QNMs of black holes encircled by a gravitating thin disk

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Outline

1. Superposition of multiple sources in GR and the SBH+disk model
2. QNMs of a perturbed black hole
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Weyl metric

Gravitational field of **static** and **axially symmetric** vacuum spacetimes is described by

$$ds^2 = -e^{2\nu}\,dt^2 + \rho^2 e^{-2\nu}\,d\phi^2 + e^{2\lambda-2\nu}(d\rho^2 + dz^2)\ , \tag{1}$$

where t, ρ, ϕ, z are the Weyl cylindrical coordinates and $\nu(\rho, z), \lambda(\rho, z)$.

Vacuum Einstein equations then

$$\Delta\nu = 0 \tag{2}$$

$$\lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2) \tag{3}$$

$$\lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z}\ , \tag{4}$$

where

- Δ is 3D Laplace operator in cylindrical coordinates (ρ, z)
- λ is integrated along some path through the vacuum region

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Multiple sources in GR

Consider **two** distinct **solutions**, described by ν_1, ν_2, and λ_1, λ_2 respectively

Their common gravitational field is given by

$$\nu = \nu_1 + \nu_2\,, \tag{5}$$
$$\lambda = \lambda_1 + \lambda_2 + \lambda_{int}\,, \tag{6}$$

where λ_{int} satisfies

$$\lambda_{\text{int},\rho} = 2\rho(\nu_{\text{Schw},\rho}\nu_{\text{disk},\rho} - \nu_{\text{Schw},z}\nu_{\text{disk},z})\,, \tag{7}$$
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SBH+disk model

We wish to study a **Schwarzschild black hole** described by

$$\nu_{\text{Schw}} = \frac{1}{2} \ln \left(\frac{R_+ + R_- - 2M}{R_+ + R_- + 2M} \right) , \tag{9}$$

$$\lambda_{\text{Schw}} = \frac{1}{2} \ln \left[\frac{(R_+ + R_-)^2 - 4M^2}{4R_+ R_-} \right] , \tag{10}$$

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$$R_\pm = \sqrt{\rho^2 + (|z| \mp M)^2} \, , \tag{11}$$

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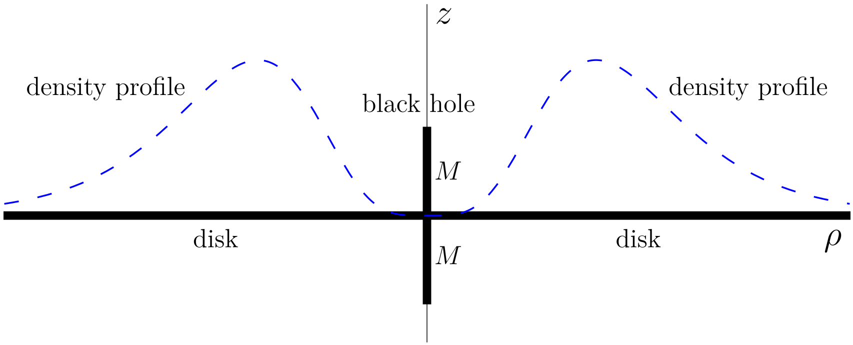

Vogt-Letelier disks

In Kotlařík and Kofroň (2022), we found

- a **family** of thin disks; potential first considered by Vogt and Letelier (2009)
- metric in **closed-forms**
- including analytical expressions for λ_{int}, when superposed with the black hole

$$\text{density} \propto \frac{\mathcal{M} b^{2m+1} \rho^{2n}}{(\rho^2 + b^2)^{m+n+3/2}}, \quad m, n \in \mathrm{N}_0, \tag{12}$$

$$\nu_{\mathrm{D}}^{(m,n)} = -(2m+1)\binom{m+n+1/2}{n} \mathcal{M} \sum_{j=0}^{m+n} \mathcal{Q}_j^{(m,n)} \frac{b^j}{r_b^{j+1}} P_j(|\cos\theta_b|), \tag{13}$$

$$r_b^2 \equiv \rho^2 + (|z| + b)^2, \qquad |\cos\theta_b| \equiv \frac{|z| + b}{r_b}, \tag{14}$$

where $\mathcal{Q}_j^{(m,n)}$ are constants, P_j are Legendre polynomials, and $\mathcal{M}$ is the total mass of the disk.

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Physical interpretation

Two simple physical interpretations

1. ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
2. two identical counter-orbiting dust streams with surface densities $(\sigma+ = \sigma- \equiv \sigma/2)$ following circular geodesics

Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \frac{\nu_{,z}(z=0^+)}{2\pi} e^{\nu-\lambda} = \sigma(\rho) e^{\nu-\lambda} \,, \quad P = \frac{\nu_{,z}(z=0^+)}{2\pi} \rho\nu_{,\rho} e^{\nu-\lambda} = \sigma(\rho)\rho\nu_{,\rho} e^{\nu-\lambda} \,.$$

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Quasinormal modes

black hole rings

$\downarrow$

emits gravitational waves

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asymptotically flat $\Rightarrow$ the whole system is dissipative

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exponential decay of the waves

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QNMs of a scalar field

How do QNMs propagates in the SBH+disk spacetime?

We take a simpler task and study QNMs of a massless scalar field ψ.

The QNMs are governed by the massless Klein-Gordon equation

$$\Box\psi = 0\,. \tag{15}$$

In the Schwarzschild coordinates

$$\rho = \sqrt{r(r-2M)}\sin\theta\,, \qquad z = (r-M)\cos\theta\,, \tag{16}$$

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Isolated Schwarzschild black hole

In the frequency domain

$$\Box\psi = \int_{-\infty}^{\infty} e^{-i\omega t}\mathcal{D}^2_\omega \psi_\omega(r,\theta,\phi)\,\mathrm{d}\omega = 0 \quad \Longrightarrow \quad \mathcal{D}^2_\omega \psi_\omega(r,\theta,\phi) = 0\ , \tag{17}$$

where the operator $\mathcal{D}^2_\omega$ reads

$$\mathcal{D}^2_\omega = f^{-1}(r)\omega^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r)\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\ ,$$

$$f(r) = 1 - \frac{2M}{r}\ . \tag{18}$$

The scalar field can be decomposed into spherical harmonics

$$\psi_\omega(r,\theta,\phi) = \sum_{m_z=-\infty}^{\infty} \sum_{\ell=|m_z|}^{\infty} \frac{\psi_{\omega\ell m_z}(r)}{r} Y_{\ell m_z}(\theta,\phi)\,, \tag{19}$$

which leads into a single radial equation for each mode

$$\frac{\mathrm{d}^2\psi_{\omega\ell m_z}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right]\psi_{\omega\ell m_z} = 0\,, \tag{20}$$

where $\mathrm{d}r_* = f^{-1}(r)\,\mathrm{d}r$ and

$$V_{\mathrm{eff}}(r) = \frac{f(r)f'(r)}{r} + \frac{\ell(\ell+1)f}{r^2}\,, \quad f'(r) \equiv \frac{\mathrm{d}f}{\mathrm{d}r}\,. \tag{21}$$

But the presence of the (axially symmetric) disk breaks this convenient property.

Cano, Fransen, and Hertog (2020): "almost separable" systems

A Schwarzschild black hole perturbed by a small deformation (gravitating light disk).

Denote $\epsilon = \mathcal{M}/M$ and expand the operator into the separable part plus first order corrections

$$\mathcal{D}^2_\omega = \mathcal{D}^2_{(0)\omega} + \epsilon\mathcal{D}^2_{(1)\omega}\,. \tag{22}$$

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The radial and angular equations are not decoupled.

However, we can separate the solution of $\mathcal{D}^2_{(0)\omega}\psi_\omega(r,\theta,\phi)$ for a certain $\ell = \ell_0, m_z = m_0$ and consistently assume

$$\psi_\omega(r,\theta,\phi) = \psi_{\omega \ell_0 m_0}(r) Y_{\ell_0 m_0}(\theta,\phi) + \epsilon \sum_{\substack{m_z=-\infty \\ m_z \neq m_0}}^{\infty} \sum_{\substack{\ell=-|m_z| \\ \ell \neq \ell_0}}^{\infty} \psi_{\omega \ell m_z}(r) Y_{\ell m_z}(\theta,\phi) \,. \tag{25}$$

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Thus

$$\mathcal{D}^2_\omega \psi_\omega(r,\theta,\phi) = \left[\mathcal{D}^2_{(0)\omega} + \epsilon \mathcal{D}^2_{(1)\omega}\right] \psi_{\omega \ell_0 m_0}(r) Y_{\ell_0 m_0}(\theta,\phi) + \\ + \epsilon \sum_{m_z=-\infty}^{\infty} \sum_{\substack{\ell=-|m_z| \\ \ell\neq\ell_0}}^{\infty} \mathcal{D}^2_{(0)\omega} \psi_{\omega \ell m_z}(r) Y_{\ell m_z}(\theta,\phi) + \mathcal{O}(\epsilon^2)\,. \quad (26)$$

and using the orthogonality of the spherical harmonics, the last term is projected out

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell_0 m_0} \mathcal{D}^2_\omega \psi_\omega(r,\theta,\phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \\ = \int_0^{2\pi}\int_0^{\pi} Y^*_{\ell_0 m_0} \left[\mathcal{D}^2_{(0)\omega} + \epsilon \mathcal{D}^2_{(1)\omega}\right] \psi_{\omega \ell_0 m_0}(r) Y_{\ell_0 m_0}(\theta,\phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi\,. \quad (27)$$

Performing the integration leads again to the radial equation

$$\frac{\mathrm{d}^2\Psi_{\omega\ell m_z}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right]\Psi_{\omega\ell m_z} = 0\,, \tag{28}$$

with the appropriately modified tortoise coordinate r_*, and

$$V_{\mathrm{eff}}(r) = V_{\mathrm{eff}}^{\mathrm{Sch}}(r) + \epsilon V_{\mathrm{eff}}^{\mathrm{corr}}(r)\,. \tag{29}$$

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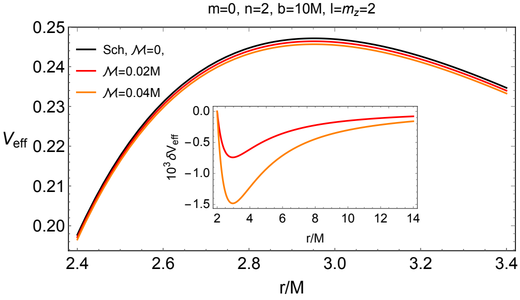
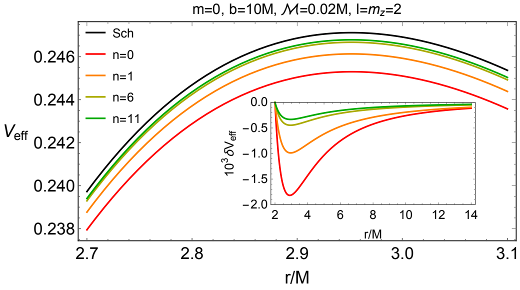
Effective potential

The disk is described by 4 parameters: 2 positive real $\mathcal{M}$, b, and a pair (m, n).

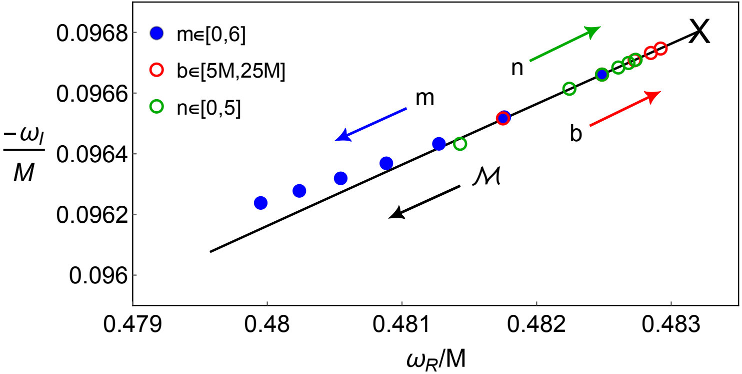
$$w^{(m,n)}(\rho) = \binom{m+n+1/2}{n} \frac{(2m+1)\mathcal{M}}{2\pi} \frac{b^{2m+1}\rho^{2n}}{(\rho^2+b^2)^{m+n+3/2}}, \qquad m, n \in \mathrm{N}_0 . \tag{30}$$



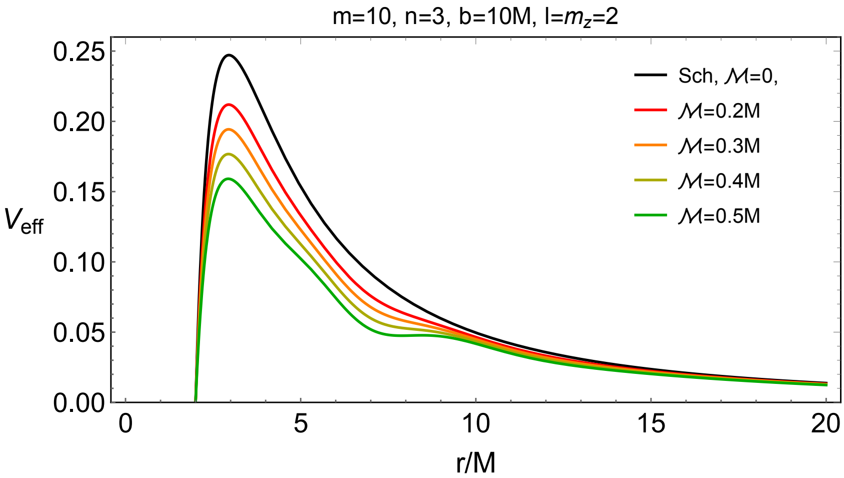


Universal behaviour?

BH immersed in spherically symmetric matter exhibits similar behavior (Cardoso et al., 2022; Konoplya, 2021).



While some quantum corrections may increase the QNMs frequencies.



m=10, n=3, b=10M, l=m_z=2
V_{eff}
0.25
0.20
0.15
0.10
0.05
0.00
0
5
10
15
20
r/M
Sch, $\mathcal{M}$=0,
$\mathcal{M}$=0.2M
$\mathcal{M}$=0.3M
$\mathcal{M}$=0.4M
$\mathcal{M}$=0.5M

Brief summary

- QNMs of a scalar field propagating in the SBH+disk background
- the effective potential is flattened by the presence of the disk
- QNMs are shifted towards the same direction in the complex plane
- hint of a universal behaviour
- it might help to disentangle environmental effects from those induced by quantum corrections

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