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Stochastic Gravitational Wave Background and Boltzmann equation

Effect of collisions off compact structures

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SUMMARY

- **Stochastic Gravitational Wave Background (SGWB)**
- **Theoretical ingredients**
  - The SGWB Stokes parameters
- **(Effective) gravitational Compton scattering**
- **SGWB Boltzmann equation for intensity and polarisation anisotropies**
  - The effect of the collision term
  - Angular power spectrum: analytic toy-model solution
- **Conclusions**

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Stochastic Gravitational Wave Background

A gravitational wave background is a **superposition of GW from unresolved sources**, throughout the entire history of the universe

Astrophysical background: random noise composed of individual deterministic signals that overlap in time (or in frequency)

Cosmological background: intrinsically stochastic rmechanisms in the early universe:

- **Inflation**
- **Phase transitions**
- **Cosmic strings**

Stochastic**:** described statistically in terms of expectation values

THEORETICAL INGREDIENTS

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GW usually expanded in plane wave basis as :

$$h_{ij}(t,\boldsymbol{x}) = \sum_{P=+,\times} \int_{-\infty}^{+\infty} \mathrm{d}f \int_{S^2} \mathrm{d}\Omega \; h_P(f,\boldsymbol{n}) e^{2\pi i f (t - \boldsymbol{n}\cdot\boldsymbol{x})} \boldsymbol{e}_{ij}^{P}(\boldsymbol{n})$$

Polarisation coefficients

Polarisation tensor basis

Statistical properties of the field described by the ***moments*** of the metric perturbations: $\langle h_P(f,\boldsymbol{n}) h_{P'}^{*}(f',\boldsymbol{n}')\rangle$

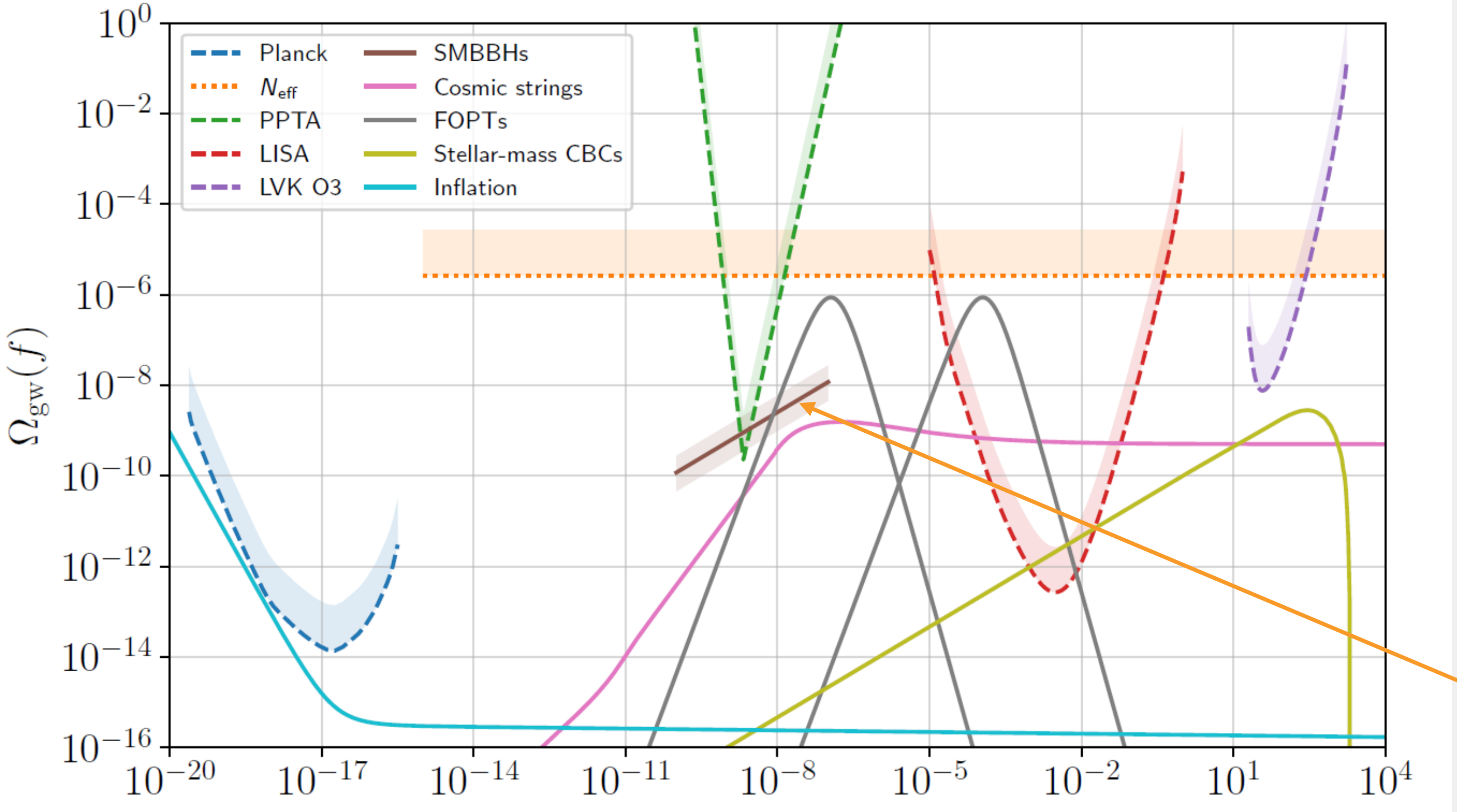
Isotropic, unpolarised background: $\langle h_P(f,\boldsymbol{n}) h_{P'}^{*}(f',\boldsymbol{n}')\rangle = \frac{1}{8\pi} S_h(f) \delta_D(f-f') \delta_{PP'} \delta_D^{(2)}(\boldsymbol{n}-\boldsymbol{n}')$

$S_h(f) \rightarrow$ (one sided) strain power spectrum

THEORETICAL INGREDIENTS

$S_h(f)$ is connected to the total fractional GW density:

$$S_h(f) = \frac{3H_0^2}{2\pi}\frac{\Omega_{gw}(f)}{f^3}$$



Renzini et al., 2022

Tipically parametrised as a power law:

$$\Omega_{gw}(f) = \Omega_\beta \left(\frac{f}{f_{ref}}\right)^\beta$$

CMB constraints for the *total* GW denisty $\Omega_{gw} \lesssim 10^{-6}$ (Pagano et al., 2016)

Eccess of red noise in PTA data (NANOGrav coll., 2020)

STOKES PARAMETERS FOR SGWB

So far we have not discussed about the polarisation content of the SGWB

Description in terms of the **density matrix** (Seto, 2008, Bartolo et al., 2018) of an ensamble of gravitons:

$$\rho(f,\boldsymbol{n}) = \begin{pmatrix} \langle h_+(f,\boldsymbol{n}) h_+^*(f',\boldsymbol{n}')\rangle & \langle h_+(f,\boldsymbol{n}) h_\times^*(f',\boldsymbol{n}')\rangle \\ \langle h_\times(f,\boldsymbol{n}) h_+^*(f',\boldsymbol{n}')\rangle & \langle h_\times(f,\boldsymbol{n}) h_\times^*(f',\boldsymbol{n}')\rangle \end{pmatrix} =$$

$$= \frac{1}{4\pi}\begin{pmatrix} (I+Q)(f,\boldsymbol{n}) & (U-iV)(f,\boldsymbol{n}) \\ (U+iV)(f,\boldsymbol{n}) & (I-Q)(f,\boldsymbol{n}) \end{pmatrix} \delta_D(f-f')\delta_D(\boldsymbol{n}-\boldsymbol{n}')$$

$\boldsymbol{I}$ → INTENSITY (related to $\Omega_{gw}(f)$)

$\boldsymbol{Q}, \boldsymbol{U}$ → LINEAR POLARISATIONS

$\boldsymbol{V}$ → CIRCULAR POLARSIATION

$\Bigg\}$ *STOKES PARAMETERS* for gravitational waves

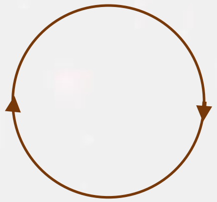
STOKES PARAMETERS FOR SGWB

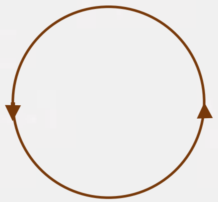
$$\boldsymbol{I} = \langle h_+(f,\boldsymbol{n})h_+^*(f',\boldsymbol{n}')\rangle + \langle h_\times(f,\boldsymbol{n})h_\times^*(f',\boldsymbol{n}')\rangle$$

$$\boldsymbol{Q} = \langle h_+(f,\boldsymbol{n})h_+^*(f',\boldsymbol{n}')\rangle + \langle h_\times(f,\boldsymbol{n})h_\times^*(f',\boldsymbol{n}')\rangle$$

$$\boldsymbol{U} = \langle h_+(f,\boldsymbol{n})h_\times^*(f',\boldsymbol{n}')\rangle + \langle h_\times(f,\boldsymbol{n})h_+^*(f',\boldsymbol{n}')\rangle$$

$$\boldsymbol{V} = i(\langle h_+(f,\boldsymbol{n})h_\times^*(f',\boldsymbol{n}')\rangle - \langle h_\times(f,\boldsymbol{n})h_+^*(f',\boldsymbol{n}')\rangle)$$





We work with the complex quantities $\boldsymbol{Q} \pm \boldsymbol{iU}$ which have ***convenient properties***

Rotation of angle α in the plane orthogonal to the GW direction:

- $\boldsymbol{I}$ and $\boldsymbol{V}$ remain unchanged (scalars)
- $(\boldsymbol{Q} \pm \boldsymbol{iU}) \rightarrow (\boldsymbol{Q} \pm \boldsymbol{iU})\mathbf{exp}[\pm \boldsymbol{is\alpha}]$: **spin-weighted**

EM (CMB): $s = 2$

GW: $s = 4$

GRAVITATIONAL COMPTON SCATTERING

Born approximation: mean free path of GW much larger than scattering length. Initial and final states described as plane waves

Scattering between graviton and rest-frame classical matter, assumed to be **non-rotating**:

$$\frac{d\sigma_{\epsilon'\epsilon}}{d\Omega} = \frac{(GM)^2}{\sin^4 \beta/2} \left|\hat{\epsilon}'_{ij}\hat{\epsilon}^{ij}\right|^2$$

INCOMING polarisation tensor

OUTGOING polarisation tensor

$\beta \rightarrow$ scattering angle

(SPG)
$$\begin{cases} h_+ = \dfrac{GM(1+\cos^2\beta)}{\sin^2\beta/2} h'_+ \\[2ex] h_\times = \dfrac{2GM\cos\beta}{\sin^2\beta/2} h'_\times \end{cases}$$

Total differential cross section for unpolarised incoming wave:

$$\frac{d\sigma}{d\Omega} = \frac{(GM)^2}{8}\frac{1+6\cos^2\beta+\cos^4\beta}{\sin^4\beta/2}$$

GRAVITATIONAL COMPTON SCATTERING

The total cross section depends on $\sin^{-4}\beta/2$: DIVERGENCE in the forward direction $\beta \to 0$

Rutherford-type dependence, **typical of long-range interactions** (related to the non-linear nature of GR)

Natural cut-off scale: **geometrics optics break-down**

Minimum scattering angle selected by the region where wave-like effects are relevant

$$r \sim \left(2\sqrt{3}R_s\lambda^2\right)^{1/3}$$

$$\beta_{\min} = \left(4\frac{R_s^2}{\sqrt{3}\lambda^2}\right)^{\frac{1}{3}}$$

For compact objects we have also $\beta_{\max} \sim 2$

$$\sigma(\lambda, M) = \frac{(GM)^2}{8}\int_0^{2\pi} \mathrm{d}\phi \int_{\beta_{\min}(\lambda)}^{\beta_{\max}} d\beta \sin\beta \, \frac{1 + 6\cos^2\beta + \cos^4\beta}{\sin^4\beta/2} = (GM)^2 \boldsymbol{\mathcal{T}(\lambda)}$$

GRAVITATIONAL COMPTON SCATTERING

Limit on the scattering region → Relation to observed wavelenght and mass of the scattering target:

$$\boxed{\lambda_{\rm obs} = \lambda(1+z) \geq 10^{-13}\left(\frac{M}{M_\odot}\right)(1+z)\ \mathrm{pc}}$$

Detector	$f_{\rm obs}$ [Hz]	$\lambda_{\rm obs}$ [pc]	Mass [$M_\odot$]
LIGO/VIRGO	$10^1 - 10^4$	$10^{-12} - 10^{-9}$	$\lesssim 1 - 10^3$
ET	$10^0 - 10^4$	$10^{-12} - 10^{-8}$	$\lesssim 1 - 10^4$
LGWA	$10^{-3} - 10^0$	$10^{-8} - 10^{-5}$	$\lesssim 10^4 - 10^7$
LISA	$10^{-5} - 10^0$	$10^{-8} - 10^{-3}$	$\lesssim 10^4 - 10^9$
PTA	$10^{-9} - 10^{-6}$	$10^{-2} - 10^1$	$\lesssim 10^{10}$

At **low frequencies**, targets of all reasonable masses (from stellar-mass objects to supermassive BH) contribute to the scattering

SGWB BOLTZMANN EQUATION

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SGWB BOLTZMANN EQUATIONS

Semi-classical limit: distribution function of gravitons in phase space $\boldsymbol{f(x^\mu, p^\mu)}$

$p^\mu = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ graviton **four-momentum** computed for some affine parameter λ along the trajectory

$f(x^\mu, p^\mu)$ can be interpreted as the *Wigner transform* of the density matrix ρ

Analogy with CMB analyses: evolution of SGWB → evolution of $\boldsymbol{f(x^\mu, p^\mu)}$ **through the Boltzmann equation**

$$\boxed{\mathcal{L}[f(x^\mu, p^\mu)] = \cancel{\mathcal{J}[f]} + \mathcal{C}[f]}$$

Collision term

Emissivity

Liouville operator: $\mathcal{L} = \dfrac{\mathrm{d}}{\mathrm{d}\lambda} = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\dfrac{\partial}{\partial x^\mu} + \dfrac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\dfrac{\partial}{\partial p^\mu}$

SGWB BOLTZMANN EQUATIONS

CMB: **Vector Radiative Transfer Equation** (VRTE) describes the rate of change in the Stokes parameters

VRTE is equivalent to Boltzmann equation for intensity and polarisations (Kosowski, 1995, Hu & White, 1997)

$\boldsymbol{S} = (I, Q + iU, Q - iU, V)^T \rightarrow$ Stokes vector. In terms of the conformal time η:

$$\boxed{\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}\eta} = \dot{\tau}(\eta)[\boldsymbol{\mathcal{F}}(\boldsymbol{S}) - \boldsymbol{S}]}$$

$$\dot{\tau}(\eta) = -a(\eta)n_{\mathrm{ph}}(\eta)\sigma(\eta, M, \lambda)$$

$$\tau(\eta) = \int_{\eta}^{\eta_0} \mathrm{d}t\, a(t)n_{\mathrm{ph}}(t)\sigma(t, M, \lambda)$$

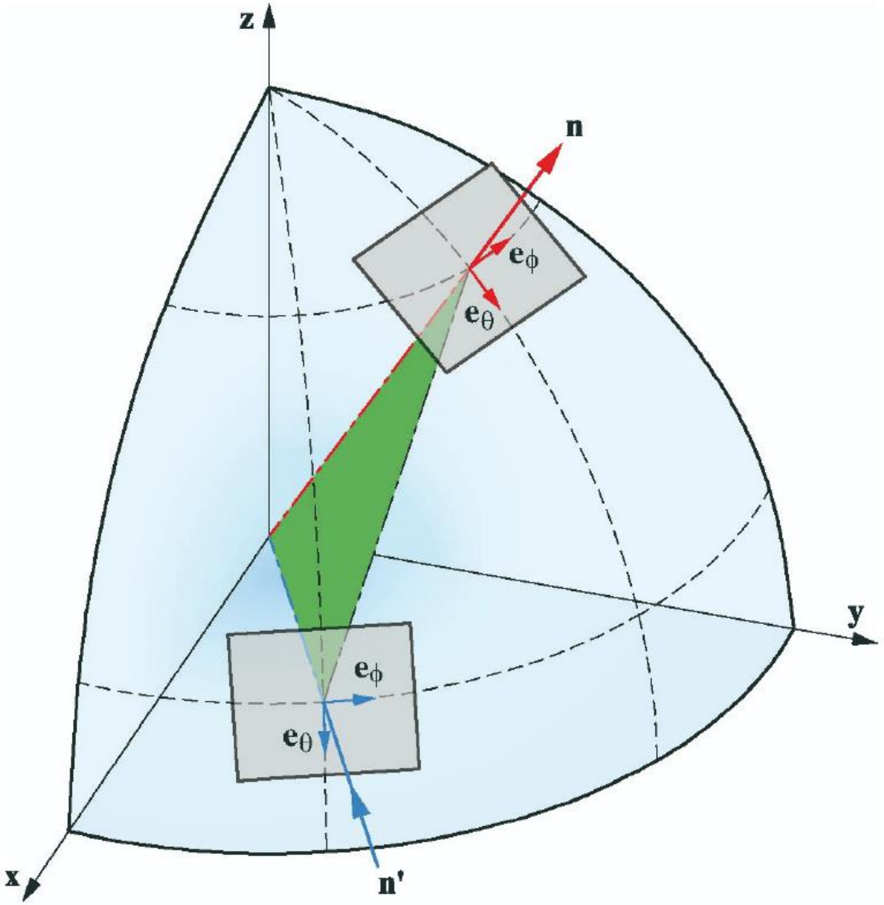
Optical depth. It controls the overall impact of the scattering

$$\boldsymbol{\mathcal{F}}(\boldsymbol{S}) = \frac{1}{\mathcal{J}} \int \mathrm{d}\Omega' \mathcal{M}(\boldsymbol{n}, \boldsymbol{n}')\boldsymbol{S}(\boldsymbol{n}')$$

Flux of outgoing radiation. $\mathcal{M}(\boldsymbol{n}, \boldsymbol{n}')$ is the scattering matrix

We work in the **scattering plane** defined by the ingoing ($\boldsymbol{n}'$) and the outgoing ($\boldsymbol{n}$) direction

$$\boldsymbol{S} = \mathcal{M}_{SPG}\boldsymbol{S}' = \frac{1}{2\sin^4(\beta/2)}(\mathcal{A}_{SPG} + \mathcal{D}_{SPG})\boldsymbol{S}'$$



The scattering matrix

$$\mathcal{A}_{SPG} = \begin{pmatrix} 1+6\cos^2\beta+\cos^4\beta & \frac{1}{2}\sin^4\beta & \frac{1}{2}\sin^4\beta & 0 \\ \sin^4\beta & \frac{1}{2}(1+\cos\beta)^4 & \frac{1}{2}(1-\cos\beta)^4 & 0 \\ \sin^4\beta & \frac{1}{2}(1-\cos\beta)^4 & \frac{1}{2}(1+\cos\beta)^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{D}_{SPG} = 4\,\mathrm{diag}(0,0,0,\cos\,\beta + \cos^3\beta)$$

To connect the SP to a generic spherical basis we need **two rotations: one for the ingoing (γ') and one for the outgoing (γ) wave**

The scattering matrix

$$\mathcal{M} = R(-\gamma)\mathcal{M}_{SPG}R(\gamma')$$

IMPORTANT: rotations bring factors $\exp(\pm 4i\gamma)$, $\exp(\pm 4i\gamma')$ to the spin-4 quantities $Q \pm iU$

Scalar quantities on a sphere are decomposed in spherical harmonics (SH)

$$A(\theta,\phi) = \sum_{\ell m} a_{\ell m} \boldsymbol{Y}_{\boldsymbol{\ell m}}(\boldsymbol{\theta},\boldsymbol{\phi})$$

Spin-weighted quantities are decomposed in *spin-weighted* SH (SWSH)

$$_s f(\theta,\phi) = \sum_{\ell m} a^{(s)}_{\ell m} \, {}_{\boldsymbol{s}}\boldsymbol{Y}_{\boldsymbol{\ell m}}(\boldsymbol{\theta},\boldsymbol{\phi})$$

Using the properties of this special functions, it is possible to write the scattering matrix in spherical basis as a sum of (scalar) SH of $\ell = 1,2,3,4$ and SWSH of $s = 4$ and $\ell = 4$

$$\mathcal{M}(\boldsymbol{n},\boldsymbol{n}') = \frac{4}{(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2}\left[\frac{8}{5}\mathcal{P}^{(0)} + \frac{64\pi}{35}\sum_{m=-2}^{2}\mathcal{P}^{(2)}_m(\boldsymbol{n},\boldsymbol{n}') + \frac{16\pi}{315}\sum_{m=-4}^{4}\mathcal{P}^{(4)}_m(\boldsymbol{n},\boldsymbol{n}') + \right.$$
$$\left. + \frac{16\pi}{105}\left(\sum_{m=-1}^{1}\mathcal{P}^{(1)}_m(\boldsymbol{n},\boldsymbol{n}') + \sum_{m=-3}^{3}\mathcal{P}^{(3)}_m(\boldsymbol{n},\boldsymbol{n}')\right)\right]$$

The scattering matrix

$$\mathcal{P}^{(0)} = \mathrm{diag}(1,0,0,0) \qquad \mathcal{P}^{(1)}_m(\boldsymbol{n},\boldsymbol{n}') = \mathrm{diag}\left(0,0,0,\boxed{70\,Y_{1m}(\boldsymbol{n})Y^*_{1m}(\boldsymbol{n}')}\right)$$

$$\mathcal{P}^{(2)}_m(\boldsymbol{n},\boldsymbol{n}') = \mathrm{diag}(Y_{2m}(\boldsymbol{n})Y^*_{2m}(\boldsymbol{n}'),0,0,0) \qquad \mathcal{P}^{(3)}_m(\boldsymbol{n},\boldsymbol{n}') = \mathrm{diag}\left(0,0,0,\boxed{3\sqrt{7}\,Y_{3m}(\boldsymbol{n})Y^*_{3m}(\boldsymbol{n}')}\right)$$

$$\mathcal{P}^{(4)}_m(\boldsymbol{n},\boldsymbol{n}') = \begin{pmatrix} Y_{4m}(\boldsymbol{n})Y^*_{4m}(\boldsymbol{n}') & \sqrt{35/2}\,Y_{4m}(\boldsymbol{n})\,{}_4Y^*_{4m}(\boldsymbol{n}') & \sqrt{35/2}\,Y_{4m}(\boldsymbol{n})\,{}_{-4}Y^*_{4m}(\boldsymbol{n}') & 0 \\ \sqrt{70}\,{}_4Y_{4m}(\boldsymbol{n})Y^*_{4m}(\boldsymbol{n}') & 35\,{}_4Y_{4m}(\boldsymbol{n})\,{}_4Y^*_{4m}(\boldsymbol{n}') & 35\,{}_4Y_{4m}(\boldsymbol{n})\,{}_{-4}Y^*_{4m}(\boldsymbol{n}') & 0 \\ \sqrt{70}\,{}_{-4}Y_{4m}(\boldsymbol{n})Y^*_{4m}(\boldsymbol{n}') & 35\,{}_{-4}Y_{4m}(\boldsymbol{n})\,{}_4Y^*_{4m}(\boldsymbol{n}') & 35\,{}_{-4}Y_{4m}(\boldsymbol{n})\,{}_{-4}Y^*_{4m}(\boldsymbol{n}') & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

CIRCULAR POLARISATION EVOLVES INDEPENDENTLY

$$\mathcal{M}(\boldsymbol{n},\boldsymbol{n}') = \frac{4}{(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2}\left[\frac{8}{5}\mathcal{P}^{(0)} + \frac{64\pi}{35}\sum_{m=-2}^{2}\mathcal{P}^{(2)}_m(\boldsymbol{n},\boldsymbol{n}') + \frac{16\pi}{315}\sum_{m=-4}^{4}\mathcal{P}^{(4)}_m(\boldsymbol{n},\boldsymbol{n}') + \right.$$

$$\left. + \frac{16\pi}{105}\left(\sum_{m=-1}^{1}\mathcal{P}^{(1)}_m(\boldsymbol{n},\boldsymbol{n}') + \sum_{m=-3}^{3}\mathcal{P}^{(3)}_m(\boldsymbol{n},\boldsymbol{n}')\right)\right]$$

CMB Thomson scattering: a comparison

Thomson scattering in the early universe between electrons and CMB photons

Large scattering targets density $n_e \sim 10^2 \mathrm{cm}^{-3}$ $(10^{57} \mathrm{pc}^{-3})$

Small cross section $\sigma_T \sim 10^{-24} \mathrm{cm}^2$ $(10^{-61}$ $\mathrm{pc}^2)$

Scattering matrix (ignoring *V*)

$$\mathcal{M}(\boldsymbol{n},\boldsymbol{n}') = \frac{2}{3}\left(\mathcal{P}^{(0)} + \frac{4\pi}{10}\sum_{m=-2}^{2}(\mathcal{P}_{CMB})_m^{(2)}(\boldsymbol{n},\boldsymbol{n}')\right)$$

$(\mathcal{P}_{CMB})_m^{(2)}$ contains SH and SWSH of $\ell = 2, s = \pm 2$

NO RUTHERFORD DIVERGENCE: convenient properties of orthogonality can be used

Gravitational Compton scattering between gravitons and massive structures

Small scattering targets density $n_{BH} \sim 10^{-14} \mathbf{pc}^{-3}$

Large cross section $\sigma(10\, M_{\odot}, 10^{-10} \mathrm{pc}) \sim 10^{-20} \mathbf{pc}^2$

Scattering matrix (ignoring *V*)

$$\mathcal{M}(\boldsymbol{n},\boldsymbol{n}') = \frac{4}{(1-\boldsymbol{n}\cdot\boldsymbol{n}')^2}\left[\frac{8}{5}\mathcal{P}^{(0)} + \right.$$

$$\left. + \frac{64\pi}{35}\sum_{m=-2}^{2}\mathcal{P}_m^{(2)}(\boldsymbol{n},\boldsymbol{n}') + \frac{16\pi}{315}\sum_{m=-4}^{4}\mathcal{P}_m^{(4)}(\boldsymbol{n},\boldsymbol{n}')\right]$$

SGWB BOLTZMANN EQUATIONS

Linearly perturbed flat FLRW metric (only scalar perturbations)

$$\boldsymbol{ds^2 = a^2(\eta)\left[-\left(1+\frac{2\Psi}{c^2}\right)c^2 d\eta^2 + \left(1-\frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j\right]},$$

Φ, Ψ gravitational potentials

Perturbation in the density matrix of gravitons: $\rho_{ij}(\eta, \boldsymbol{x}, \boldsymbol{n}, q) = \rho_{ij}^{(0)}(\eta, q) + \rho_{ij}^{(1)}(\eta, \boldsymbol{x}, \boldsymbol{n}, q)$

$q \rightarrow$ comoving momentum modulus

$\rho_{ij}^{(0)}(\eta, q) \rightarrow$ diagonal matrix connected to the background energy density $\Omega_{GW}(\eta, q)$

$\rho_{ij}^{(1)}(\eta, \boldsymbol{x}, \boldsymbol{n}, q)$ can be written in terms of the Stokes parameters brightness perturbations

$$\Delta_I, \qquad \Delta_{Q\pm iU}, \qquad \Delta_V$$

SGWB BOLTZMANN EQUATIONS

$$\mathcal{I}^{(0)}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{32}{5}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}\cdot\mathbf{n}')^2}\Delta_I(\mathbf{n}')\,,$$

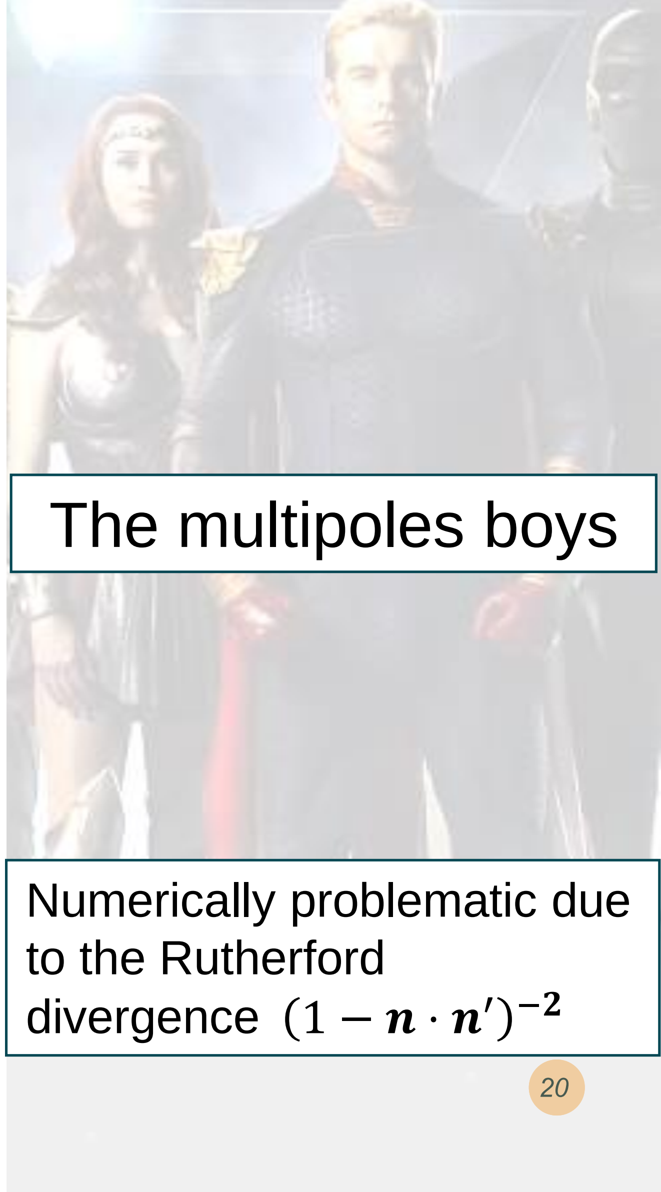
$$\mathcal{I}_m^{(2)}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{256\pi}{35}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}'\cdot\mathbf{n})^2}\Delta_I(\mathbf{n}')Y^*_{2m}(\mathbf{n}')\,,$$

$$\mathcal{I}_m^{(4)}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{64\pi}{315}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}'\cdot\mathbf{n})^2}\Delta_I(\mathbf{n}')Y^*_{4m}(\mathbf{n}')\,,$$

$$\mathcal{L}_m^{(4)\,\pm}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{64\pi}{315}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}'\cdot\mathbf{n})^2}\Delta_{Q\pm iU}(\mathbf{n}')\,{}_{\pm4}Y^*_{4m}(\mathbf{n}')\,,$$

$$\mathcal{V}_m^{(1)}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{32\pi}{3}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}'\cdot\mathbf{n})^2}\Delta_V(\mathbf{n}')Y^*_{1m}(\mathbf{n}')\,,$$

$$\mathcal{V}_m^{(3)}(\mathbf{n}) = \frac{1}{\mathcal{T}}\frac{16\pi}{5\sqrt{7}}\int \frac{\mathrm{d}\Omega'}{(1-\mathbf{n}'\cdot\mathbf{n})^2}\Delta_V(\mathbf{n}')Y^*_{3m}(\mathbf{n}')\,.$$



The multipoles boys

Numerically problematic due to the Rutherford divergence $(1-\boldsymbol{n}\cdot\boldsymbol{n}')^{-2}$

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SGWB BOLTZMANN EQUATIONS

Intensity:

$$(\partial_\eta + n^i\partial_i)\Delta_I(\mathbf{n}) - 4\left[\dot{\Phi} - n^i\partial_i\Psi\right] = -a\sigma n_{\rm ph}\Big[\Delta_I(\mathbf{n}) - \mathcal{I}^{(0)}(\mathbf{n}) - \sum_{m=-2}^{2} Y_{2m}(\mathbf{n})\mathcal{I}^{(2)}_m(\mathbf{n}) - \sum_{m=-4}^{4} Y_{4m}(\mathbf{n})\times$$
$$\left(\mathcal{I}^{(4)}_m(\mathbf{n}) + \sqrt{\frac{35}{2}}\mathcal{L}^{(4)\,+}_m(\mathbf{n}) + \sqrt{\frac{35}{2}}\mathcal{L}^{(4)\,-}_m(\mathbf{n})\right)\Big]$$

Linear polarisations:

$$(\partial_\eta + n^i\partial_i)\Delta_{Q\pm iU}(\mathbf{n}) = -a\sigma n_{\rm ph}\left[\Delta_{Q\pm iU}(\mathbf{n}) - \sum_{m=-4}^{4} {}_{\pm4}Y_{4m}(\mathbf{n})\left(\sqrt{70}\mathcal{I}^{(4)}_m(\mathbf{n}) + 35\mathcal{L}^{(4)\,+}_m(\mathbf{n}) + 35\mathcal{L}^{(4)\,-}_m(\mathbf{n})\right)\right]$$

Circular polarisation:

$$(\partial_\eta + n^i\partial_i)\Delta_V(\mathbf{n}) = -a\sigma n_{\rm ph}\left[\Delta_V(\mathbf{n}) - \sum_{m=-1}^{1} Y_{1m}(\mathbf{n})\mathcal{V}^{(1)}_m(\mathbf{n}) - \sum_{m=-3}^{3} Y_{3m}(\mathbf{n})\mathcal{V}^{(3)}_m(\mathbf{n})\right]$$

IMPACT OF THE COLLISION TERM

IMPACT OF THE COLLISION TERM

- Unpolarised incoming radiation
- We work in the frame where the outgoing direction is aligned along the z-axis: $\boldsymbol{n}\cdot\boldsymbol{n}' = \cos\beta$
- Rotation $R(\Theta,\Phi)$ to go back to generic (laboratory) frame

Much simpler intergals:

SH decomposition: $\Delta_I'(\beta,\gamma) = \sum_{\ell m} \tilde{a}_{\ell m} Y_{\ell m}(\beta,\gamma)$

$$\tilde{\Delta}_I = (GM)^2 \int \mathrm{d}\Omega_z \frac{\Delta_I'(\beta,\gamma)\boxed{(1+6\cos^2\beta+\cos^4\beta)}}{\boxed{2\sin^4(\beta/2)}}$$

- **It does not depend on $\boldsymbol{\gamma}$**! ONLY SH with $m=0$ survives.
- Rutherford factor: **ALL MULTIPOLES** are scattered!

$$\tilde{\Delta}_{Q\pm iU} = (GM)^2 \int \mathrm{d}\Omega_z \frac{\Delta_I'(\beta,\gamma)\sin^4\beta}{2\sin^4(\beta/2)} \boxed{\exp[\pm i4\gamma]}$$

- **It depends on** $\exp[\pm i4\gamma]$. ONLY SH with $m=\pm 4$ survives. Polarisation generated if $\ell \geq 4$!
- The integral **converges** for small β

IMPACT OF THE COLLISION TERM

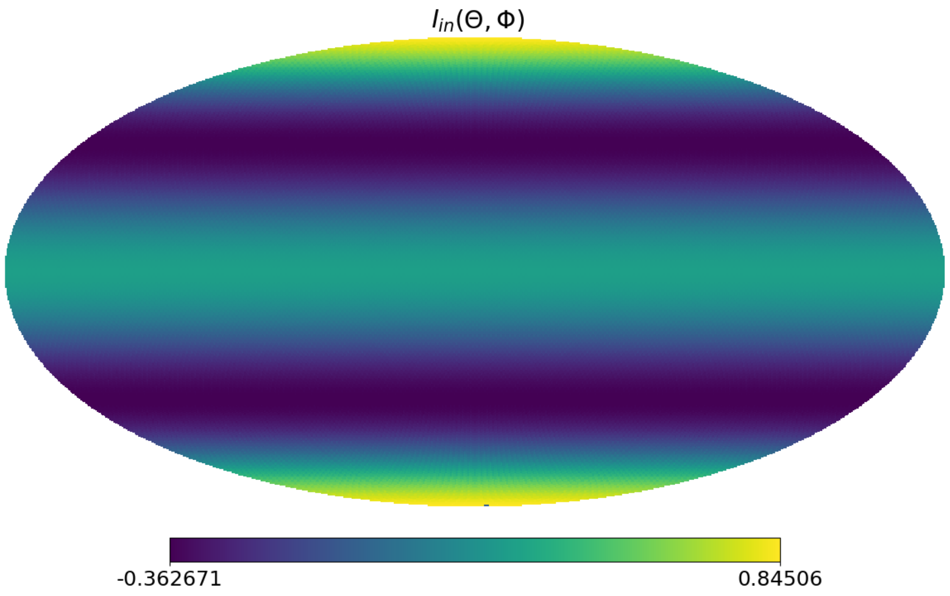
Let's assume that the scattering happens against a (uniform) distribution of target located in a small redshift interval at $z \sim 1$ (see e.g. Cusin et al., 2018)

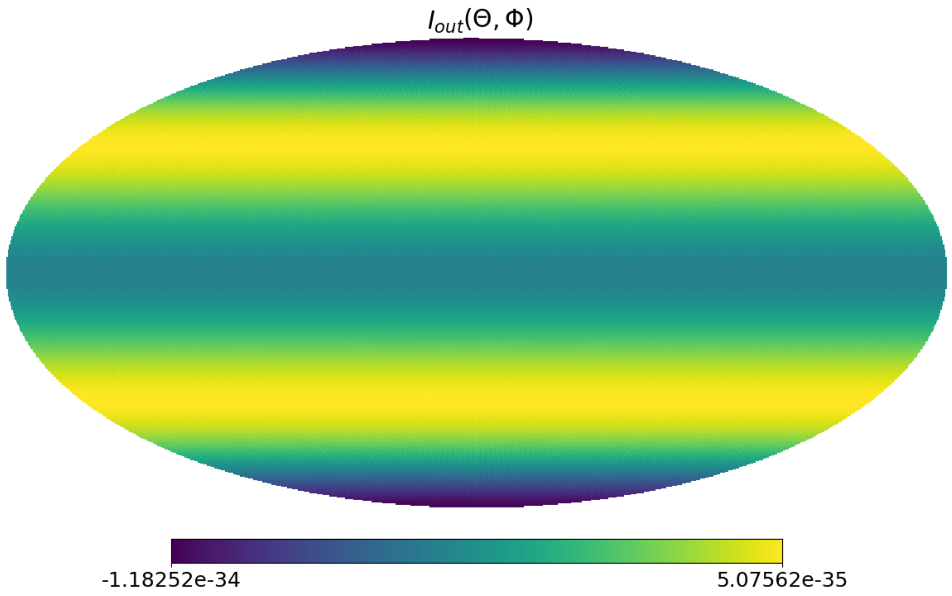
Derivative of the optical depth: $\dot{\tau}\left(z = 1, \lambda = 10^{-10}\text{pc},\ M = 10M_{\odot}\right) \sim 10^{-33}$

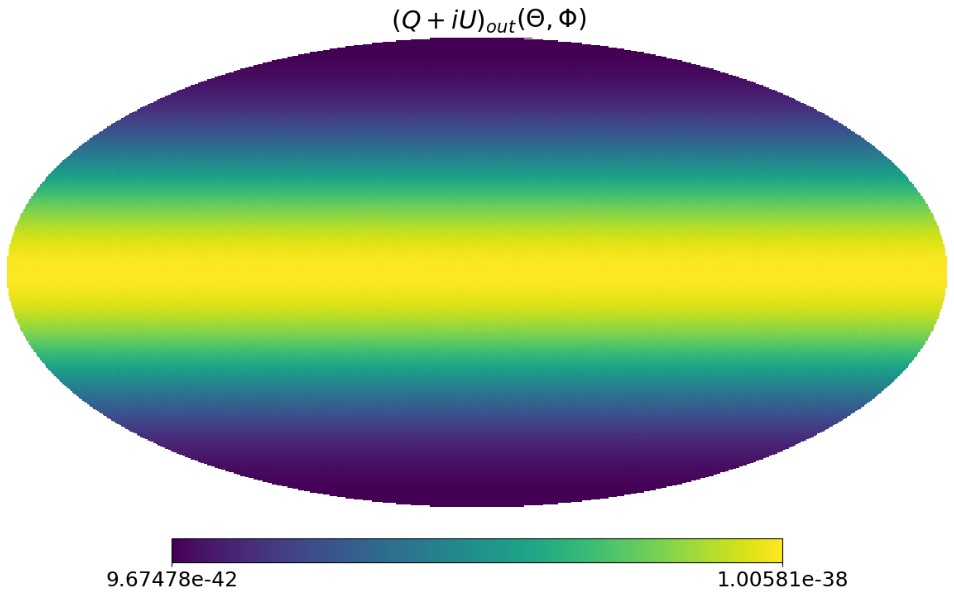
Incoming radiation: pure exadecapole intensity perturbation in laboratory frame

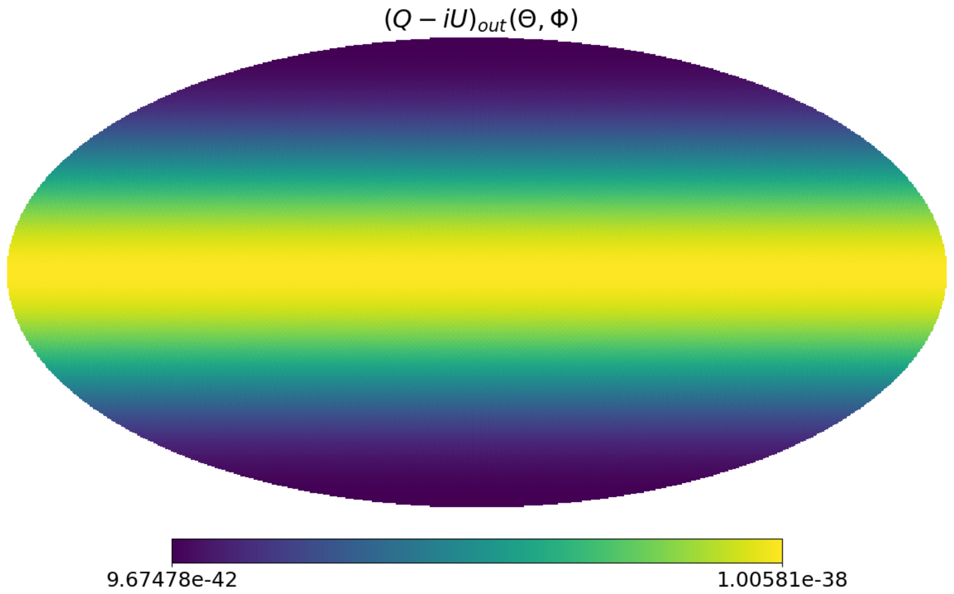
$$\Delta_I'(\theta\,,\phi) = Y_{40}(\theta,\phi) = \frac{3}{16\sqrt{\pi}}(35\cos^4\theta - 30\cos^2\theta + 3\,)$$

And the output of the RHS of Boltzmann equations is…


$I_{in}(\Theta, \Phi)$
-0.362671
0.84506


$I_{out}(\Theta, \Phi)$
-1.18252e-34
5.07562e-35


$(Q + iU)_{out}(\Theta, \Phi)$
9.67478e-42
1.00581e-38


$(Q - iU)_{out}(\Theta, \Phi)$
9.67478e-42
1.00581e-38

IMPACT OF THE COLLISION TERM

For intensity, the RHS of Boltzmann equation depends on the **frequency** of the incoming wave

$f_{\rm obs}$ [Hz]	$\lambda_{\rm obs}$ [pc]	$\dot{\tau}(\lambda, M = 10\, M_\odot)$	R.H.S (I)	R.H.S $(Q \pm iU)$	Detector
10^{2}	10^{-10}	$\sim 10^{-33}$	$\sim 10^{-33}$	$\sim 10^{-38}$	LIGO/VIRGO, ET
10^{-3}	10^{-5}	$\sim 10^{-27}$	$\sim 10^{-27}$	$\sim 10^{-38}$	LGWA, LISA
10^{-7}	10^{-1}	$\sim 10^{-22}$	$\sim 10^{-21}$	$\sim 10^{-38}$	PTA
10^{-11}	10^{3}	$\sim 10^{-16}$	$\sim 10^{-16}$	$\sim 10^{-38}$	−−
10^{-14}	10^{6}	$\sim 10^{-12}$	$\sim 10^{-12}$	$\sim 10^{-38}$	−−

Still, extremely small effect even for cosmological wavelenghts!

EPILOGUE: A TOY MODEL SOLUTION

EPILOGUE: A TOY MODEL SOLUTION

Initial conditions

$$\Delta_I(\eta_0) = \left[\Delta_I(\eta_{in}) + 4\Psi(\eta_{in})\right] e^{ik\mu(\eta_{in}-\eta_0)-\tau(\eta_{in})} + \\ + \int_{\eta_{in}}^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \left[4(\dot{\Phi}+\dot{\Psi}) - \dot{\tau}(4\Psi + \mathcal{C}_I)\right] - 4\Psi(\eta_0)\,,$$

$$\Delta_{Q\pm iU}(\eta_0) = \Delta_{Q\pm iU}(\eta_{in}) e^{ik\mu(\eta_{in}-\eta_0)-\tau(\eta_{in})} - \int_{\eta_{in}}^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \dot{\tau} \mathcal{C}_{Q\pm iU}$$

Initial conditions

$$\mathcal{C}_I = \mathcal{I}^{(0)} + \sum_{m=-2}^{2} Y_{2m}\mathcal{I}_m^{(2)} + \sum_{m=-4}^{4} Y_{4m}\left(\mathcal{I}_m^{(4)} + \sqrt{\frac{35}{2}}\mathcal{L}_m^{(4)+} + \sqrt{\frac{35}{2}}\mathcal{L}_m^{(4)-}\right)$$

$$\mathcal{C}_{Q\pm iU} = \sum_{m=-4}^{4} \pm 4 Y_{4m}\left(\sqrt{70}\mathcal{I}_m^{(4)} + 35\mathcal{L}_m^{(4)+} + 35\mathcal{L}_{\imath}^{(4)-}\right).$$

We work in Fourier space (all the terms depend on the wavenumber $\boldsymbol{k}$)

$\mu = \hat{\boldsymbol{k}} \cdot \boldsymbol{n}$ is the cosine between the wave direction and the wave number

$\tau(\eta_0) = 0$ by definition

EPILOGUE: A TOY MODEL SOLUTION

Assumptions:

- initial condition unpolarised, monopole intensity perturbation: all the angular dependence in $\boldsymbol{\mu}$
- We neglect scalar perturbations of the metric: focus only on the collision term
- The scattering happens only in a time interval centered on $\eta^* = \eta(z^*)$ **small enough** that the other quantities don't vary considerably. We set $z^* = 1$

Let's define the inital power spectrum of the intensity perturbations at some time $\boldsymbol{\eta_{in}}$:

$$\langle \Delta_I(\eta_{in}, q, \mathbf{k}) \Delta_I^*(\eta_{in}, q, \mathbf{k}') \rangle = \frac{2\pi^2}{k^3} P_{in}(q,k) (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

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EPILOGUE: A TOY MODEL SOLUTION

Real space angular power spectra:

obtained from the coefficients of expansion in SH of observed intensity and polarisations anisotropies

$$\langle \Delta_{I,\ell m} \Delta^*_{I,\ell' m'} \rangle = \delta_{\ell,\ell'} \delta_{m,m'} 4\pi \int_0^\infty \frac{\mathrm{d}k}{k} P_{in}(k,q) \sum_{\ell_1,\ell_2} \sum_{\ell_3,\ell_4} (-i)^{\ell_1+\ell_2-\ell_3-\ell_4} (2\ell_1+1)(2\ell_2+1) \times$$

$$(2\ell_3+1)(2\ell_4+1) j_{\ell_1}[k(\eta_{in}-\eta^*)] j_{\ell_3}[k(\eta_{in}-\eta^*)] j_{\ell_2}[k(\eta^*-\eta_0)] j_{\ell_4}[k(\eta^*-\eta_0)] \times$$

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \ell_3 & \ell_4 & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \left[e^{-2\tau(\eta_{in})} + \boxed{\left(\frac{\tau(\eta^*)}{1+\tau(\eta^*)} \right)^2 \frac{\mathcal{K}_{\ell_1} \mathcal{K}_{\ell_3}}{\mathcal{T}^2} + \frac{2}{\mathcal{T}} \frac{\tau(\eta^*)}{1+\tau(\eta^*)} e^{-\tau(\eta_{in})} \mathcal{K}_{\ell_1}} \right]$$

Scattering contribution: $\boldsymbol{G^I_{\ell_1 \ell_3}}$

$$\mathcal{K}_\ell = 4\pi \int_{\cos(\beta_{\max})}^{\cos(\beta_{\min})} \frac{dx}{(1-x)^2} (1 + 6x^2 + x^4) \mathcal{P}_\ell(x)$$

EPILOGUE: A TOY MODEL SOLUTION

Real space angular power spectra:

obtained from the coefficients of expansion in SH of observed intensity and polarisations anisotropies

$$\langle \Delta_{Q\pm iU,\ell m}\Delta^{*}_{Q\pm iU,\ell' m'}\rangle = \frac{512}{35}\frac{4\pi}{\mathcal{T}^2}\int_0^\infty \frac{\mathrm{d}k}{k} P_{in}(k,q)\sum_{\ell_1,\ell_2}\sum_{\ell_3,\ell_4}(-i)^{\ell_1+\ell_2-\ell_3'-\ell_4}(2\ell_1+1)(2\ell_2+1)\times$$

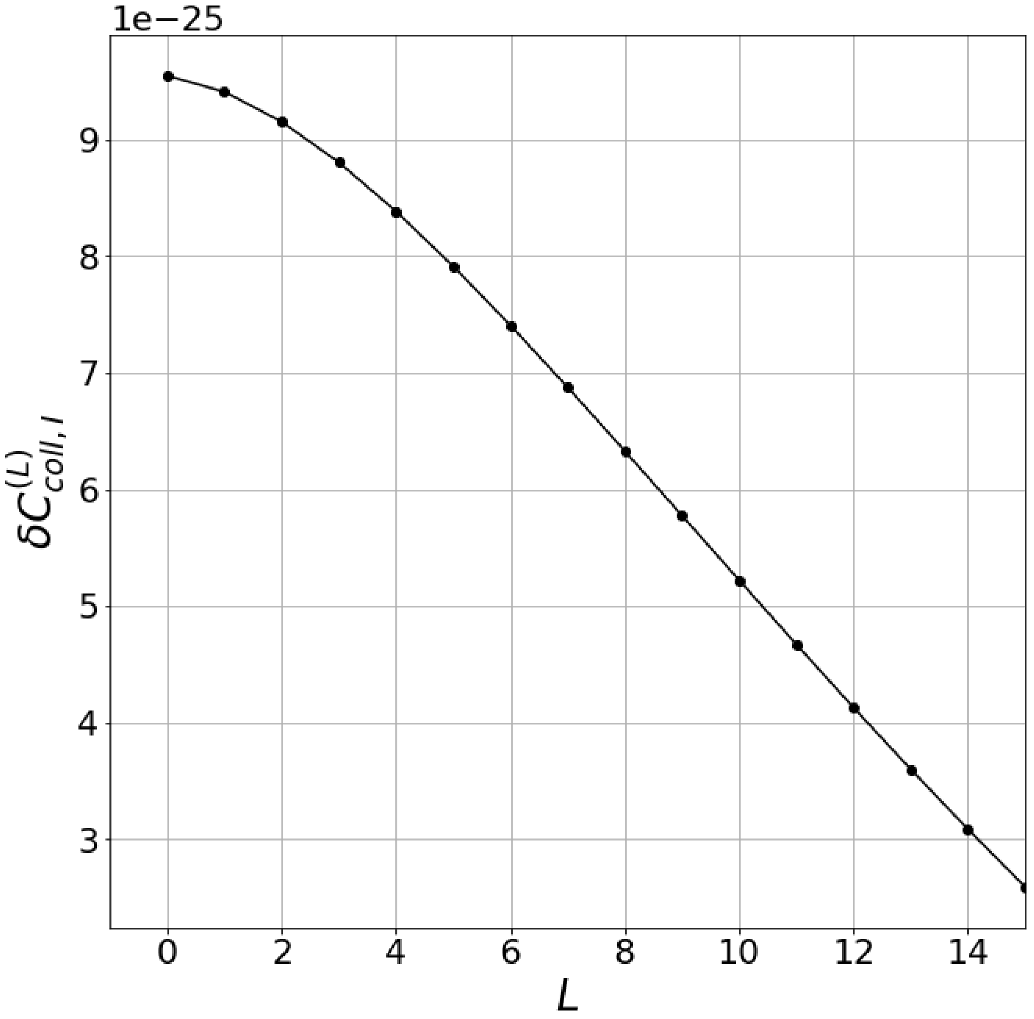
$$(2\ell_3+1)(2\ell_4+1)j_{\ell_1}[k(\eta_{in}-\eta^*)]j_{\ell_3}[k(\eta_{in}-\eta^*)]j_{\ell_2}[k(\eta^*-\eta_0)]j_{\ell_4}[k(\eta^*-\eta_0)]\times$$

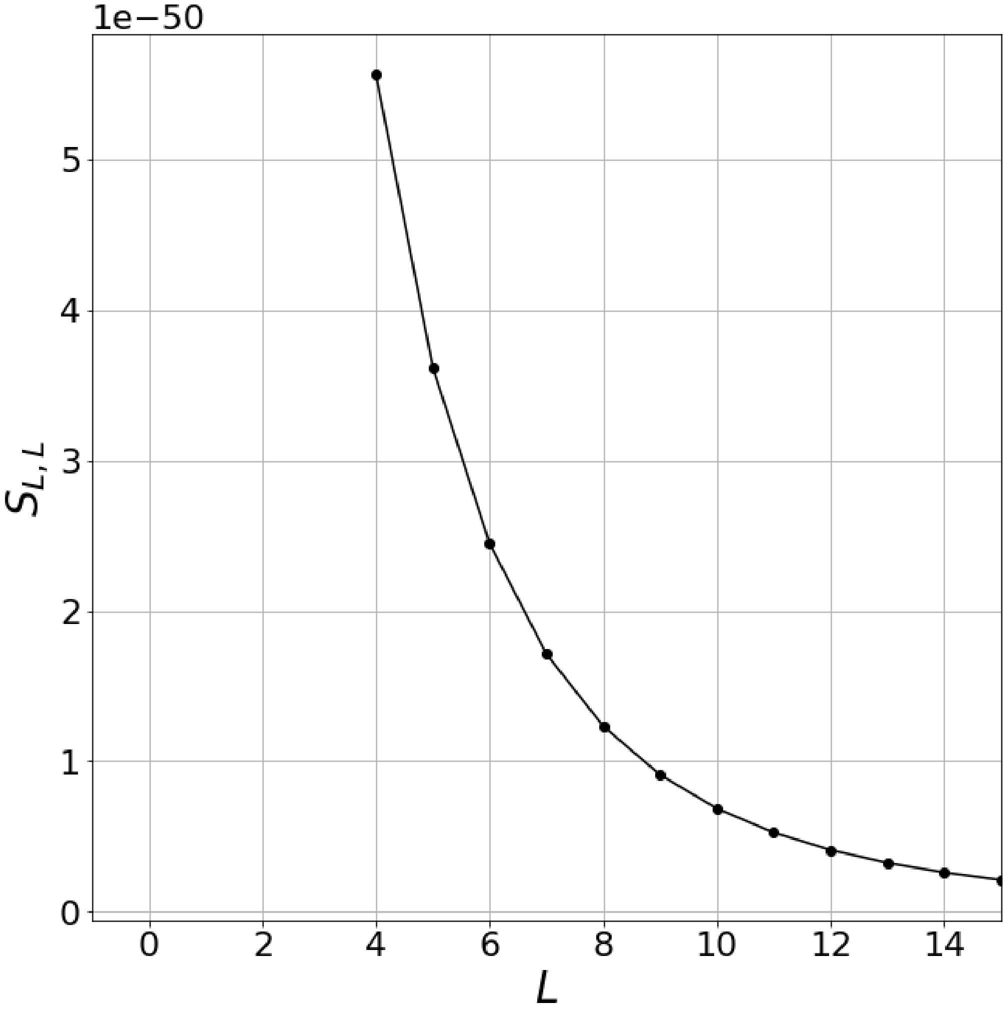
$$\begin{pmatrix}\ell_1 & \ell_2 & \ell\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_3 & \ell_4 & \ell\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ \pm 4 & \mp 4 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell_3 & \ell_4\\ \mp 4 & \pm 4 & 0\end{pmatrix}\boxed{\left[\left(\frac{\tau(\eta^*)}{1+\tau(\eta^*)}\right)^2 \mathcal{K}^{\mp 4}_{\ell_1}\mathcal{K}^{\mp 4}_{\ell_3}\right]}\delta_{\ell,\ell'}\delta_{m,m'}$$

Scattering contribution: $\boldsymbol{S}_{\ell_1\ell_3}$

$$\mathcal{K}^{\mp 4}_{\ell} = 2\pi(40320)^{\mp 1/2}\int_{\cos(\beta_{\max})}^{\cos(\beta_{\min})}\frac{dx}{(1-x)^2}\sqrt{\frac{(\ell\pm 4)!}{(\ell\mp 4)!}}\mathcal{P}^{\mp 4}_{\ell}(x)\mathcal{P}^{\pm 4}_{4}(x))$$

EPILOGUE: A TOY MODEL SOLUTION





$\delta C^{(L)}_{\text{coll},I} = G^{I}_{L,L} \exp[2\tau(\eta_{in})]$ relative difference between scattered and unscattered intensity angular PS

CONCLUSIONS

- We have presented a set **of Boltzmann equations** describing the evolution of intensity and polarisation anisotropies in the SGWB

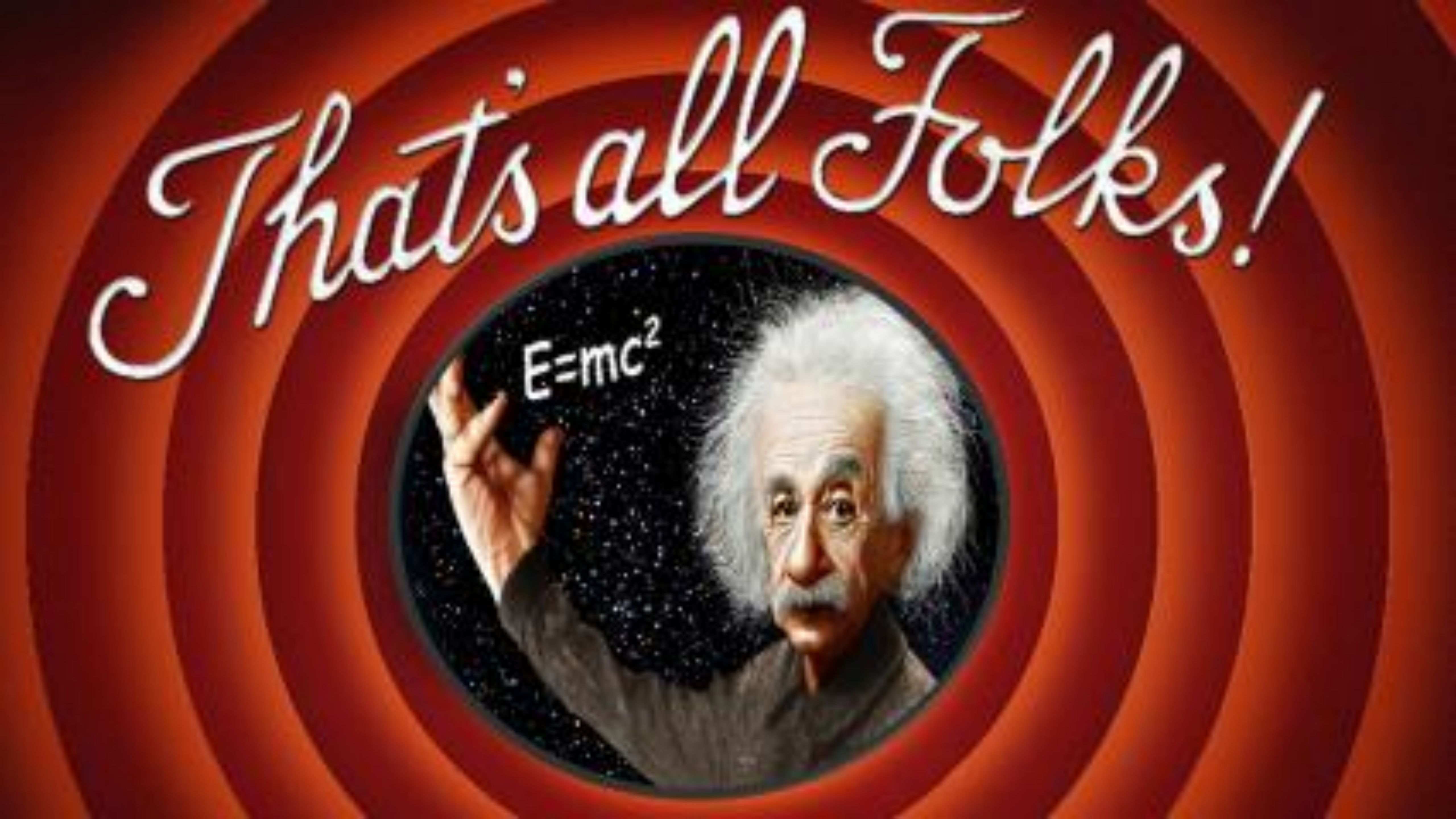
- We have included a **collision term accounting for gravitational Compton scattering** in the WF regime

- The scattering **affects all multipoles in the intensity**, provided that $m = 0$
- The scattering **generates polarisation** only if $m = \pm 4 \qquad (\ell \geq 4)$

- The contribution of the collision term **is several order of magnitude smaller than the Liouville term**, in agreement with other studies (e.g. Cusin et al., 2018)

What next?

- Solve the equations considering an incoming polarised flux
- Implement a numerical solver to correclty estimate the interplay between intensity and polarisation during the propagation over the LSS
- Extend to non-standard scenarios (different polarisation modes, different scattering contribution)


That's all Folks!
E=mc²