Expanding impulsive gravitational waves

Robert Švarc

in collaboration with **David Kofroň** and **Michal Karamazov**

Institute of Theoretical Physics

CzechLISA, December 6, 2022

Outline

1. **Motivation**
   - General relativity
   - Gravitational waves: real physics!
   - Explicit (unrealistic) toy-models
2. **Expanding impulsive waves**
   - Spacetime geometry
   - Geodesics
3. **Sources of expanding impulses**
   - Complex mapping $h(Z)$
   - One string
   - Two strings
4. **Conclusions**

General relativity: Einstein's theory of gravity

Gravity – **universal interaction** – inherent property of the '**arena**' (spacetime)

- natural application of differential geometry: spacetime $\leftrightarrow$ Lorentzian manifold
- WANTED: metric tensor g_{ab} (dead or alive)

Einstein's field equations

$$R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

' *geometry* = *energy* and *momentum* '

(credit: ligo.org)

Non-linearity: *matter* $\to$ *space curvature* $\to$ *matter motion* $\to$ *curvature changed* $\to$ **etc.**

Corollary: ripples in the spacetime curvature propagating with the speed of light $\leftrightarrow$ **gravitational waves**

Solutions: exact spacetimes $\times$ perturbative models $\times$ numerical simulations

Free fall: geodesic motion $\leftrightarrow$ $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$

Gravitational waves: Sci-fi? No!

February 11, 2016, Washington, D.C.

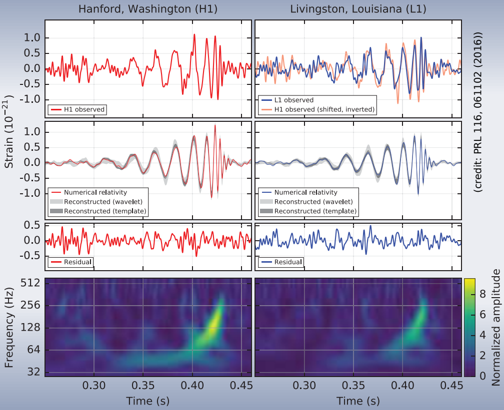
Ladies and gentlemen,

We have detected gravitational waves!

We did it!

David Reitze (LIGO Executive Director)

Rainer Weiss and Kip Thorne



event: **GW150914**

detected: **LIGO**, September 14, 2015, 09:50:45 UTC

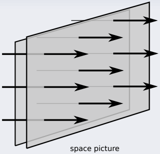
lag: **6,9 ms H1** behind **L1**

Exact toy-models: Sandwich waves

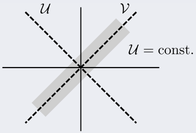
Flat Minkowski space + ‘something’: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \text{‘something’}$

- *Non-expanding:* $\mathcal{U} = \frac{1}{\sqrt{2}}(t - z)$ and $\mathcal{V} = \frac{1}{\sqrt{2}}(t + z) \quad \rightarrow$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} + \text{‘something’}$$



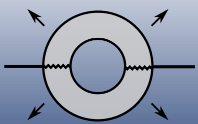
space picture



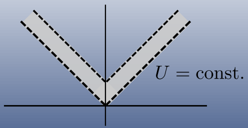
spacetime picture

- *Expanding:* $\mathcal{U} = Z\bar{Z}\,V - U, \quad \mathcal{V} = V, \quad \eta = ZV \quad$ where $\eta = \frac{1}{\sqrt{2}}(x + \mathrm{i}y) \quad \rightarrow$

$$\mathrm{d}s^2 = 2V^2\,\mathrm{d}Z\mathrm{d}\bar{Z} + 2\,\mathrm{d}U\,\mathrm{d}V + \text{‘something’}$$



space picture



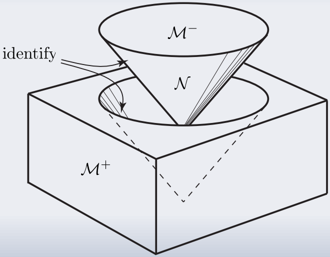
spacetime picture

Cut and paste construction of expanding impulses

Minkowski space in the null cone foliation:

$$\mathrm{d}s^2 = 2\,\frac{V^2}{|1+\epsilon Z\bar{Z}|^2}\,\mathrm{d}Z\mathrm{d}\bar{Z} + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2$$

Cut along the null cone:



Identification via the Penrose junction condition:

$$[\,U=0_-,\,V,\,Z,\,\bar{Z}\,]_{\mathcal{M}^-} \equiv \left[\,U=0_+,\,\frac{1+\epsilon\,h\bar{h}}{1+\epsilon Z\bar{Z}}\,\frac{V}{|h'|},\,h(Z),\,\bar{h}(\bar{Z})\right]_{\mathcal{M}^+}$$

Continuous line element

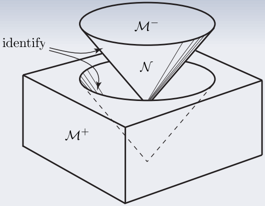
Flat Minkowski space:

$$\mathrm{d}s^2 = -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2\,\mathrm{d}\eta\,\mathrm{d}\bar{\eta}$$

Explicit cut and paste procedure:

- for $U < 0$

$$\mathcal{U} = \frac{Z\bar{Z}}{p}\,V - U$$
$$\mathcal{V} = \frac{V}{p} - \epsilon U$$
$$\eta = \frac{Z}{p}\,V$$



- for $U > 0$

$$\mathcal{U} = B\,V - E\,U$$
$$\mathcal{V} = A\,V - D\,U$$
$$\eta = C\,V - F\,U$$

Resulting metric: $\mathcal{M}^-$ and $\mathcal{M}^+$ re-attached along **null cone** $U = 0$ with a 'warp'

$$\mathrm{d}s^2 = 2\left|\frac{V}{p}\,\mathrm{d}Z + U_+(U)\,p\,\bar{H}\,\mathrm{d}\bar{Z}\right|^2 + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2$$

where

$$U_+ \equiv U_+(U) = \begin{cases} 0 & \text{if} \quad U \le 0 \\ U & \text{if} \quad U \ge 0 \end{cases} \qquad \text{and} \qquad H(Z) = \frac{1}{2}\left[\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2\right]$$

Basic properties of expanding impulses

Continuous metric:

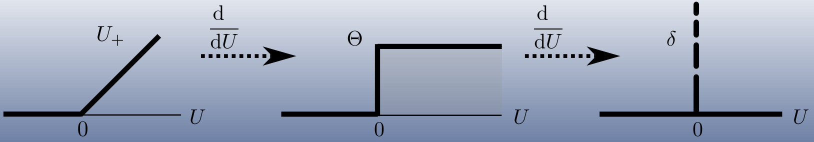
$$ds^2 = 2\left|\frac{V}{p}\,\mathrm{d}Z + U_+(U)\,p\,\bar{H}\,\mathrm{d}\bar{Z}\right|^2 + 2\,\mathrm{d}U\,\mathrm{d}V - 2\epsilon\,\mathrm{d}U^2$$

Penrose junction conditions: satisfied on $U = 0$

Curvature: discontinuity in the derivatives of the metric $\rightarrow$ impulsive components

- Ricci tensor: $\Phi_{22} = \frac{p^4 H\bar{H}}{V^2}\,U\,\delta(U)$
- Weyl tensor: $\Psi_4 = \frac{p^2 H}{V}\,\delta(U)$

where $p = 1 + \epsilon Z\bar{Z}$ and $\epsilon = -1, 0, +1$

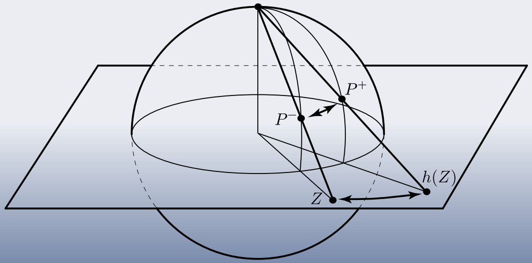


Geometric nature of junction condition

Penrose junction condition: geometric nature

$$\Big[\, U=0_-\,,\ V,\ Z,\ \bar{Z}\,\Big]_{\mathcal{M}^-}\ \equiv\ \Big[\, U=0_+,\ \tfrac{1+\epsilon\, h\bar{h}}{1+\epsilon Z\bar{Z}}\,\tfrac{V}{|h'|},\ h(Z),\ \bar{h}(\bar{Z})\,\Big]_{\mathcal{M}^+}$$

- the impulse on $U=0$ $\leftrightarrow$ sphere expanding with the speed of light: $x^2+y^2+z^2=t^2$
- $h(Z)$ is an arbitrary *holomorphic function*
- identification of points in complex plane $Z\to h(Z)$ $\leftrightarrow$ *stereographic projection*



Geodesics: global uniqueness and C^1-regularity

J. Podolský, C. Sämann, R. Steinbauer, R. Švarc: *The global uniqueness and C^1-regularity of geodesics in expanding impulsive gravitational waves*, Class. Quantum Grav. **33** (2016) 195010

Theorem: *Existence and Uniqueness.*
For the entire class of expanding impulsive waves on any background of constant curvature described by the continuous form of the metric with smooth H we have: Given any point $\mathcal{P}$ and any direction $v \in T_{\mathcal{P}}M$ there exists a *unique* $\mathcal{C}^1$-solution γ in the sense of Filippov to the geodesic equations with this initial data.

Moreover, if such a geodesic meets the impulsive wave located at $\mathcal{N} = \{U = 0\}$ at all, it is either one of its null generators or it hits it in *isolated points*.

Corollary: *Preservation of causal character.*
The geodesics γ of the theorem above satisfy $g(\dot\gamma, \dot\gamma) =$ const., and, in particular, the causal character of γ can be defined globally.

Corollary: *Crossing the expanding impulse.*
The geodesics of the theorem above that start off the wave surface $\mathcal{N} = \{U = 0\}$ and hit it at all, do so in isolated points either

- *transversally* and pass from the 'outside' to the 'inside', or vice versa, or
- *tangentially*, in which case they are spacelike and come from the 'outside' and revert to the 'outside' again.

Remark: *Uniqueness for non-smooth H*

If the function H has singularities then, given arbitrary initial data, the geodesic equation possesses locally defined unique $\mathcal{C}^1$-solutions in any region where H is sufficiently smooth.

Background coordinates: matching conditions on $U = 0$

Geodesics are straight lines outside the impulse: *matching across the impulse needed*

- begin with unique globally defined $\mathcal{C}^1$-geodesics in the continuous coordinates
- evaluate the positions and velocities at the instant of interaction $U_i = U(\tau_i) = 0$
- apply transformations **in front of** and **behind** the impulse

Refraction formulae: *background geodesics crossing the spherical impulse*

• positions:

$$x_i^- = |h'|\frac{Z_i + \bar{Z}_i}{h + \bar{h}} x_i^+ \qquad y_i^- = |h'|\frac{Z_i - \bar{Z}_i}{h - \bar{h}} y_i^+$$

$$z_i^- = |h'|\frac{Z_i\bar{Z}_i - 1}{|h|^2 - 1} z_i^+ \qquad t_i^- = |h'|\frac{Z_i\bar{Z}_i + 1}{|h|^2 + 1} t_i^+$$

• velocities:

$$\begin{aligned}
\dot{x}_i^- &= a_x\dot{x}_i^+ + b_x\dot{y}_i^+ + c_x\dot{z}_i^+ + d_x\dot{t}_i^+ \\
\dot{y}_i^- &= a_y\dot{x}_i^+ + b_y\dot{y}_i^+ + c_y\dot{z}_i^+ + d_y\dot{t}_i^+ \\
\dot{z}_i^- &= a_z\dot{x}_i^+ + b_z\dot{y}_i^+ + c_z\dot{z}_i^+ + d_z\dot{t}_i^+ \\
\dot{t}_i^- &= a_t\dot{x}_i^+ + b_t\dot{y}_i^+ + c_t\dot{z}_i^+ + d_t\dot{t}_i^+
\end{aligned}$$

where the coefficients a, b, c, d are complicated functions of Z_i, $h(Z_i)$ and its derivatives

Construction of a specific impulse

Junction condition: $h(Z)$ *vs.* **Curvature**: $H(Z)$

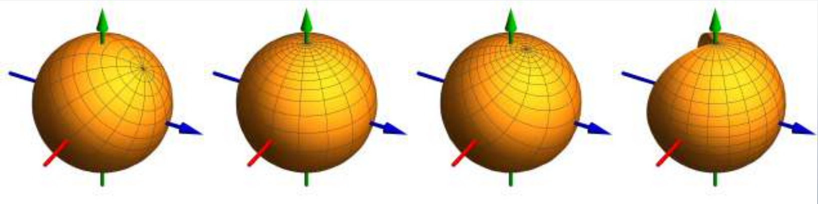
Möbius transformation: leaves $H(Z)$ unchanged

$$h(Z): \quad Z \mapsto \frac{aZ+b}{cZ+d}$$

Beyond linear fractions $\mapsto$ *non-trivial Schwarzian derivative* $H(Z) \mapsto$ *topological defects*

Decomposition of $h(Z)$: elementary operations

- spatial rotations $\mathcal{R}_{\{\phi,\theta,\psi\}}Z$ parameterized by the Euler angles $\{\phi,\ \theta,\ \psi\}$
- boost in the z direction $\mathcal{B}_{\{w\}}Z$ parameterized by w
- the simplest string-like structure $\mathcal{S}_{\{\delta\}}Z = Z^{1-\delta}$: cuts out the wedge $2\pi\delta$ around z-axis



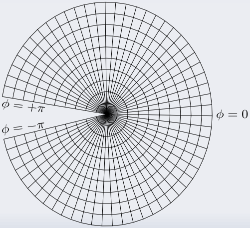
$\mathcal{R}_{\{0,-\frac{\pi}{3},0\}}(Z)$ $\mathcal{B}_{\{1/3\}}Z$ $\mathcal{B}_{\{1/3\}}\mathcal{R}_{\{0,-\frac{\pi}{3},0\}}(Z)$ $\mathcal{S}_{\{1/4\}}Z$

Specific function $h(Z)$: one string

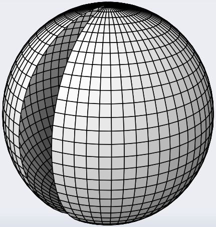
Assume: *complex mapping*

$$h(Z) = Z^{1-\delta}$$

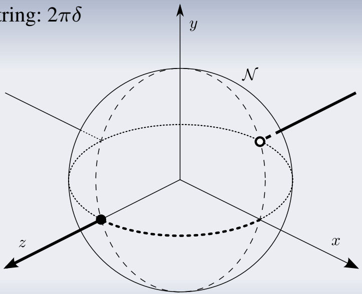
where $\delta \in [0, 1)$ characterizes *deficit angle* induced by a string: $2\pi\delta$



complex plane



Riemann sphere



space picture

Metric function $H(Z)$: *Schwarzian derivative* of $h(Z)$ i.e. $H(Z) = \dfrac{\frac{1}{2}\delta(1-\frac{1}{2}\delta)}{Z^2}$

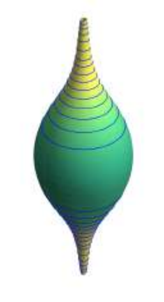
Specific function $h(Z)$: one string – initial data

Initial data for geodesics: static observers in front of the impulse

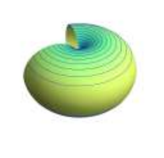
- interacting simultaneously at $t_i^+ = \text{const}$ $\mapsto$ emerging at different t_i^-
- interacting at specific $t_i^+ =$ $\mapsto$ emerging simultaneously at $t_i^- = \text{const}$

$\mathcal{M}^+, t_i^+ = \text{const}$

$\mapsto$

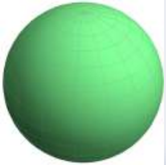


$\mathcal{M}^-$



$\mathcal{M}^+$

$\mapsto$



$\mathcal{M}^-, t_i^- = \text{const}$

Colour changes show the *time shift* and the *lines of constant* $t_i^\pm$ are thick blue lines.

Specific function $h(Z)$: boosted one string

Perform: string creation along z axis $\mapsto$ perpendicular rotation $\mapsto$ boost $\mapsto$ backward rotation

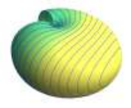
i.e. $$h(Z) = \mathcal{R}_{\{0,-\frac{\pi}{2},0\}}\mathcal{B}_{\{w\}}\mathcal{R}_{\{0,\frac{\pi}{2},0\}}\mathcal{S}_{\{\delta\}}Z$$

Initial data for geodesics:

$\mathcal{M}^+, t_i^+ = \text{const}$

$\mapsto$

$\mathcal{M}^-$



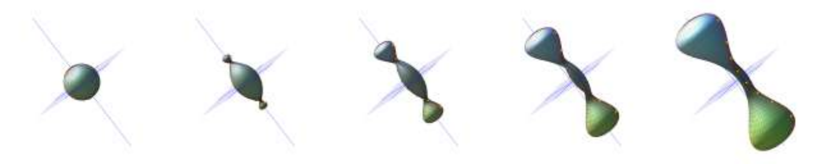
$\mathcal{M}^+$

$\mapsto$

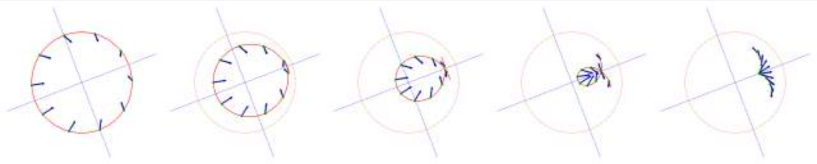
$\mathcal{M}^-, t_i^- = \text{const}$

Boosted one string: geodesic motion

Initial data: static *in front of* the impulse and emerging at $t_i^- = \text{const}$



$t = t_i^-$ $\quad$ $t = t_i^- + 0.4$ $\quad$ $t = t_i^- + 0.8$ $\quad$ $t = t_i^- + 1.2$ $\quad$ $t = t_i^- + 1.6$



$t = t_i^-$ $\quad$ $t = t_i^- + 0.4$ $\quad$ $t = t_i^- + 0.8$ $\quad$ $t = t_i^- + 1.2$ $\quad$ $t = t_i^- + 1.6$

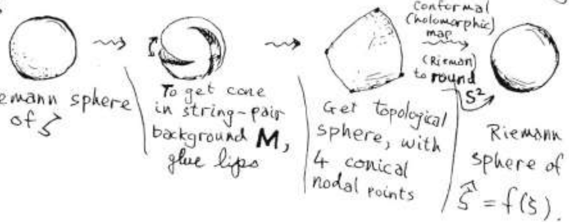
- the test observers gain nontrivial velocities
- asymmetry induced by the string motion in the x direction

Specific function $h(Z)$: 2 strings

Y. Nutku, R. Penrose: *On impulsive gravitational waves*, Twistor Newsletter **34** (1992) 9–12

A more involved situation is provided by a pair of strings that collide

Here the function h is obtained in the following way:



It seems to be hard to find f explicitly for this case, but a Riemann theorem ensures that f exists.

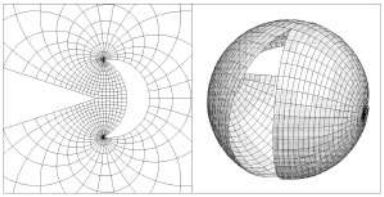
Specific function $h(Z)$: 2 strings

J. Podolský, J.B. Griffiths: *The collision and snapping of cosmic strings generating spherical impulsive gravitational waves*, Class. Quant. Grav. **17** (2000) 1401–13

Two cosmic strings:

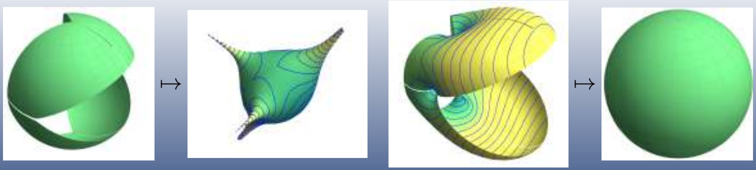
Located: *along* y and z

$$h(Z) = \left(\frac{\mathrm{i}Z^{1-\delta} - 1}{Z^{1-\delta} - \mathrm{i}} \right)^{1-\varepsilon}$$



Problem: $h(Z) \neq Z$ for $\delta = 0 = \varepsilon$
Solution: *additional rotation*

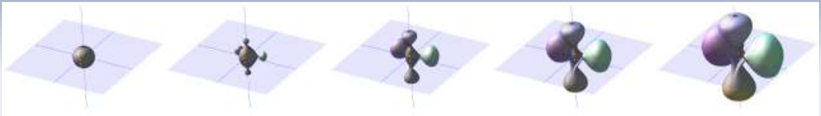
Initial data:



$\mathcal{M}^+, t_i^+ = \text{const}$ $\mapsto$ $\mathcal{M}^-$ $\mathcal{M}^+$ $\mapsto$ $\mathcal{M}^-, t_i^- = \text{const}$

Two strings: geodesic motion

Initial data: static *in front of* the impulse and emerging at $t_i^- = \text{const}$



$t = t_i^-$ $t = t_i^- + 0.4$ $t = t_i^- + 0.8$ $t = t_i^- + 1.2$ $t = t_i^- + 1.6$

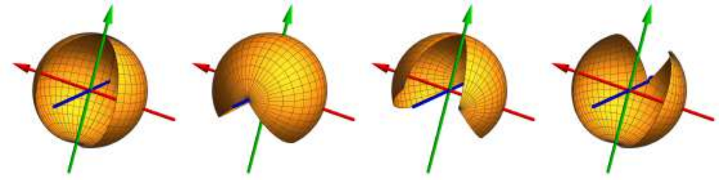
$t = t_i^-$ $t = t_i^- + 0.4$ $t = t_i^- + 0.8$ $t = t_i^- + 1.2$ $t = t_i^- + 1.6$

- test observers dragged by the ends of strings
- asymmetry: different values of deficit angles and ‘non-penpendicular’ strings positions

Two strings: peculiar case

Perform: string $\mapsto$ perpendicular rotation $\mapsto$ string $\mapsto$ backward rotation

i.e. $$h(Z) = \mathcal{R}_{\{0,-\frac{\pi}{2},0\}}\, \mathcal{S}_{\{\epsilon\}}\, \mathcal{R}_{\{0,\frac{\pi}{2},0\}}\, \mathcal{S}_{\{\delta\}}\, Z$$

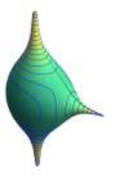


$\mathcal{S}_{\{\delta\}}$ $\quad$ $\mathcal{R}_{\{0,\frac{\pi}{2},0\}}$ $\quad$ $\mathcal{S}_{\{\epsilon\}}$ $\quad$ $\mathcal{R}_{\{0,-\frac{\pi}{2},0\}}$

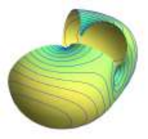
Initial data:

$\mathcal{M}^+, t_i^+ = \text{const}$

$\mapsto$



$\mathcal{M}^-$



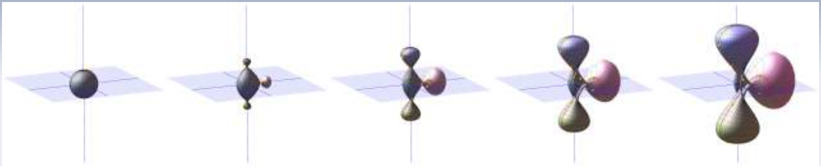
$\mathcal{M}^+$

$\mapsto$

$\mathcal{M}^-, t_i^- = \text{const}$

Peculiar two string case: geodesic motion

Initial data: static *in front of* the impulse and emerging at $t_i^- = \text{const}$



$t = 0$ $\quad t = t_i^- + 0.4$ $\quad t = t_i^- + 0.8$ $\quad t = t_i^- + 1.2$ $\quad t = t_i^- + 1.6$

$t = 0$ $\quad t = t_i^- + 0.4$ $\quad t = t_i^- + 0.8$ $\quad t = t_i^- + 1.2$ $\quad t = t_i^- + 1.6$

- test observers dragged by the ends of strings
- asymmetry: different values of deficit angles and only *three* string pieces

Our contribution

We were mainly interested in:

- *expanding* impulsive gravitational waves propagating on a flat background
- construction of specific complex mappings $h(Z)$ encoding the source properties
- geometric description of resulting spacetimes
- effects of the expanding impulse on free test particles

For more detail see:

- J. Podolský, R. Švarc: *Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant*, Phys. Rev. D **81** (2010) 124035
- M. Karamazov: *Impulsive gravitational waves*, Bachelor Thesis, Charles University (2015)
- J. Podolský, C. Sämann, R. Steinbauer, R. Švarc: *The global uniqueness and C^1-regularity of geodesics in expanding impulsive gravitational waves*, Class. Quantum Grav. **33** (2016) 195010
- D. Kofroň, M. Karamazov, R. Švarc: *An interpretation of spacetimes with expanding gravitational impulses generated by a pair of snapped cosmic strings* to be submitted


... thank you for your attention ...