -\int |\partial_u \overset{(-1)}{\check{h}}_{AB}|^2 \sin\theta d\theta d\phi$$



Questions...

- Given that cosmological observations suggest a positive Λ,

  How to study gravitational radiation?

  - In (generalized) Harmonic gauge
  - In Bondi gauge

- Solution of linearised EE in terms of source integral?
- Linearised fields in terms of source quadrupole moments?
- Power radiated quadrupole formula?

Is there any significant observable effect in orbital decay and orbital phase computation in a binary system?

How small are correction term? Order of magnitude?

Questions...

how to generalise Bondi-Sachs's formalism?

Asymptotic fall-off condition for the gravitational fields?

Asymptotic symmetries?

Mass-loss formula for $\Lambda > 0$?

Analogous 'News tensor'?

Surprisingly this remained unsolved for 60 years!! Lots of Non-trivialities !!

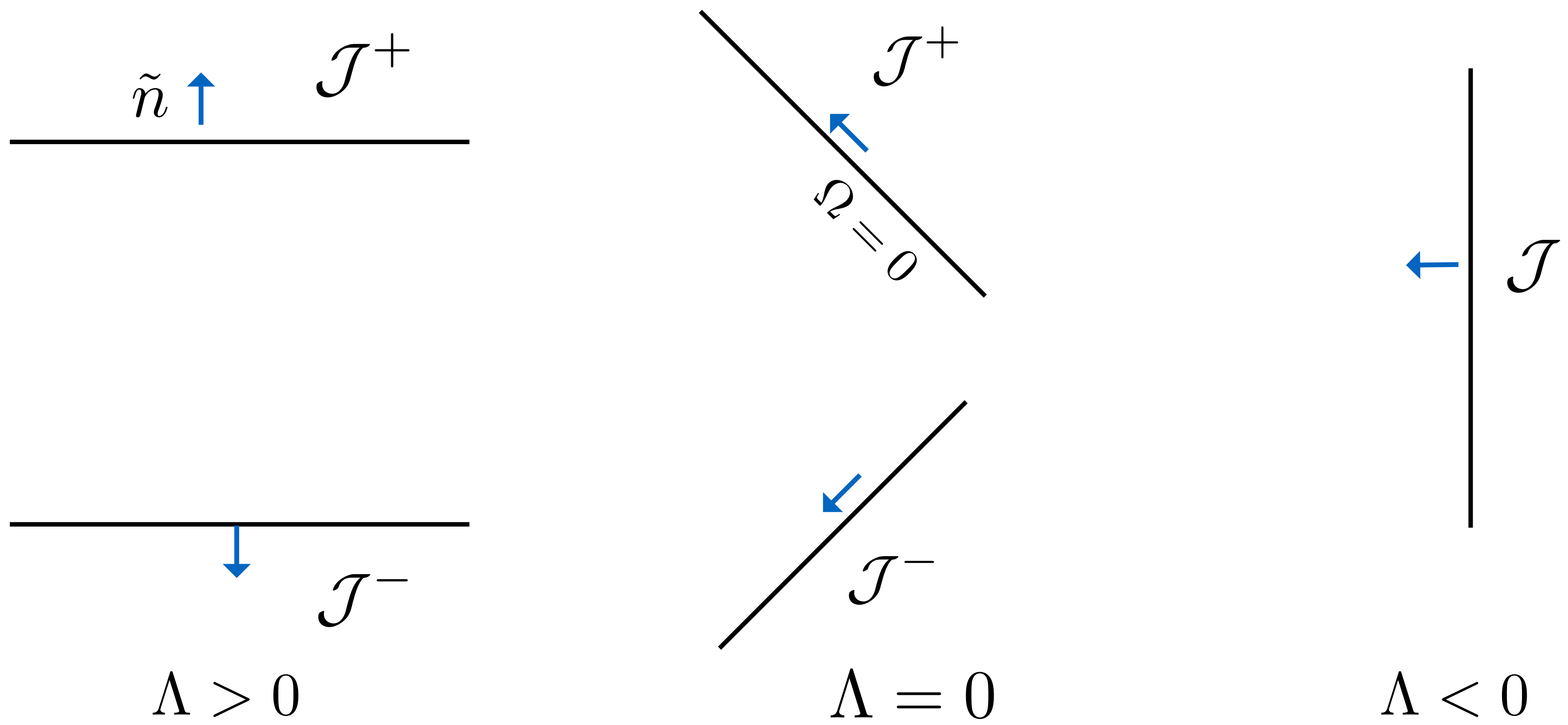
A simpler version of the problem - Bondi-Sach's formalism for linearised gravitational fields on de Sitter background.

How to reconcile Bondi Gauge to generalised Harmonic gauge?

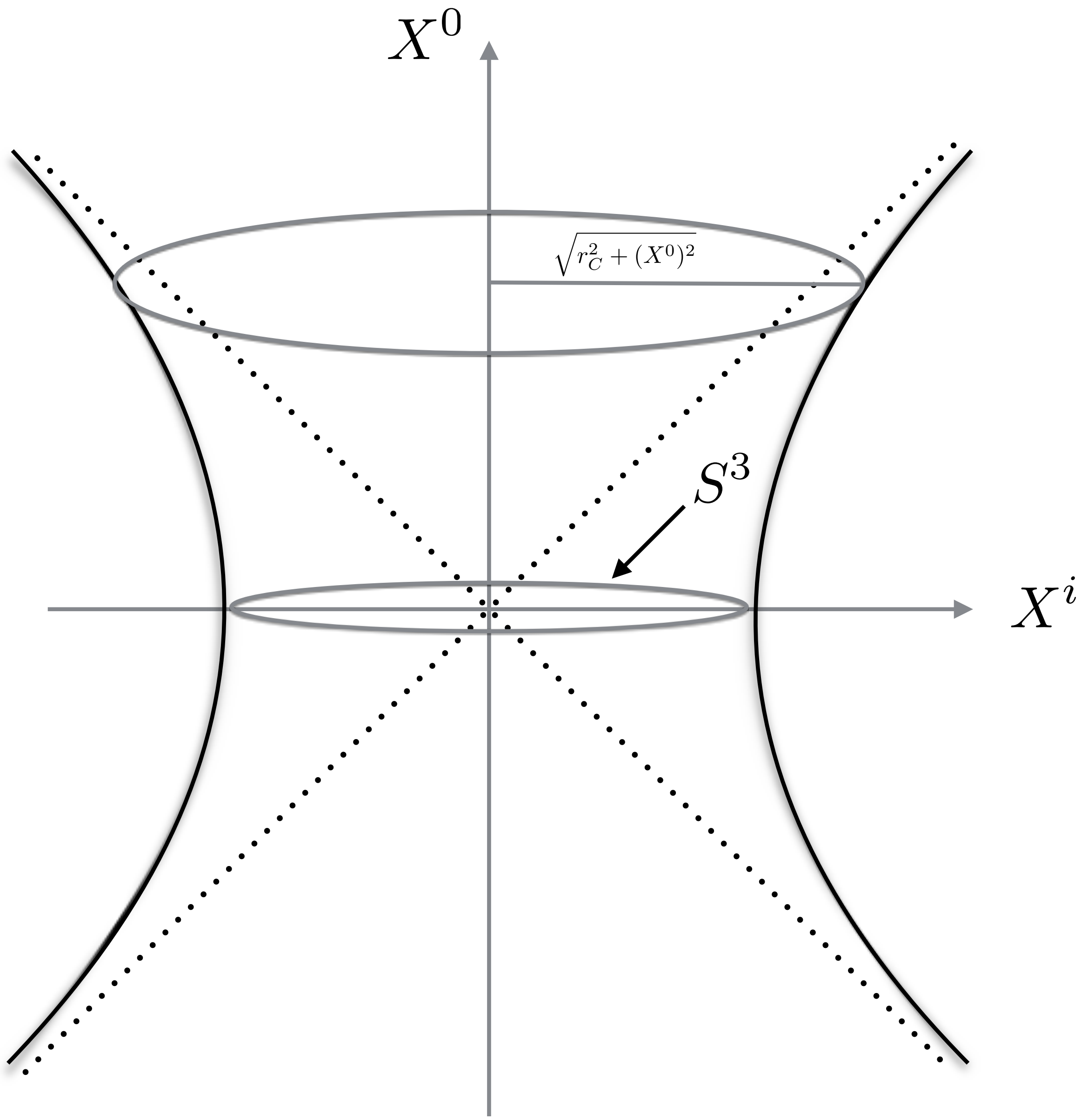
Non-triviality for positive Λ

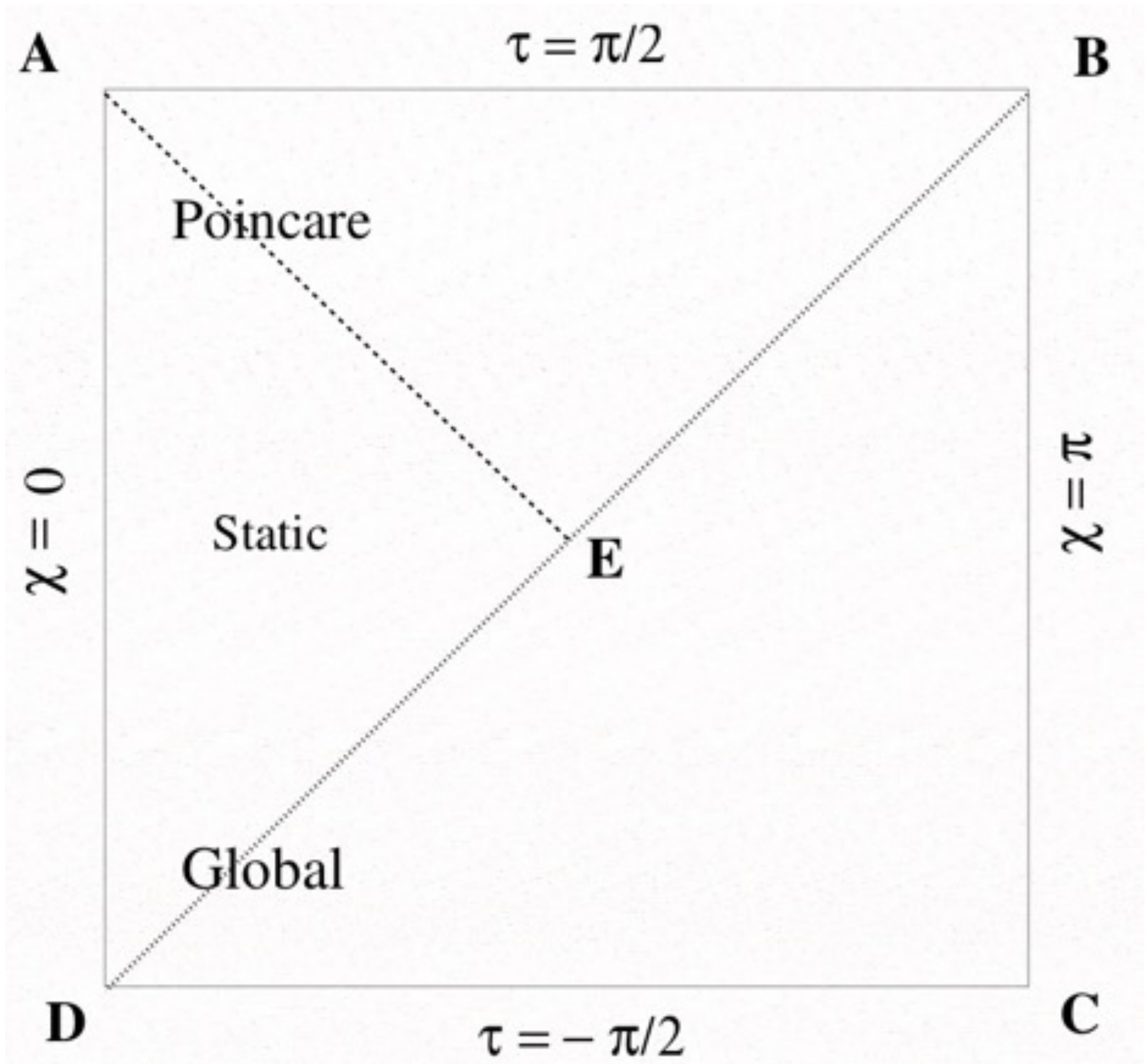
- Standard framework does not extend from $\Lambda = 0$ to $\Lambda > 0$.

- Structure of null infinity alters, for $\Lambda > 0$: space-like
  $\Lambda < 0$: time-like

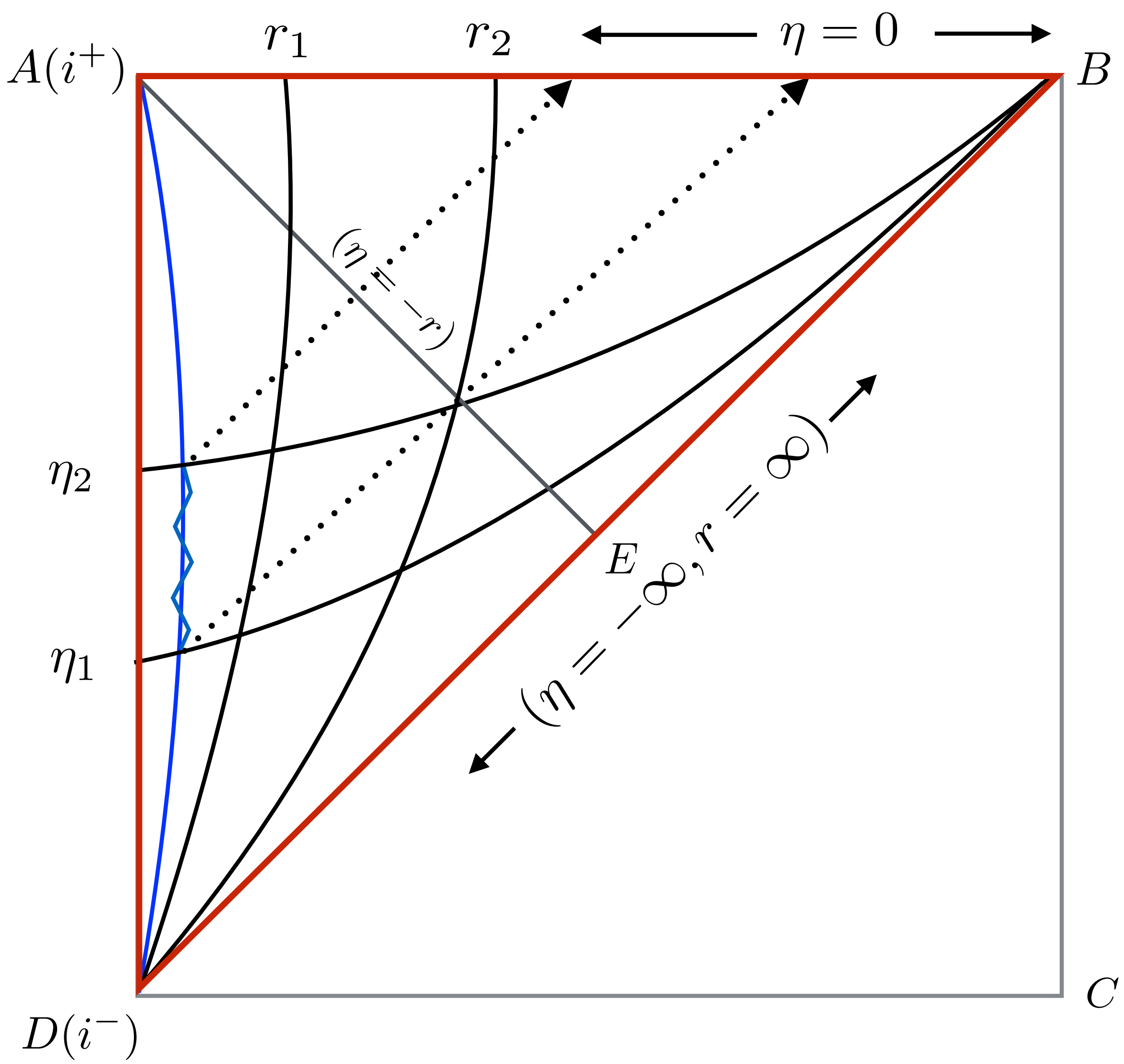


de Sitter space-time and patches





Conformal chart



$$ds^2_{\text{conformal}} = \frac{3}{\Lambda \eta^2}\left[-d\eta^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$a^2 := \frac{1}{\eta^2 H^2}, H = \sqrt{\frac{\Lambda}{3}}\ , \quad \eta = -H^{-1}e^{-Ht}$$

Linearised equation

Choose a background metric : $\bar{g}_{\mu\nu}(x)$, $\quad g_{\mu\nu}(x) := \bar{g}_{\mu\nu}(x) + \epsilon h_{\mu\nu}(x)$

Define : $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h \quad , \quad h := h_{\alpha\beta}\bar{g}^{\alpha\beta}, \quad B_\mu := \bar{\nabla}_\alpha \tilde{h}^\alpha_\mu$

Linearised equation :

$$
\begin{aligned}
8\pi T_{\mu\nu} \quad = \quad & \frac{1}{2}\left[-\bar{\Box}\tilde{h}_{\mu\nu} + \{\bar{\nabla}_\mu B_\nu + \bar{\nabla}_\nu B_\mu - \bar{g}_{\mu\nu}(\bar{\nabla}^\alpha B_\alpha)\}\right] \\
& + \frac{\Lambda}{3}\left[\tilde{h}_{\mu\nu} - \tilde{h}\bar{g}_{\mu\nu}\right]
\end{aligned}
$$

Gauge condition: $B_\mu = \frac{2\Lambda\eta}{3}\tilde{h}_{0\mu}$, simplifies the equation.

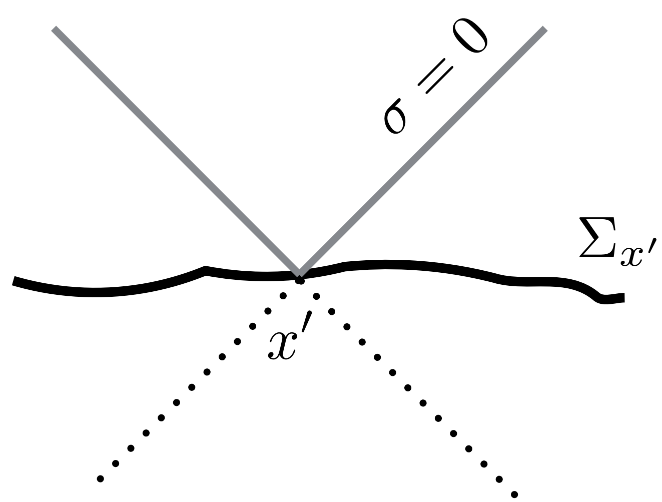
- $\chi_{\mu\nu} := a^{-2}\tilde{h}_{\mu\nu}$, with $\quad \partial^\alpha \chi_{\alpha\mu} + \frac{1}{\eta}\left(2\chi_{0\mu} + \delta^0_\mu \chi_\alpha{}^\alpha\right) = 0$
- Residual gauge freedom $\Longrightarrow$ $\chi_{0i} = 0 = \hat{\chi}(:= \chi_{00} + \chi_i{}^i)$

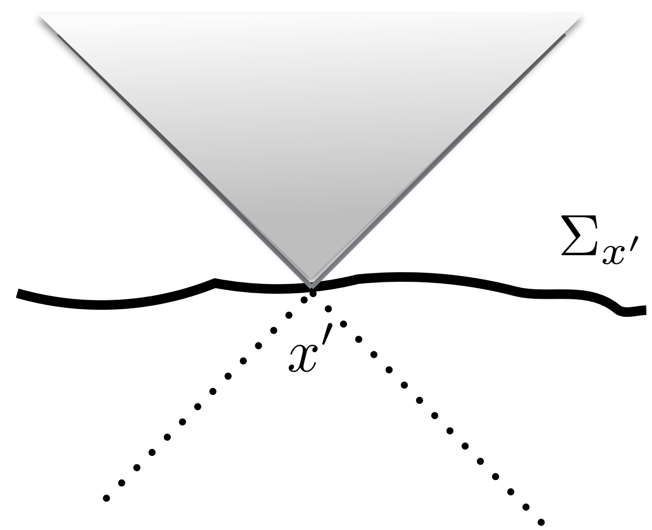
$$\therefore \Box\chi_{ij} + \frac{2}{\eta}\partial_0\chi_{ij} = -16\pi T_{ij}$$

Retarded Green function - decoupled eqns

$$\left(\Box + \frac{2}{\eta}\partial_0\right) G_R(\eta, x; \eta', x') = -\frac{\Lambda}{3}\eta^2\delta^4(x - x') \ ,$$

$$G_R(\eta, x; \eta' x') = \frac{\Lambda}{3}\frac{1}{4\pi}\left(\frac{\eta\eta'}{|x - x'|}\delta(\eta - \eta' - |x - x'|) + \theta(\eta - \eta' - |x - x'|)\right) .$$





$$\Longrightarrow \quad \chi_{ij}(\eta, x) = 4\int d^3x' \frac{1}{|x - x'|} T_{ij}(\eta', x')|_{\eta' = \eta - |x - x'|} + 4\int d^3x' \int_{-\infty}^{\eta - |x - x'|} \frac{d\eta'}{\eta'} \frac{\partial T_{ij}(\eta', x')}{\partial \eta'}$$

- Having discussed the propagation of waves in de Sitter space, we need two more things to generalise the Einstein Quadrupole formula in de Sitter
- Source moments
- Particular solution of inhomogeneous wave equation in terms of source moments

Source moments

Mass moment :

$$Q^{ij}(t) \quad := \quad \int_{(t)} d^3x \; a^3(t) \; \rho(t, \vec{x}) \; \bar{x}^i \bar{x}^j \; .$$

Pressure moment :

$$\bar{Q}^{ij}(t) \quad := \quad \int_{(t)} d^3x \; a^3(t) \; p(t, \vec{x}) \; \bar{x}^i \bar{x}^j \; .$$

$$\bar{x}^{\underline{i}} \quad := \quad f^{\underline{i}}_{\alpha} x^{\alpha} = -(\eta H)^{-1} \delta^{\underline{i}}_{\;j} x^j = a(t) x^{\underline{i}} \quad ;$$

$$f^{\alpha}_{\;\underline{m}} \quad := \quad -H\eta \; \delta^{\alpha}_{\;\underline{m}}, \; \eta := -H^{-1} \; e^{-Ht};$$

$$\rho : \quad = \quad T_{\alpha\beta} f^{\alpha}_{\;\underline{0}} f^{\beta}_{\;\underline{0}} \; , \; P_{\underline{ij}} \; := \; T_{\alpha\beta} f^{\alpha}_{\;\underline{i}} f^{\beta}_{\;\underline{j}}, \; p := P_{\underline{ij}} \delta^{\underline{ij}}$$

Inhomogeneous solution

- Using conservation equation of de Sitter background in tetrad frame, relate source integral with moment variables.

$$
\begin{aligned}
\chi_{ij}(t,r) \quad \approx \quad & \frac{2}{r\ \bar{a}}\left\{\partial_t^2 Q_{ij} - 2H\partial_t Q_{ij} + H\partial_t \bar{Q}_{ij}\right\} \\
& -2H\left\{\partial_t^2 Q_{ij} - 3H\partial_t Q_{ij} + H\partial_t \bar{Q}_{ij} + 2H^2 Q_{ij} - H^2 \bar{Q}_{ij}\right\} \\
& -2H^2\left\{\partial_{t'} Q_{ij} - 2H Q_{ij} + H\bar{Q}_{ij}\right\}\Big|_{-\infty}
\end{aligned}
$$

A. Ashtekar, B. Bonga and A. Kesavan, Phys. Rev. D 92, 044011 (2015).

G. Date and S. J. Hoque, Phys. Rev. D 94, 064039 (2016).

Energy Propagation

- We use covariant phase space approach of Lee and Wald for linearised gravity in de Sitter
- We introduce symplectic structure on the space of linearised solutions
- For a linearised solution h, energy flux is

$$E_T = \omega(h, \pounds_T h)$$

Energy flux in dS background

- Time translation generator : $T^{\mu} = -H(\eta, x^{i})$

- Conservation equation :

$$0 = \int_{v} d^4x \sqrt{\bar{g}} \bar{\nabla}_{\mu} \omega^{\mu} = \int_{v} d^4x \; \partial_{\mu}(\sqrt{\bar{g}} \omega^{\mu}) = \int_{\partial v} \; d\Sigma_{\mu} \omega^{\mu},$$

- Energy flux across 3-dimensional hyper-surface : $\int_{\Sigma} d\Sigma_{\mu} \omega^{\mu}$

Flux : constant physical radial distance

$r_{phy} = |a|r := \rho$:Trajectory of killing observer.

$$\Rightarrow \quad \int_{\Sigma_\rho} d\Sigma_\alpha \omega^\alpha \quad = \quad \int_{-\infty}^{+\infty} d\tau \int_{S^2} d\Omega \left[\frac{1}{8\pi}\right] \mathcal{R}^{tt}_{ij}\mathcal{R}^{tt}_{kl}\ \delta^{ik}\delta^{jl}$$

$$\mathcal{R}^{tt}_{mn} \ := \ \left[\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}\right]^{tt}(t_{ret}) \ ,$$

G. Date and S. J. Hoque, Phys. Rev. D 96, 044026 (2017).

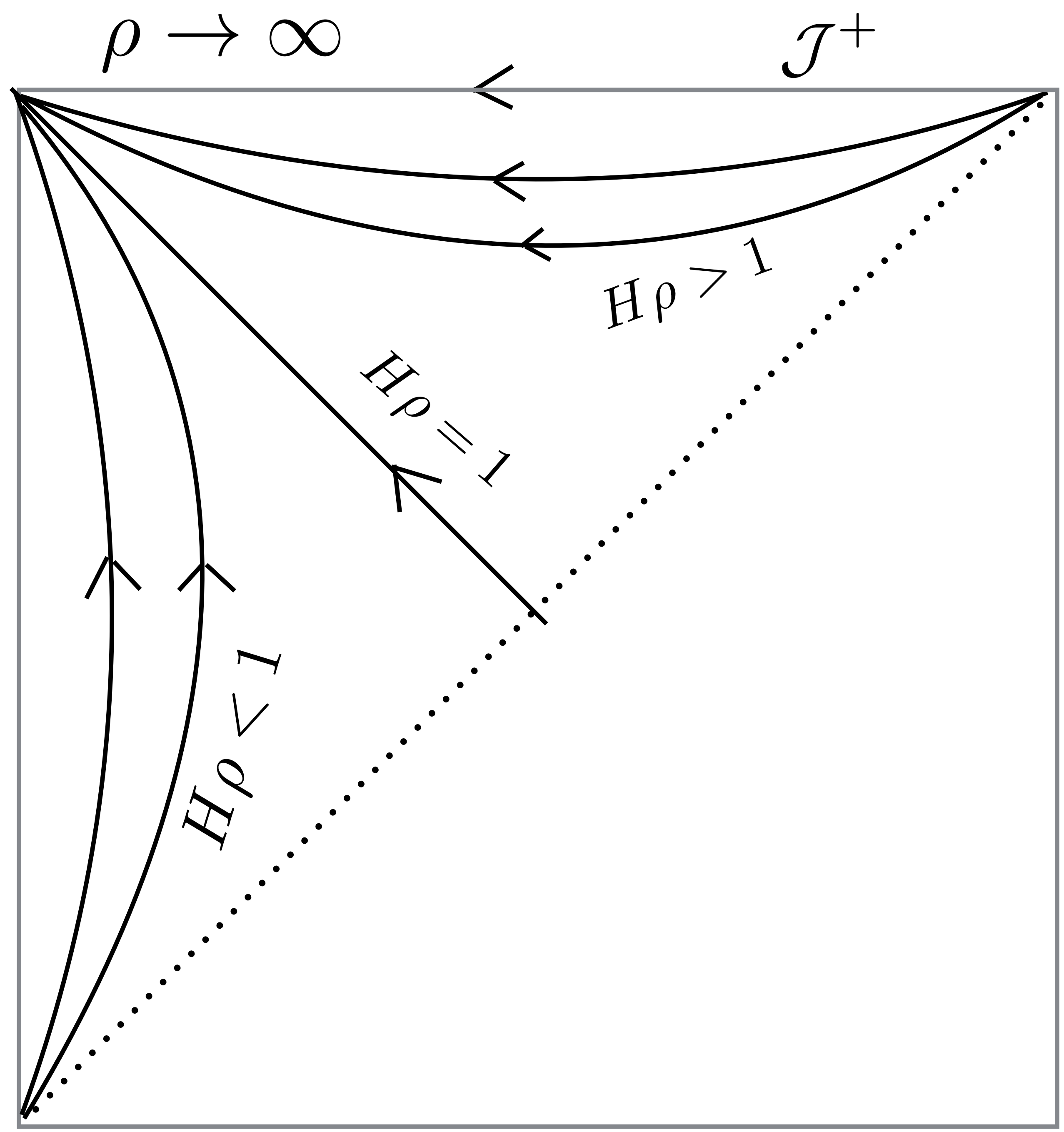
S. J. Hoque and A. Virmani, Gen.Rel.Grav. 50, 40 (2018).

A. Ashtekar, B. Bonga and A. Kesavan, Phys. Rev. D 92, 10432 (2015).

Remarks :

- Full flux integral is independent of physical radial distance, for large enough ρ

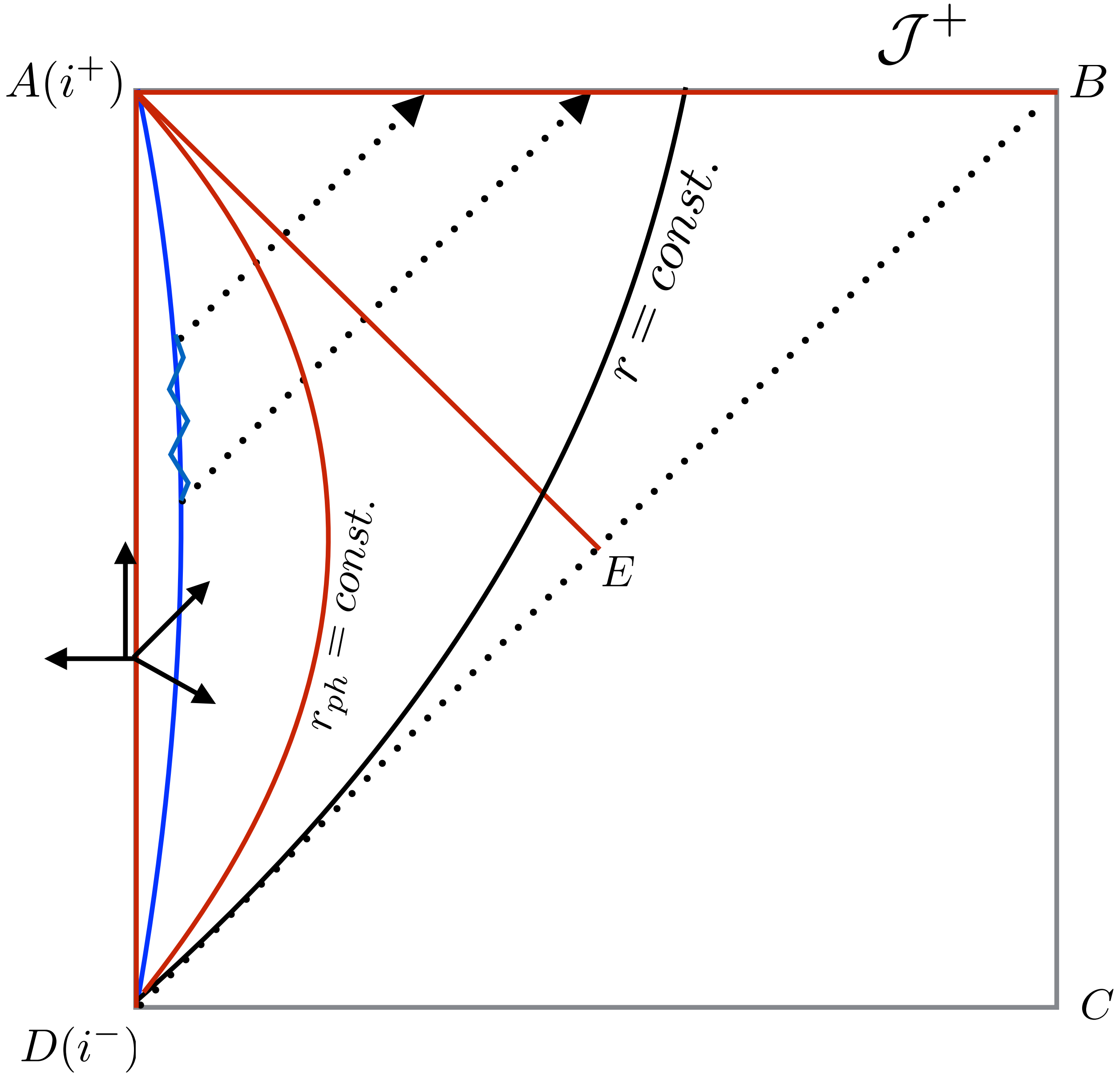
- Analogous to r independence of flux in flat background

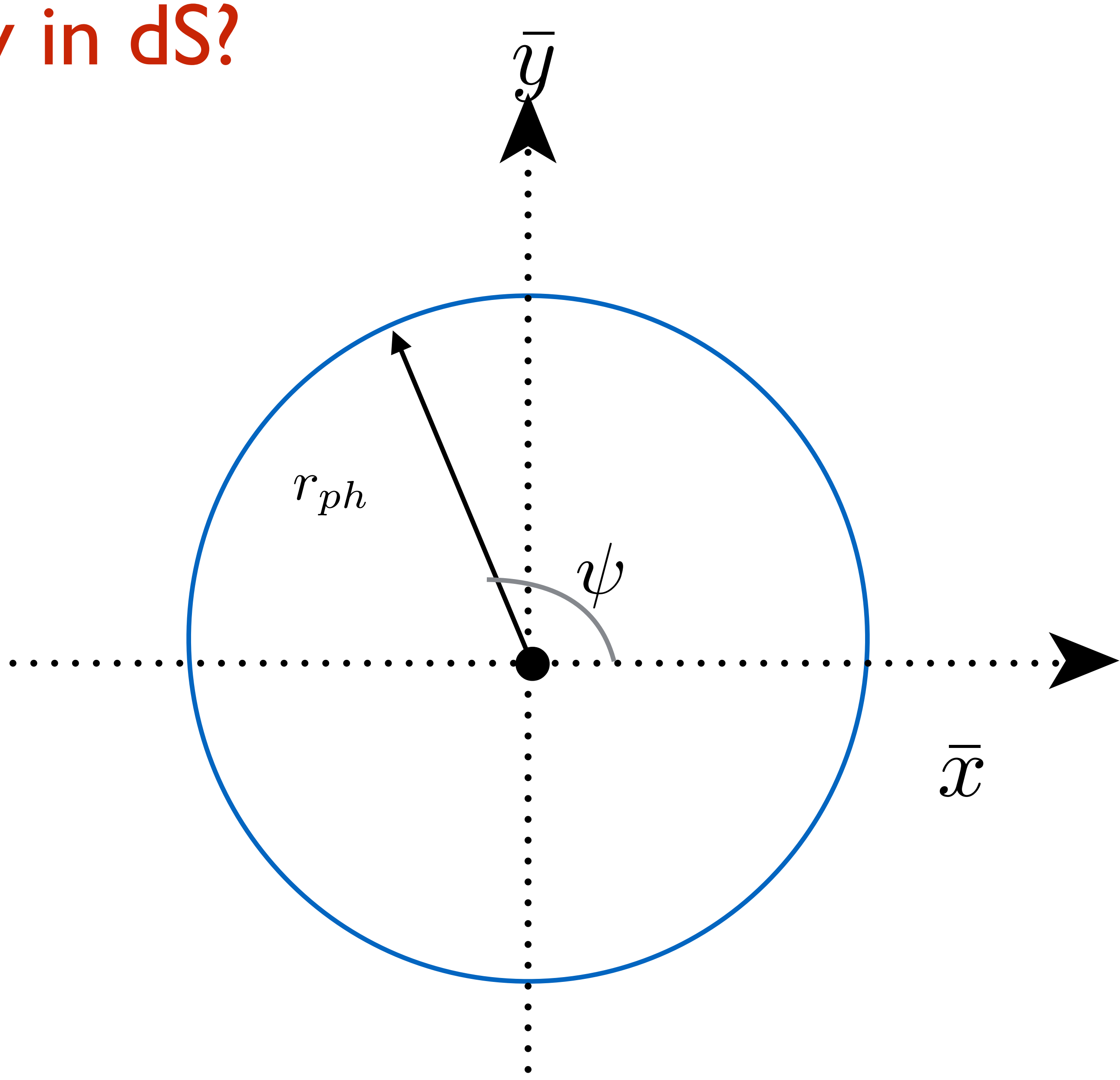


- $$\lim_{\rho \to \infty} \int_{\Sigma_\rho} d\Sigma_\mu \omega^\mu = \int_{\Sigma(H\rho=1)} d\Sigma_\mu \omega^\mu$$

Modelling of compact binary in de Sitter background

How to model the compact binary in dS?





$$r_{ph} = const.$$

Modelling of source in de Sitter background

- $r = const.$ surface does not have compact support.
- $r_{ph} = const.$ represents circular orbit in de Sitter background.
- Moments are defined in tetrad frame
- In CM frame the system is equivalent to an effective one body problem with reduced mass μ .
- Attach a tetrad to the centre of circular binary .
- Mass quadrupole moment :

$$Q^{ij} = \mu r_{ph}^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi & 0 \\ \sin\psi\cos\psi & \sin^2\psi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Power radiated by binary

For weakly stressed system neglecting pressure moment terms,

$$\mathcal{P} = \frac{32}{5G}\left(\frac{GM_c\,\omega_{gw}}{2}\right)^{10/3}\left[1 + \;5H^2\omega_{gw}^{-2} + 4H^4\omega_{gw}^{-4}\right]$$

$$H^2\omega_{gw}^{-2} \sim (\lambda/L_B)^2 \sim 10^{-42} - 10^{-44}$$

for LIGO, $f \sim 10^2 - 10^3\mathrm{Hz}$

$\sim 10^{-28} - 10^{-20}$ for LISA, $f \sim 10^{-4} - 1\ \mathrm{Hz}$

$\sim 10^{-18} - 10^{-12}$ for PTA, $f \sim 10^{-6} - 10^{-9}\ \mathrm{Hz}$

S.J. Hoque and A. Aggarwal,
IJMPD 28, 1950025 (2019)

Gravitational phasing formula

- In the weak field limit on de Sitter background the potential is Newtonian.

$$E = -\frac{GM\mu}{2R} \quad , \qquad \frac{dE}{dt} = -\left(\frac{G^2 M_c^5}{32}\right)^{1/3} \frac{2}{3} \omega_{gw}^{-1/3} \dot{\omega}_{gw}$$

- Including back reaction of gravitational waves in presence of cosmological constant one obtains a phasing formula

$$\begin{aligned} \phi &= \int d\omega_{gw} (\dot{\omega}_{gw})^{-1} \omega_{gw} \\ &= \frac{15}{12} 2^{-1/3} (GM_c)^{-5/3} \omega_{gw}^{-5/3} \left(-\frac{1}{5} + \frac{5}{11} H^2 \omega_{gw}^{-2} - \frac{21}{17} H^4 \omega_{gw}^{-4} + \cdots \right) + C \end{aligned}$$

Summary and outlook

- Discussed linearised gravitational waves in de Sitter background in a generalised Harmonic gauge.
- Obtain field in terms of quadrupole moment of the source.
- Power radiated quadrupole formula and its application to binary system

- Interesting to explore how cosmological constant affects waveform. It will give a bound on cosmological constant from current observations.
- Gravitational radiation in dS in Penrose's conformal language, linearisation stability and global hyperbolicity.

Ongoing work with P. Krtouš and C. Peón-Nieto

- Gravitational memory effect in de Sitter.

Ongoing work with G. Compère

- Gravitational waves in FLRW space-time.

 Ongoing work with B. Bonga and B. Schneider

1. Sk Jahanur Hoque and Amitabh Virmani, The Kerr-de Sitter spacetime in Bondi coordinates, Classical and Quantum Gravity 38, 225002 (2021).

2. Sumanta Chakraborty, Sk Jahanur Hoque and Roberto Oliveri, Gravitational multipole moments for asymptotically de Sitter spacetimes, Phys. Rev. D 104, 064019 (2021).

3. P.T. Chruściel, Sk Jahanur Hoque, Maciej Maliborski and Tomasz Smołka, On the canonical energy of weak gravitational elds with a cosmological constant $\Lambda \in \mathbb{R}$, The European Physical Journal C 81, 696 (2021) .

4. P.T. Chruściel, Sk Jahanur Hoque and Tomasz Smołka, Energy of Weak gravitational waves in spacetimes with a positive cosmological constant , Phys. Rev. D 103, 064008 (2021).

5. Aneesh Prema Balakrishnan, Sk Jahanur Hoque and Amitabh Virmani, Conserved charges in asymptotically de Sitter spacetimes, Classical and Quantum Gravity 36, 205008 (2019).

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7 . Sk Jahanur Hoque and Amitabh Virmani, On Propagation of Energy Flux in de Sitter Spacetime, Gen.Rel.Grav. 50, 40 (2018).

8. Ghanashyam Date and Sk Jahanur Hoque, Cosmological Horizon and the Quadrupole Formula in de Sitter background, Phys. Rev. D 96, 044026 (2017).

9. Ghanashyam Date and Sk Jahanur Hoque, Gravitational waves from compact sources in a de Sitter background, Phys. Rev. D 94, 064039 (2016).

Thank you

Outline

- We wish to generalise Bondi's mass loss formula for linearised gravitational field with a positive cosmological constant in covariant phase-space formalism.

- Discuss $\Lambda = 0$ limit.

- asymptotic fall off condition for fields.

- We will work in Bondi frame.

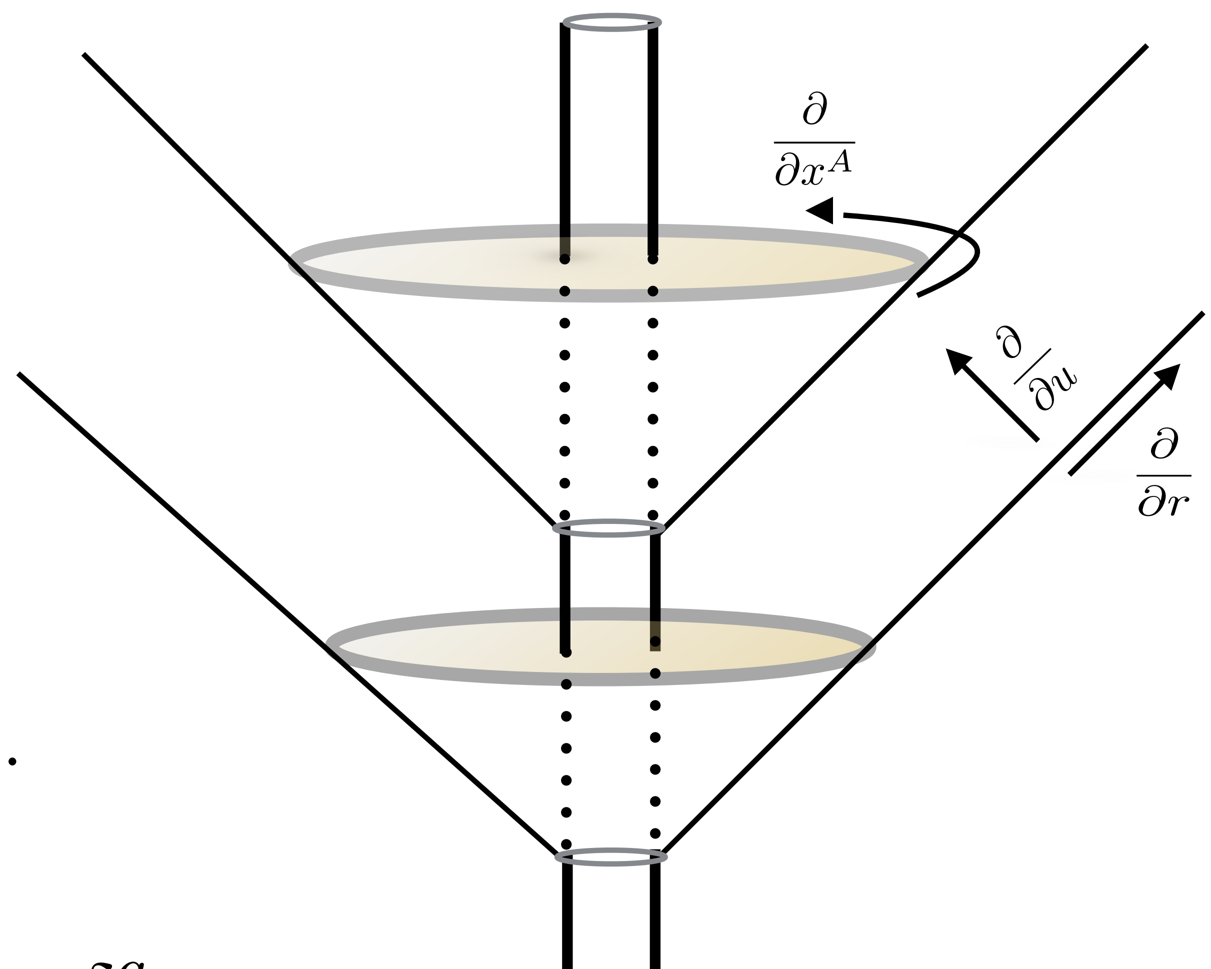
Set up for Bondi coordinates

- We construct Bondi coordinates for de Sitter.

- Bondi coordinates are based on a family of outgoing null hypersurfaces.

- Hypersurfaces $u = const$ are null.

$$\implies g^{ab}\partial_a u \partial_b u = 0 \implies g^{uu} = 0$$



- Two angular coordinates x^A, are constant along null rays.

$$\implies g^{ab}\partial_a u \partial_b x^A = 0 \implies g^{uA} = 0.$$

- g^{ab} and g_{ab} are related by $g^{ac}g_{cb} = \delta^a_b$

$$\implies g_{rr} = 0 = g_{rA}$$

Metric in Bondi-Sachs coordinates

- Metric in Bondi-Sachs coordinates,

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

- r varies along null rays, chosen to be an areal coordinate;

$$\det g_{AB} = r^4 \sin^2\theta$$

- We will explore Einstein equations for linearized fields on de Sitter background.

background $g_{ab}(\lambda = 0) := \bar{g}_{ab}$; perturbation $h_{ab} := \left.\frac{dg_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0}$

Bondi gauge for linearized theory

- In Bondi coordinates de Sitter Background metric takes the form,

$$\bar{ds}^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) du^2 - 2dudr + r^2 \mathring{\gamma}_{AB} dx^A dx^B$$

- Bondi gauge condition for linearized fields

$$h_{rr} = 0 = h_{rA}, \qquad \mathring{\gamma}^{AB} h_{AB} = 0$$

- We wish to explore linearised Einstein equation with Bondi metric,

$$E_{ab} := R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0$$

Einstein equations: systems of hierarchical PDEs

- Four independent hyper surface equations, $E^u_a = 0$

- $E^u_r = 0 \implies \partial_r \beta = \frac{r}{16}\gamma^{AC}\gamma^{BD}(\partial_r \gamma_{AB})(\partial_r \gamma_{CD})$

  Linearisation: $\partial_r \delta\beta = 0 \implies \delta\beta = \delta\beta(u, x^A)$

Using gauge $\delta\beta = 0 = \delta g_{ur}$, $h_{ab} \mapsto h_{ab} + \mathcal{L}_\xi \bar{g}_{ab}$

- $E^u_A = 0 \implies \partial_r[r^4 e^{-2\beta}\gamma_{AB}(\partial_r U^B)] = 2r^4 \partial_r \left(\frac{1}{r} D_A \beta\right) - r^2 \gamma^{EF} D_E(\partial_r \gamma_{AF})$

  Linearisation: $\partial_r[r^4 \partial_r (r^{-2}\delta g_{uA})] = r^2 \mathring{D}^F \partial_r (r^{-2}\delta g_{AF})$

- $$E^u_u = 0 \implies 2e^{-2\beta}(\partial_r V) = \mathcal{R} - 2\gamma^{AB}[D_A D_B \beta + D_A\beta D_B \beta] + \frac{e^{-2\beta}}{r^2} D_A[\partial_r(r^4 U^A)] - \frac{r^4}{2} e^{-2\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B) - 2\Lambda r^2$$

  Linearisation: $2\partial_r \delta V = \delta\mathcal{R} - \frac{1}{r^2}\mathring{D}^A[\partial_r(r^2 \delta g_{uA})]$

Solution for h_{uA}

Given the ansatz : $$h_{AB} = r^2\left(\overset{(0)}{\check{h}}_{AB} + \frac{\overset{(-1)}{\check{h}}_{AB}}{r} + \frac{\overset{(-2)}{\check{h}}_{AB}}{r^2} + \frac{\overset{(-3)}{\check{h}}_{AB}}{r^3} + \ldots\right)$$

solve $$\partial_r[r^4\partial_r(r^{-2}\delta g_{uA})] = r^2\mathring{D}^F\partial_r(r^{-2}\delta g_{AF})$$?

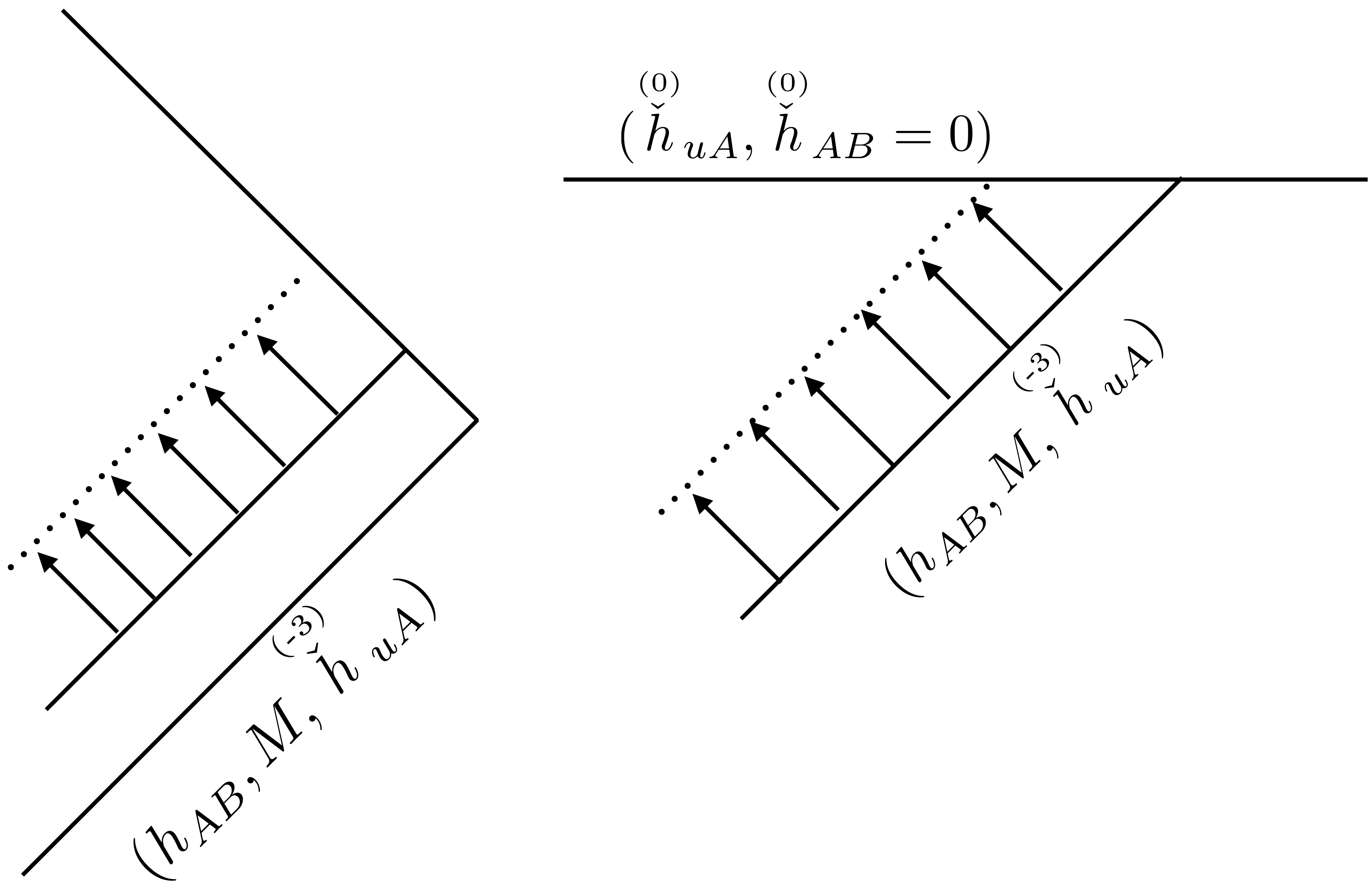
$$h_{uA} = r^2\left(\overset{(0)}{\check{h}}_{uA} + \frac{1}{2}\mathring{D}^B\overset{(-1)}{\check{h}}_{AB}\, r^{-2} + \left(\overset{(-3)}{\check{h}}_{uA} + \frac{2}{9}\mathring{D}^B\overset{(-2)}{\check{h}}_{AB}(3\ln r + 1)\right)r^{-3} + \ldots\right)$$

- Similarly, solve for V
- Given the ansatz h_{AB}, hypersuface equations $E^u_a = 0$ fix the asymptotic fall off condition for other components of field.

Evolution equation for $\check{h}_{AB}$

- Traceless symmetric parts of $E_{AB} = 0$ gives evolution equation for $\check{h}_{AB} := r^{-2} h_{AB}$

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2 \check{h}_{AB}))] = 0$$



Non-polyhomogenous de Sitter

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2\check{h}_{AB}))] = 0$$

Asymptotic analysis of this equation gives non trivial equations

$$\frac{\Lambda}{3}\overset{(-2)}{\check{h}}_{AB} = 0$$

- NO log term in de Sitter. De Sitter is non-polyhomogenous!!
- To get rid of log term one needs to set $\overset{(-2)}{\check{h}}_{AB} = 0,$ for flat space-time. In Bondi's paper this condition is termed as outgoing radiation condition.
- For de Sitter this is a consequence of equation of motion.
- This result is true for full non-linear theory also.

G. Compère, A. Fiorucci, R Ruzziconi - 2019

A. Pole, K. Skenderis, M. Taylor -2019

Asymptotic symmetry group is NOT BMS

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2 \check{h}_{AB}))] = 0$$

Asymptotic analysis of this equation also gives,

$$\partial_u \overset{(0)}{\check{h}}_{AB} = \frac{\Lambda}{3} \overset{(-1)}{\check{h}}_{AB} + (\mathring{D}_A \overset{(0)}{\check{h}}_{uB} + \mathring{D}_B \overset{(0)}{\check{h}}_{uA} - \mathring{\gamma}_{AB} \mathring{D}^C \overset{(0)}{\check{h}}_{uC})$$

- $\overset{(0)}{\check{h}}_{AB}$ and $\overset{(0)}{\check{h}}_{uA}$ can not be zero simultaneously by a gauge transformation!! Asymptotic symmetry group of de Sitter is not BMS.

- Whether this gauge condition is achieved by any physical space-time is difficult.

S. J. Hoque, A. Virmani - 2021

Asymptotic expansion of linearised fields

$$h_{AB} = r^2 \Big(\underbrace{\overset{(0)}{\check{h}}{}_{AB}}_{=0} + \frac{\overset{(-1)}{\check{h}}{}_{AB}}{r} + \underbrace{\overset{(-2)}{\check{h}}{}_{AB}}_{=0} r^{-2} + \frac{\overset{(-3)}{\check{h}}{}_{AB}}{r^3} + \ldots \Big),$$

$$h_{uA} = r^2 \Big(\overset{(0)}{\check{h}}{}_{uA} + \frac{1}{2} \mathring{D}^B \overset{(-1)}{\check{h}}{}_{AB} r^{-2} + \overset{(-3)}{\check{h}}{}_{uA} r^{-3} + \ldots \Big),$$

$$h_{uu} = r \mathring{D}^A \overset{(0)}{\check{h}}{}_{uA} + \frac{M}{r} - \frac{1}{2r^2} \mathring{D}^A \overset{(-3)}{\check{h}}{}_{uA} + \ldots$$

$$h_{ur} = 0$$

Evolution equations for integration constant

- $E_{uu} = 0$, gives the evolution equation for h_{uu}

$$2\partial_u M = \partial_u \mathring{D}^A \mathring{D}^B \overset{(\text{-}1)}{\check{h}}_{AB} - \Lambda \mathring{D}^A \overset{(\text{-}3)}{\check{h}}_{uA}$$

- $E_{uA} = 0$, gives the evolution equation for h_{uA}

$$3\partial_u \overset{(\text{-}3)}{\check{h}}_{uA} = \mathring{D}_A M + \frac{1}{2}(\mathring{D}^B \mathring{D}_A \mathring{D}^C \overset{(\text{-}1)}{\check{h}}_{CB} - \triangle_{\mathring{\gamma}} \mathring{D}^C \overset{(\text{-}1)}{\check{h}}_{CA}) - \Lambda \mathring{D}^B \overset{(\text{-}3)}{\check{h}}_{AB}$$

Energy in the linearised theory

- The Hamiltonian for the linearised theory associated with a hyper surface Σ, and a vector field X reads,

$$\tilde{\mathcal{H}}[\Sigma, X] := \int_\Sigma \tilde{\mathcal{H}}^\mu d\Sigma_\mu$$
$$= \frac{1}{2}\left(\int_\Sigma \omega^\mu(\tilde{\phi}, \mathcal{L}_X \tilde{\phi})\ d\Sigma_\mu - \int_{\partial\Sigma} X^{[\sigma} \tilde{\pi}_A^{\mu]} \tilde{\phi}^A d\Sigma_{\sigma\mu} \right)$$

$\tilde{\phi}$ is linearised field, $\tilde{\pi}^\mu$ is associated canonical conjugate momenta.

- When X is a time-translational symmetry of background, the numerical value of the integration is identified with the total energy of the field contained in Σ.

- Our approach differs from Ashtekar's group work in boundary terms.

Energy flux in the linearised theory

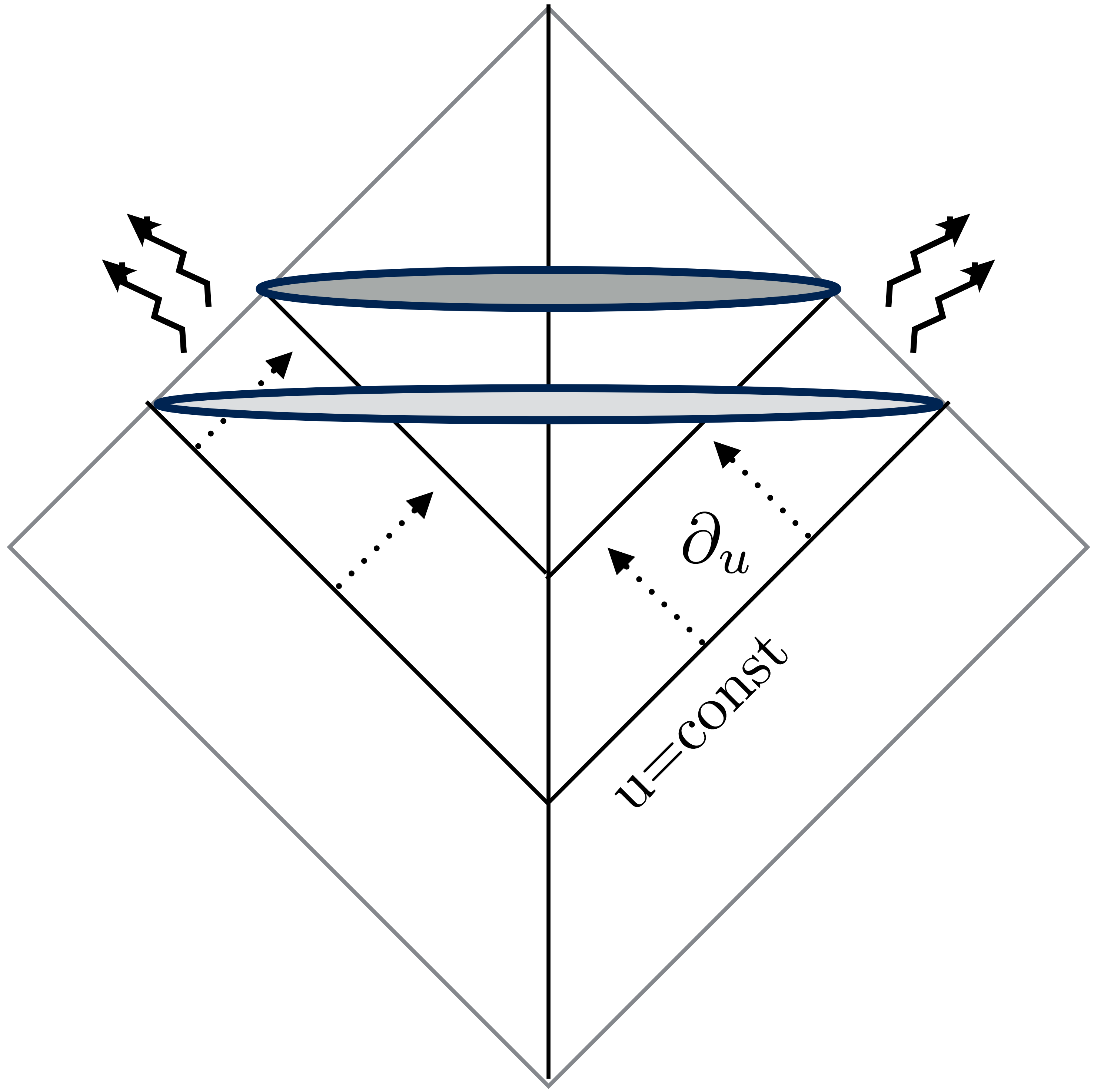
- Consider a family of hyper surfaces labelled by τ and define,

$$
\begin{aligned}
\frac{d\tilde{\mathcal{H}}[\Sigma_\tau, X]}{d\tau} &= \frac{1}{2}\frac{d}{d\tau}\int_\Sigma \omega^\mu(\tilde{\phi}, \mathcal{L}_X\tilde{\phi})\, d\Sigma_\mu - \frac{1}{2}\int_{\partial\Sigma}\mathcal{L}_X\left(X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\tilde{\phi}^A\right)d\Sigma_{\sigma\mu} \\
&= -\frac{1}{2}\int_{\partial\Sigma} X^{[\sigma}\omega^{\mu]}(\tilde{\phi}, \mathcal{L}_X\tilde{\phi})\, d\Sigma_{\sigma\mu} \\
&\quad -\frac{1}{2}\int_{\partial\Sigma}\left(X^{[\sigma}\mathcal{L}_X\tilde{\pi}_A{}^{\mu]}\tilde{\phi}^A + X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\mathcal{L}_X\tilde{\phi}^A\right)d\Sigma_{\sigma\mu}\,.
\end{aligned}
$$

Using, $\omega^\mu(\tilde{\phi}, \mathcal{L}_X\tilde{\phi}) := \mathcal{L}_X\tilde{\phi}^A\tilde{\pi}_A{}^\mu - \tilde{\phi}^A\mathcal{L}_X\tilde{\pi}_A{}^\mu$

$$
\frac{d\tilde{\mathcal{H}}(\Sigma_\tau, X)}{d\tau} = -\int_{\partial\Sigma_\tau} X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\mathcal{L}_X\tilde{\phi}^A d\Sigma_{\sigma\mu}.
$$

The integrand represents the flux of the energy through $\partial\Sigma$ when Σ is dragged along the flow of X.



∂_u
u=const

Canonical energy for gravitational field

$$E_c[h, \mathcal{C}_{u,R}] = \frac{1}{64\pi}\int_{\mathcal{C}_{u,R}} \bar{g}^{BE}\bar{g}^{FC}(\partial_u h_{BC}\partial_r h_{EF} - h_{BC}\partial_r\partial_u h_{EF}) r^2 \sin\theta dr d\theta d\phi$$
$$- \frac{1}{32\pi}\int_{S(R)} \bar{P}^{r(bc)d(ef)} h_{bc}\bar{\nabla}_d h_{ef} \; r^2 \sin\theta d\theta d\phi$$

where:

h_{ab} solution of linearised vacuum Einstein equations,

$\mathcal{C}_u$ light cone $u = \mathrm{const}$ emanating from $r = 0$

$\mathcal{C}_{u,R}$ light cone truncated at radius $r = R$

$S(R)$ sphere of radius R

Boundary term in canonical energy

- In Bondi gauge boundary integral of $E_C(h, \mathcal{C}_{u,R})$ becomes,

$$-\frac{\Lambda R}{192\pi}\int_{S^2} \mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(\text{-}1)}{\check{h}}{}_{AC}\overset{(\text{-}1)}{\check{h}}{}_{BD}\ \sin\theta d\theta d\phi$$

$$-\frac{1}{64\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(\text{-}1)}{\check{h}}{}_{AC}\overset{(\text{-}1)}{\check{\partial}}{}_{u}\check{h}_{BD} - 6\mathring{\gamma}^{AB}\overset{(0)}{\check{h}}{}_{uA}\overset{(\text{-}3)}{\check{h}}{}_{uB})\sin\theta d\theta d\phi$$

Renormalised energy and flux

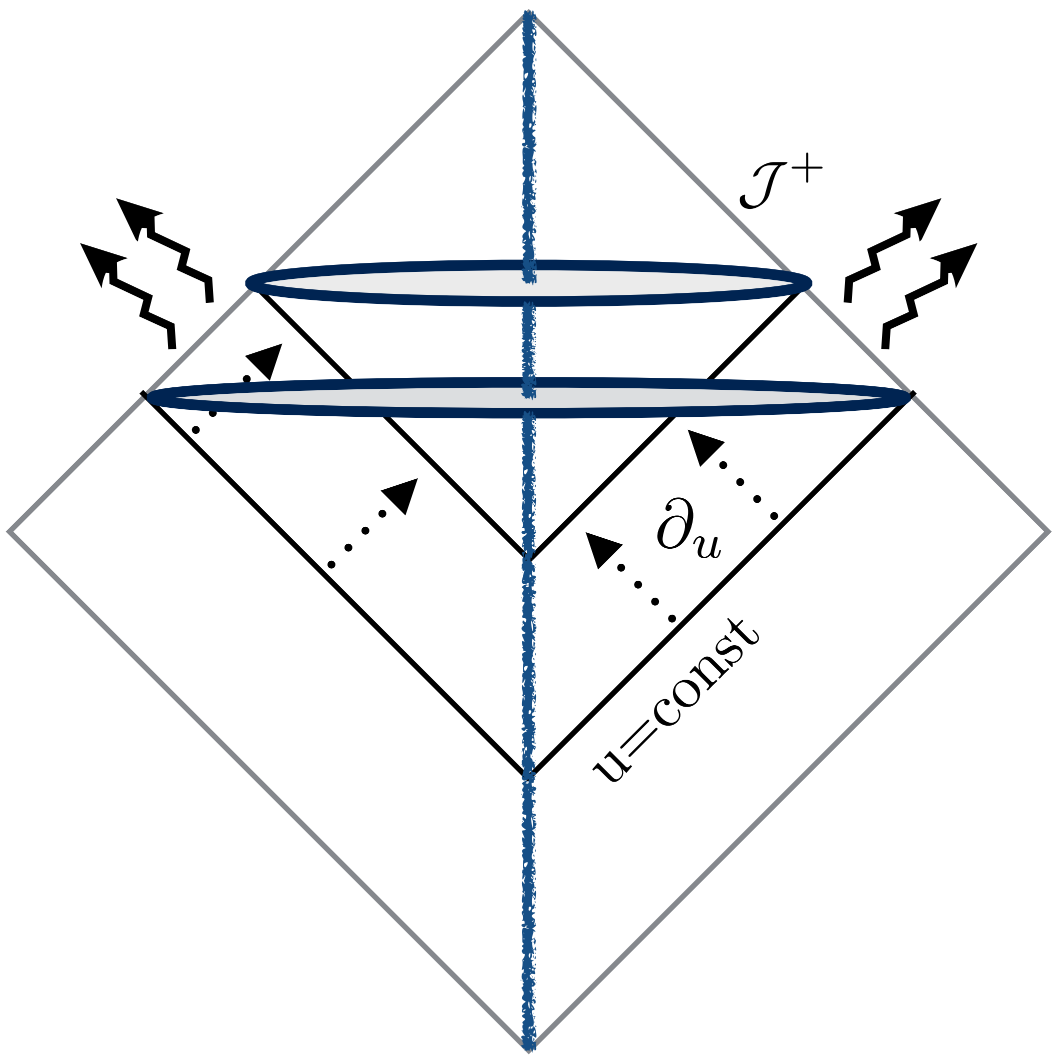
We propose to introduce a renormalised canonical energy

$$\begin{aligned}\hat{E}_c[h,\mathcal{C}_u] \quad &:= \quad \frac{1}{64\pi}\int_{\mathcal{C}_u} g^{BE}g^{FC}(\partial_u h_{BC}\partial_r h_{EF} - h_{BC}\partial_r\partial_u h_{EF})r^2\sin\theta dr d\theta d\phi \\ &- \quad \frac{1}{64\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(-1)}{\check{h}}{}_{AC}\partial_u\overset{(-1)}{\check{h}}{}_{BD} - 6\mathring{\gamma}^{AB}\overset{(0)}{\check{h}}{}_{uA}\overset{(-3)}{\check{h}}{}_{uB})\sin\theta d\theta d\phi\end{aligned}$$

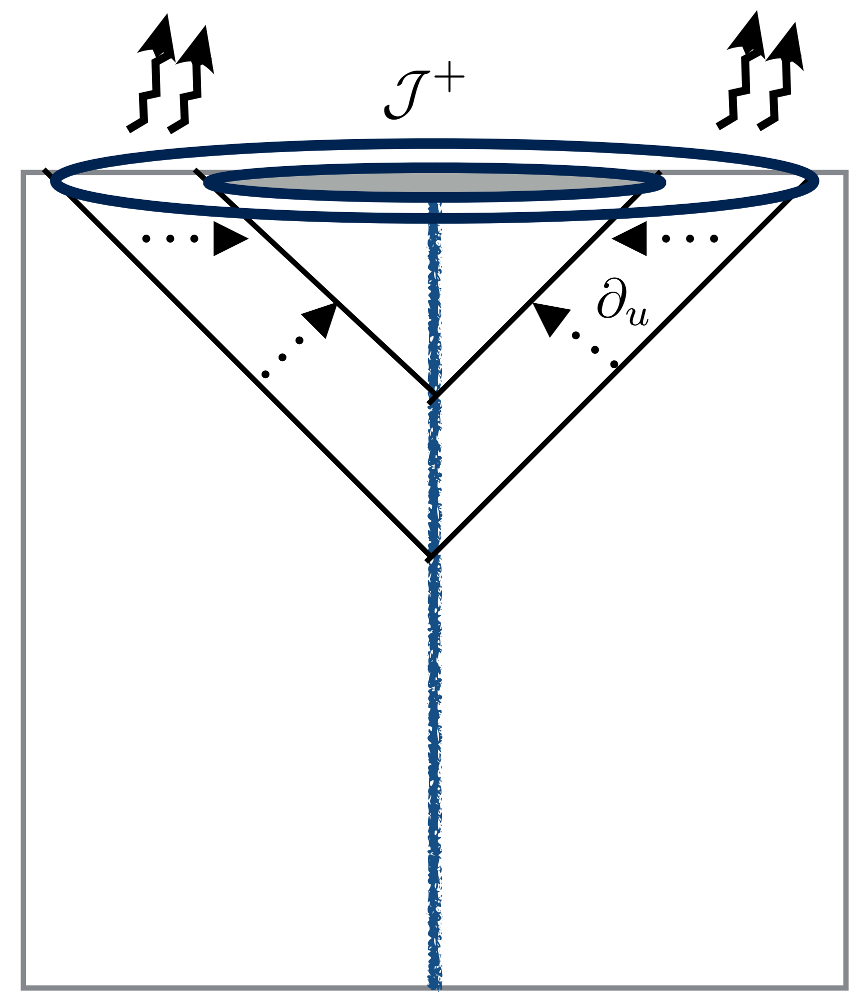
which has its own flux formula

$$\frac{d\hat{E}_c[h,\mathcal{C}_u]}{du} = -\frac{1}{32\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\partial_u\overset{(-1)}{\check{h}}{}_{AC}\partial_u\overset{(-1)}{\check{h}}{}_{BD} - 6\mathring{\gamma}^{AB}\overset{(-3)}{\check{h}}{}_{uA}\partial_u\overset{(0)}{\check{h}}{}_{uB})\sin\theta d\theta d\phi$$

For $\Lambda = 0$, we obtain linearised version of Bondi's mass-loss formula.



$\Lambda = 0$



$\Lambda > 0$

Summary

- Bondi-Sachs coordinates are constructed for de Sitter.
- NO log term in de Sitter
- Asymptotic fall off condition for linearised gravitational field have been obtained in Bondi frame. Qualitatively different from $\Lambda = 0$ case.
- Due to different fall-off asymptotic symmetry group is not BMS
- A definition of candidate Energy and Energy flux have been obtained for linearised fields.
- Interesting to generalise Bondi-Sachs formalism for FLRW case.
- How our solutions are related to other linearised solutions on dS background and quadrupole formula in de Sitter background.

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The residual gauge transformations are thus defined by a u-parameterised family of vector fields $\xi^A(u,\cdot)$ on S^2 together with

$$\partial_u \xi^u(u,x^A) = \frac{\mathring{D}_B \xi^B(u,x^A)}{2}\,, \tag{3.44}$$

and (3.25). Explicitly:

$$\begin{aligned}\zeta &= \left(\int \frac{\mathring{D}_B \xi^B(u,x^A)}{2} du + \mathring{\xi}^u(x^A)\right)\partial_u + \frac{1}{2}\left(\Delta_{\mathring{\gamma}}\xi^u - r\mathring{D}_B\xi^B\right)\partial_r \\ &\quad + (\xi^B(u,x^A) - \frac{1}{r}\mathring{D}^B\xi^u(u,x^A))\partial_B\,,\end{aligned} \tag{3.45}$$

with an arbitrary function $\mathring{\xi}^u(x^A)$.

Covariant phase-space: Linearised Lagrangian

- Given a Lagrangian density $\mathcal{L}(\phi, \partial\phi)$, the field equations are

$$\mathcal{E}_A := \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial {\phi^A}_\mu} \right) - \frac{\partial \mathcal{L}}{\partial \phi^A} = 0 \quad ; \quad \phi^A_\mu := \partial_\mu \phi^A$$

- Consider, ϕ as a one parameter family of field configuration.

background: $\phi(\lambda = 0) = \bar{\phi}$, Linearised field: $\tilde{\phi} := \frac{d\phi}{d\lambda}|_{\lambda=0}$

Covariant phase-space: Linearised Lagrangian

- Linearised equation:

$$\partial_\mu\left(\pi_A{}^\mu{}_B{}^\nu\partial_\nu\tilde{\phi}^B+\pi_A{}^\mu{}_B\tilde{\phi}^B\right)=\left(\pi_B{}^\mu{}_A\partial_\mu\tilde{\phi}^B+\pi_{AB}\tilde{\phi}^B\right)+\frac{d\mathcal{E}_A}{d\lambda}$$

with,

$$\pi_A{}^\mu:=\frac{\partial\mathcal{L}}{\partial\phi^A{}_\mu}\,,\qquad\pi_A:=\frac{\partial\mathcal{L}}{\partial\phi^A}\,,$$

$$\pi_A{}^\mu{}_B{}^\nu:=\frac{\partial^2\mathcal{L}}{\partial\phi^A{}_\mu\partial\phi^B{}_\nu}\,,\;\pi_A{}^\mu{}_B:=\frac{\partial^2\mathcal{L}}{\partial\phi^A{}_\mu\partial\phi^B}\,,\;\pi_{AB}:=\frac{\partial^2\mathcal{L}}{\partial\phi^A\partial\phi^B}$$

- Linearised Lagrangian density,

$$\tilde{\mathcal{L}}=\frac{1}{2}\pi_A{}^\mu{}_B{}^\nu\partial_\mu\tilde{\phi}^A\partial_\nu\tilde{\phi}^B+\pi_A{}^\mu{}_B\partial_\mu\tilde{\phi}^A\tilde{\phi}^B+\frac{1}{2}\pi_{AB}\tilde{\phi}^A\tilde{\phi}^B+\frac{d\mathcal{E}_A}{d\lambda}\tilde{\phi}^A$$

Covariant phase-space: Hamiltonian density

- Hamiltonian density for $\mathcal{L}$ and a vector field X,

$$\mathcal{H}^\mu[X] = \frac{\partial \mathcal{L}}{\partial {\phi^A}_\mu} \mathcal{L}_X \phi^A - X^\mu \mathcal{L}$$

$$\swarrow$$

$$\theta^\mu(\phi, \mathcal{L}_X \phi)$$

$$\delta \mathcal{L} = E_A(\phi)\delta\phi^A + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial {\phi^A}_\mu} \delta \phi^A \right)$$

- Corresponding Hamiltonian density for Linearised Lagrangian,

$$\tilde{\mathcal{H}}^\mu[X] = \frac{\partial \tilde{\mathcal{L}}}{\partial {\tilde{\phi}^A}_\mu} \mathcal{L}_X \tilde{\phi}^A - X^\mu \tilde{\mathcal{L}}$$

Covariant phase-space: presymplectic current

- Consider two parameter family of field configuration

$\phi^A(\lambda, \tau)$,

$$\omega^\mu\left(\frac{d\phi}{d\lambda}, \frac{d\phi}{d\tau}\right) := \frac{d\phi^A}{d\tau}\frac{d\pi^\mu_A}{d\lambda} - \frac{d\phi^A}{d\lambda}\frac{d\pi^\mu_A}{d\tau}$$

- In Wald-Zoupas terminology,

$$\omega^\mu(\phi, \delta_1\phi, \delta_2\phi) := \delta_1\theta^\mu(\phi, \delta_2\phi) - \delta_2\theta^\mu(\phi, \delta_1\phi)$$

- On linearised field equations, presymplectic current is conserved.

$$\partial_\mu\omega^\mu = \frac{d\mathcal{E}_A}{d\lambda}\frac{d\phi^A}{d\tau} - \frac{d\mathcal{E}_A}{d\tau}\frac{d\phi^A}{d\lambda}$$

Relation between Hamiltonian and presymplectic current

- From the conservation of presymplectic current, naively one would expect flux should be related to ω^i

$$\partial_t \int \omega^t d^3x = -\int \omega^i n_i d^2S$$

- We wish to ask how canonical energy of linearised theory is related to presymplectic current.

How does flux law related to presymplectic current?

- The relation between canonical energy of linearised theory and presymplectic current can be established by taking the second variation of Hamiltonian vector density in original theory.

First variation of Hamiltonian

$$\mathcal{H}^\mu[X] := \frac{\partial \mathcal{L}}{\partial {\phi^A}_\mu} \mathcal{L}_X \phi^A - X^\mu \mathcal{L}$$
$$:= \pi^\mu_A \mathcal{L}_X \phi^A - X^\mu \mathcal{L}$$

$$\frac{d\mathcal{H}^\mu[X]}{d\lambda} = \mathcal{L}_X \phi^A \frac{d\pi^\mu_A}{d\lambda} - \mathcal{L}_X \pi^\mu_A \frac{d\phi^A}{d\lambda} + 2\partial_\sigma \left(X^{[\sigma} \pi^{\mu]}_A \frac{d\phi^A}{d\lambda} \right)$$
$$+ \mathcal{H}^\mu[\frac{dX}{d\lambda}] + X^\mu \mathcal{E}_A \frac{d\phi^A}{d\lambda}$$

- Vector field X does not depend on the field configurations: $\dfrac{dX}{d\lambda} = 0$

Second variation of Hamiltonian

$$\left.\frac{d^2\mathcal{H}^\mu}{d\lambda^2}\right|_{\lambda=0} = \mathcal{L}_X\tilde{\phi}^A\tilde{\pi}^\mu_A + \mathcal{L}_X\phi^A\frac{d}{d\lambda}\tilde{\pi}^\mu_A - \mathcal{L}_X\tilde{\pi}^\mu_A\tilde{\phi}^A - \mathcal{L}_X\pi^\mu_A\frac{d}{d\lambda}\tilde{\phi}^A$$

$$+ 2\partial_\sigma\left(X^{[\sigma}\tilde{\pi}^{\mu]}_A\tilde{\phi}^A - X^{[\sigma}\pi^{\mu]}_A\frac{d\tilde{\phi}^A}{d\lambda}\right)$$

- second variation of Hamiltonian can also be written in terms linearised Hamiltonian density,

$$\left.\frac{d^2\mathcal{H}^\mu}{d\lambda^2}\right|_{\lambda=0} = \mathcal{L}_X\phi^A\frac{d\tilde{\pi}^\mu_A}{d\lambda} - \mathcal{L}_X\pi^\mu_A\frac{d\tilde{\phi}^A}{d\lambda} + 2\tilde{\mathcal{H}}^\mu - 2\partial_\sigma\left(X^{[\mu}\pi^{\sigma]}_A\frac{d\tilde{\phi}^A}{d\lambda}\right)$$

- Comparing these two,

$$\tilde{\mathcal{H}}^\mu[X] = \frac{1}{2}\Big(\underbrace{\mathcal{L}_X\tilde{\phi}^A\tilde{\pi}^\mu_A - \mathcal{L}_X\tilde{\pi}^\mu_A\tilde{\phi}^A}_{\omega^\mu(\tilde{\phi},\mathcal{L}_X\tilde{\phi})}\Big) + \partial_\sigma\left(X^{[\sigma}\tilde{\pi}^{\mu]}_A\tilde{\phi}^A\right)$$

Summary and questions

- A definition of candidate Energy and Energy flux have been obtained for linearised fields.

- Asymptotic fall off condition for linearised gravitational fields are Qualitatively different from $\Lambda = 0$ case.

- Proposed renormalised energy and flux in the limit $\Lambda = 0$ become classical Bondi quantities.

Our works can be extended in several directions:

- adding matter fields, generalisation to FLRW case
- Implication of new term in radiation reaction, and in the gravitational wave observations.
- How our solutions are related to other linearised solutions on dS background.

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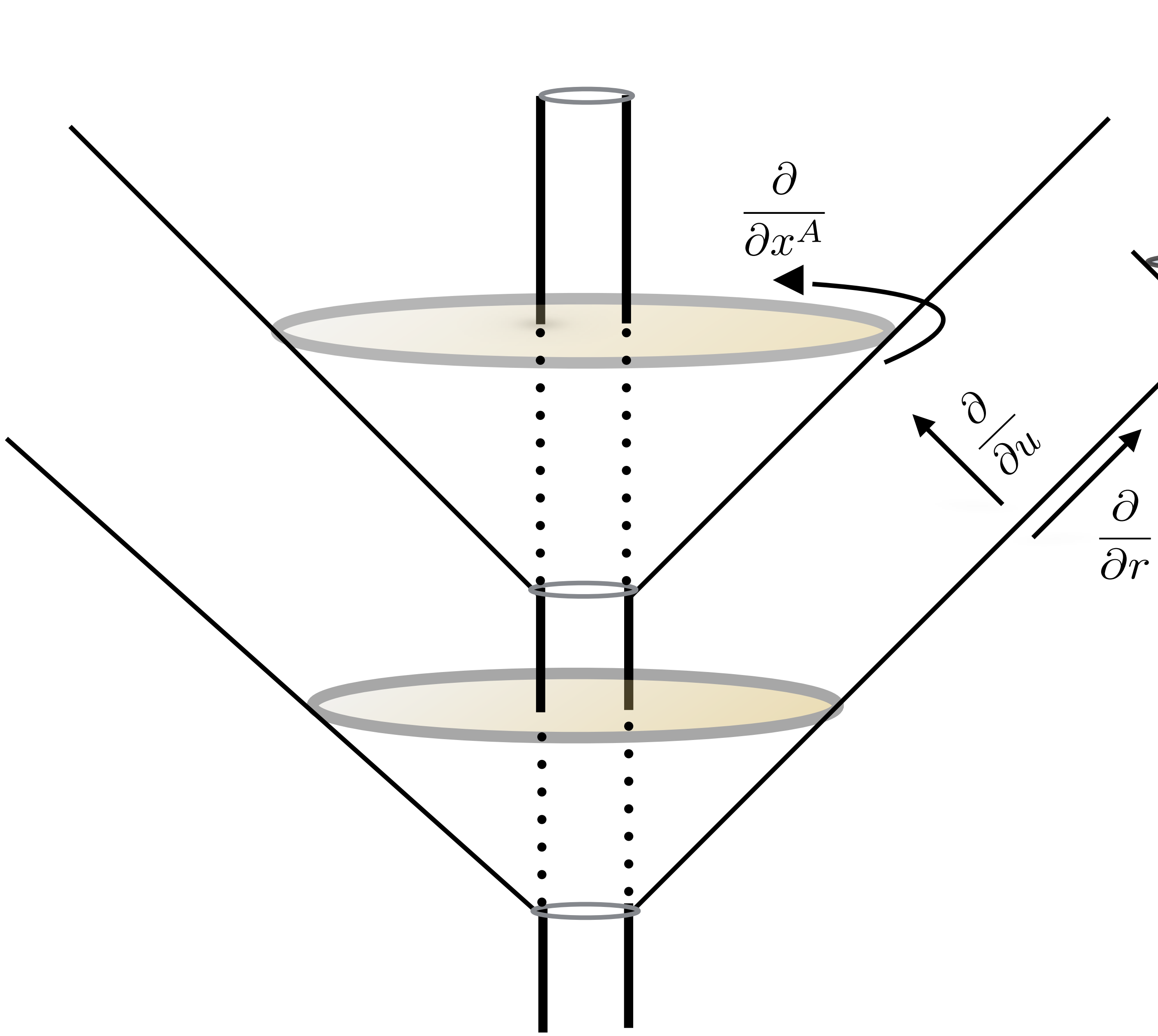
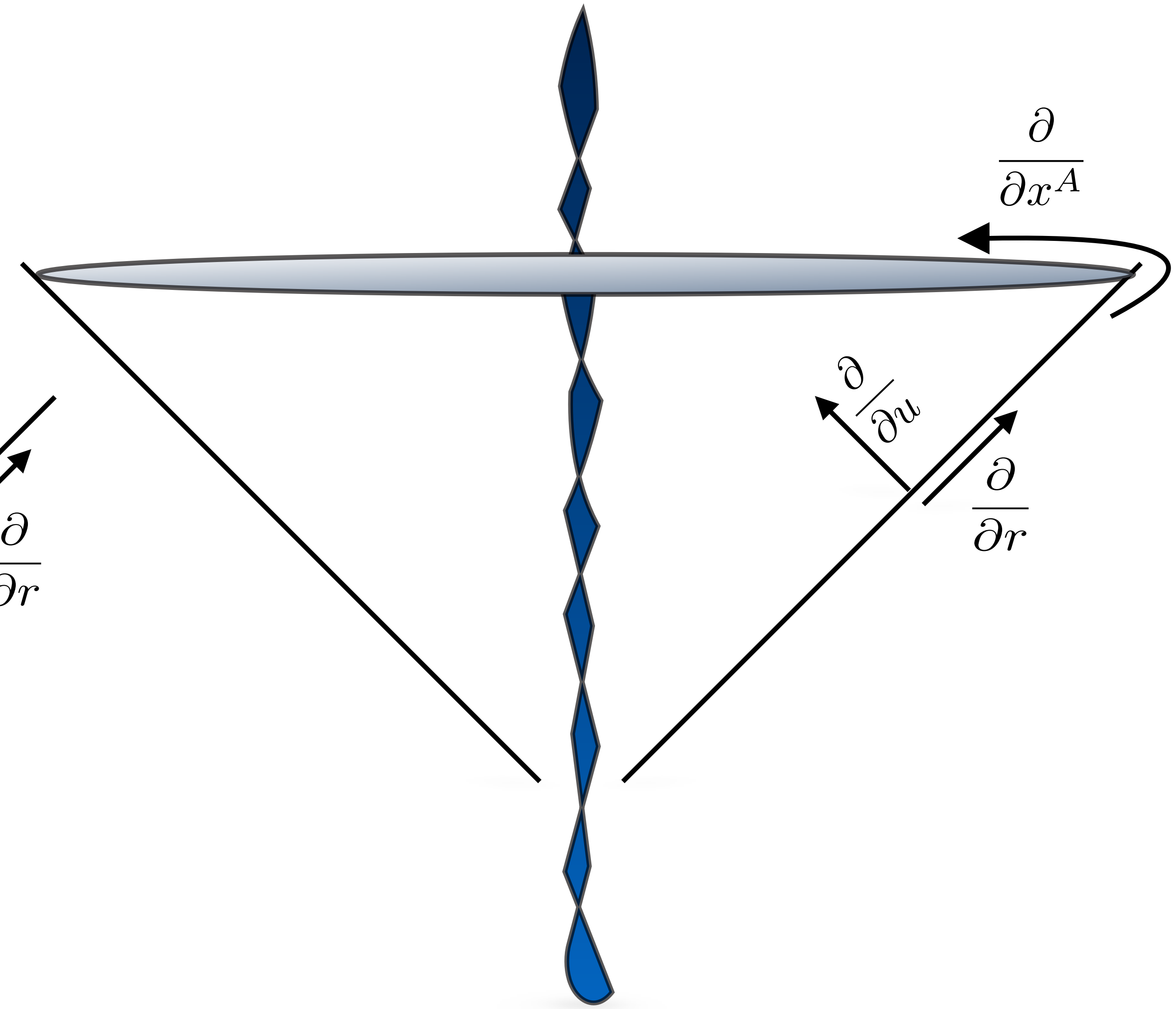


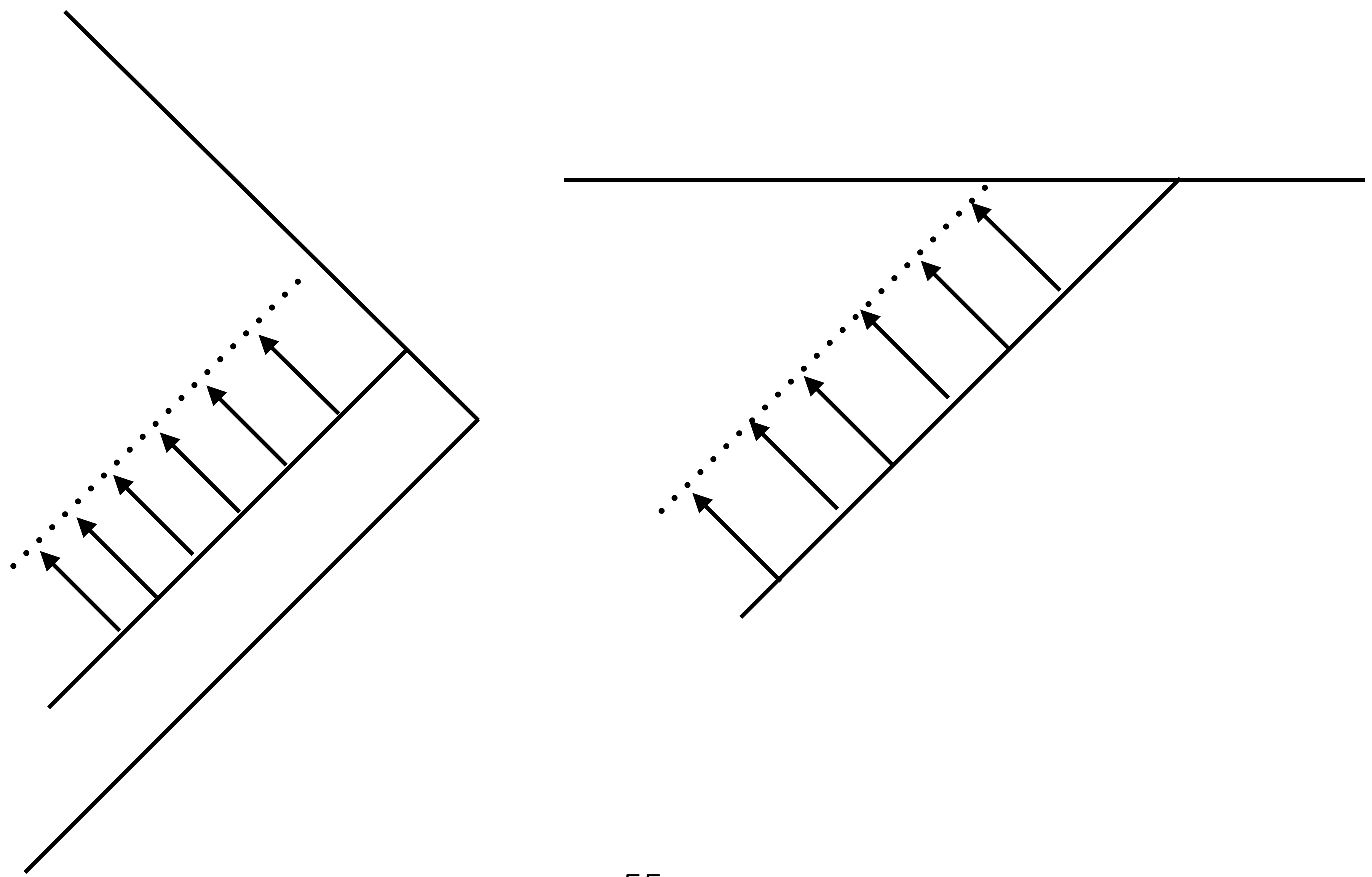
$\frac{\partial}{\partial x^A}$
$\partial/\partial u$
$\frac{\partial}{\partial r}$



$\frac{\partial}{\partial x^A}$
$\partial/\partial u$
$\frac{\partial}{\partial r}$

Based on: Phys.Rev.D 103 (2021) 6, 064008 with Piotr T. Chruściel, Tomasz Smolka,

EPJC, 81, 696(2021) with Piotr T. Chruściel, Tomasz Smolka, Maciej Maliborski



Outline

- A simpler version of the problem - linearised field in de Sitter background.

- We wish to generalise Bondi's mass loss formula for linearised gravitational field with a positive cosmological constant in covariant phase-space formalism.

- Discuss $\Lambda = 0$ limit.

- Asymptotic fall off condition for fields.

- We will work in Bondi frame.

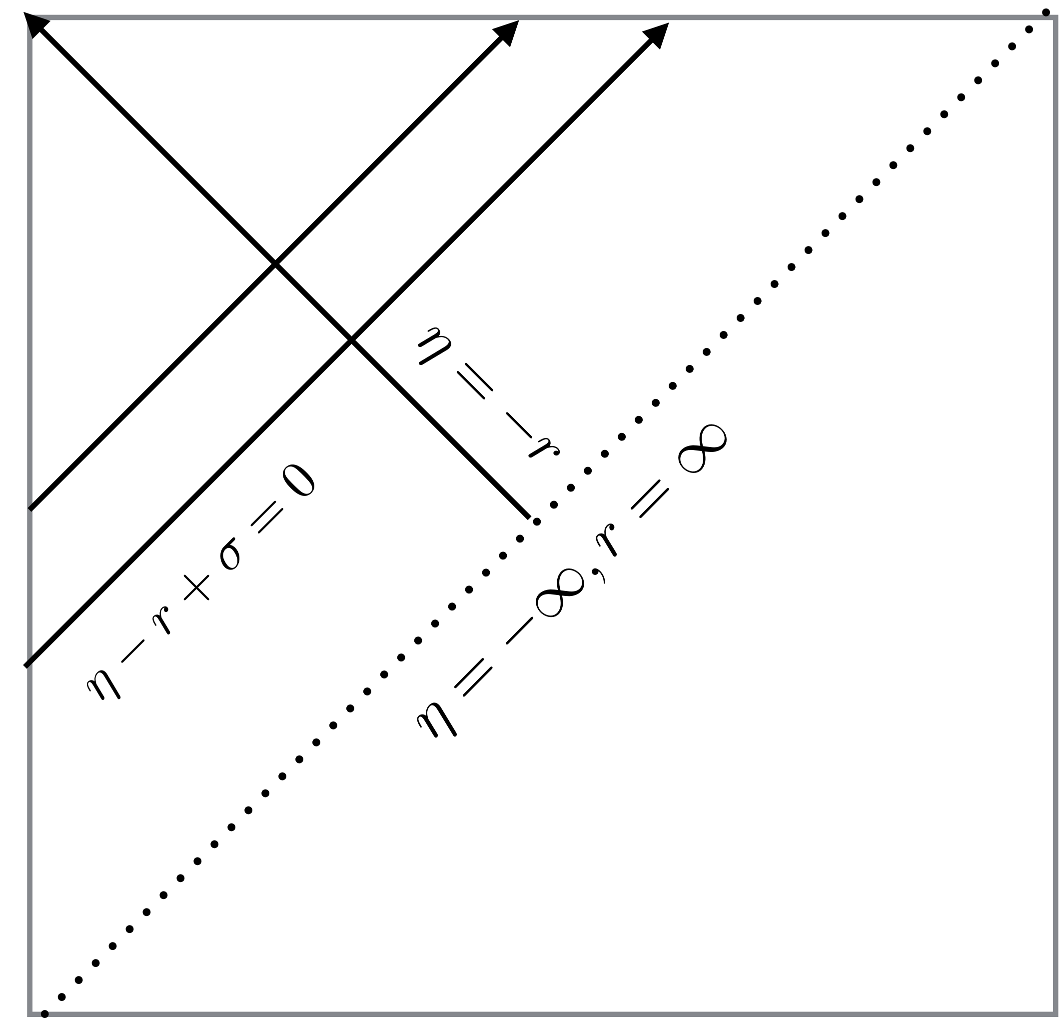
$$\int_{\Sigma_\rho} d\Sigma_\alpha \omega^\alpha = \int_{-\infty}^{+\infty} d\tau \int_{S^2} d\Omega \; r^2 a^3 \Big(H\rho \; \omega^0 + \frac{\omega^i x_i}{r} \Big)$$

$$= H\rho^2 \int_{-\infty}^{+\infty} d\tau \int_{S^2} d\Omega \Big[\, \big(\frac{d}{d\tau}\chi_{ij}^{tt}\big)\big(r\partial_\eta \chi_{kl}^{tt} + \eta \partial_r \chi_{kl}^{tt}\big) \Big] \delta^{ik}\delta^{jl}$$

and for rapidly varying source, $\partial_r \chi_{ij} \approx -\partial_\eta \chi_{ij}$

Flux through null hyper surfaces

Null hyper surfaces : $\eta + \epsilon r + \sigma = 0$



Null normals : $n_\mu = \gamma(1, \epsilon \hat{x}_i)$

$$\int d\Sigma_\alpha \omega^\alpha = \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d\Omega\ r^2 a^2\ \gamma \left(\omega^0 + \epsilon \frac{\omega^i x_i}{r} \right)$$

$$= \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d\Omega\ [\gamma H] \left[\frac{(1+\epsilon)}{8\pi} \frac{\eta^2}{\eta - r} \mathcal{R}^{tt}_{ij} \mathcal{R}^{tt}_{kl} \right] \delta^{ik} \delta^{jl}$$

Remarks :

- Identifying null normal of cosmological horizon with Killing vector, $\gamma = -(Hr)^{-1}$, flux matches with $r_{phy} = const$ hyper surface.

- Vanishing flux across outgoing null hypersurface

  $\Longrightarrow$ sharp energy propagation

- Radiated power can be defined on cosmological horizon

$$\mathcal{P}(\tau) := \frac{dE}{d\tau} = \frac{1}{8\pi}\int_{S^2} d\Omega\, \mathcal{R}^{tt}_{ij}\mathcal{R}^{tt}_{kl}\, \delta^{ik}\delta^{jl}$$

$$\mathcal{R}^{tt}_{mn} := \left[\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}\right]^{tt}(t_{ret}) \ ,$$

LINEARISED THEORY

Choose a background metric : $\bar{g}_{\mu\nu}(x)$

Define perturbation as : $g_{\mu\nu}(x) := \bar{g}_{\mu\nu}(x) + \epsilon h_{\mu\nu}(x)$

Gauge transformations : $\delta h_{\mu\nu} := \mathcal{L}_\xi \bar{g}_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$

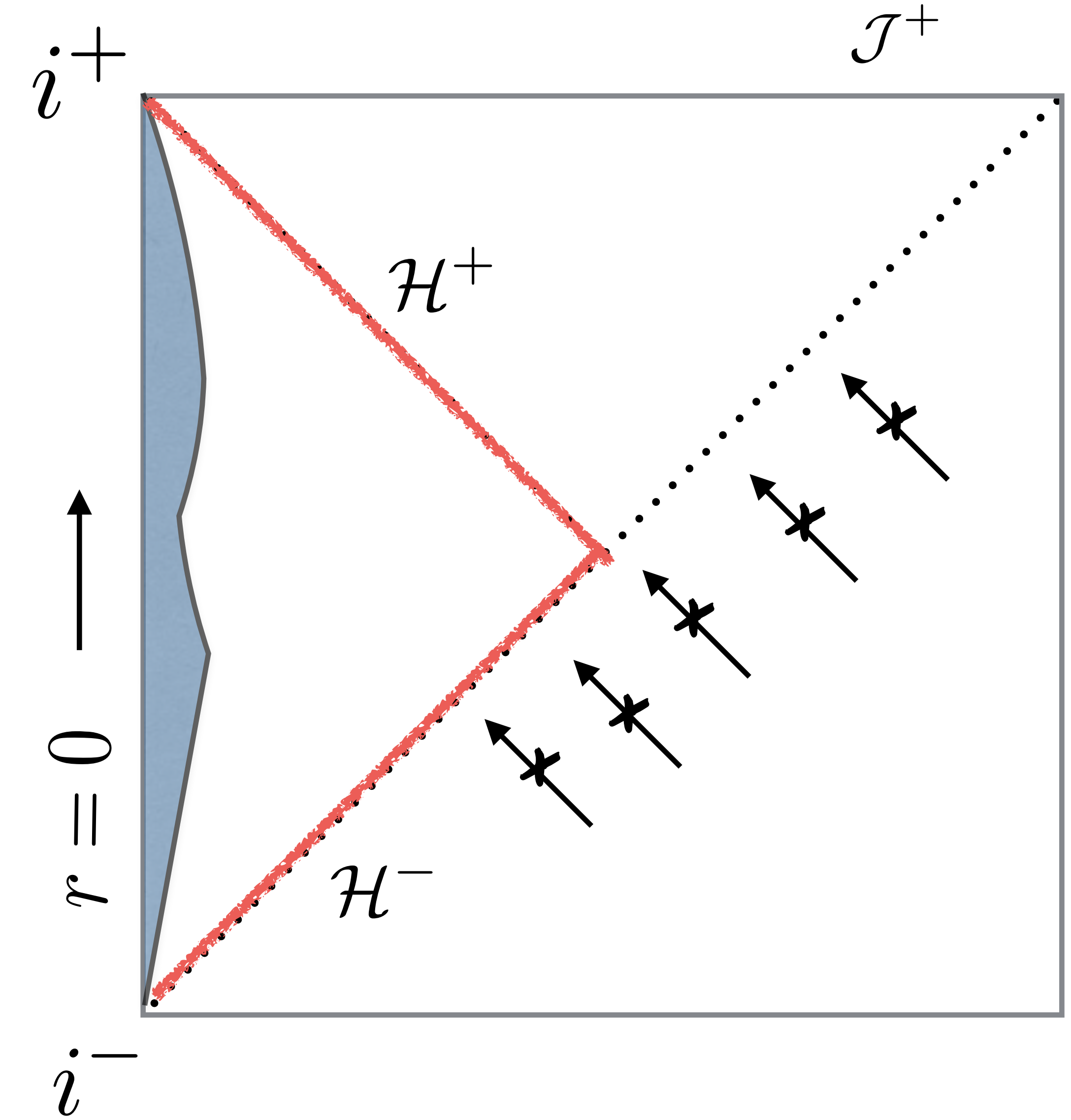
Physical perturbations : solutions of the linearised Einstein solution modulo the gauge transformations.

For explicit calculation we need to choose coordinates, choose a gauge, identify region of interest and compute observables.

$$\chi^{TT}_{ij} \approx \Lambda_{ij}{}^{kl}\chi_{kl}, \quad \Lambda_{ij}{}^{kl} := \frac{1}{2}(P_i{}^k P_j{}^l + P_i{}^l P_j{}^k - P_{ij}P^{kl}) \ , \ P_i{}^j := \delta_i{}^j - \hat{x}_i \hat{x}^j .$$

Cosmological Horizon : effective null infinity

- Observer at finite physical distance away from source must remain confined within the cosmological horizon.
- For rapidly varying compact source quadrupole power can be evaluated at the cosmological horizon



- No incoming radiation + conservation

$\Longrightarrow$ Energy flux at cosmological horizon exactly matches with that of at $\mathcal{J}^+$

S. J. Hoque and A. Virmani
in preparation