Gravitational waves from quasielliptic compact binary systems in massless scalar-tensor theories

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- Massless scalar-tensor gravity [$\Leftrightarrow$ Damour-Esposito-Farèse gravity] is simple and well-posed
- Strongly constrained by Solar System tests and binary pulsar observations, but we are interested in constraining it with GWs as well [BH-NS or NS-NS systems $\Rightarrow$ different regime !]
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- Massless scalar-tensor gravity [$\Leftrightarrow$ Damour-Esposito-Farèse gravity] is simple and well-posed
- Strongly constrained by Solar System tests and binary pulsar observations, but we are interested in constraining it with GWs as well [BH-NS or NS-NS systems $\Rightarrow$ different regime !]
- Simple theory, so good toy model to study features of PN in the presence of a scalar field: can then be extended to more complex theories, e.g. scalar-Gauss-Bonnet, etc.

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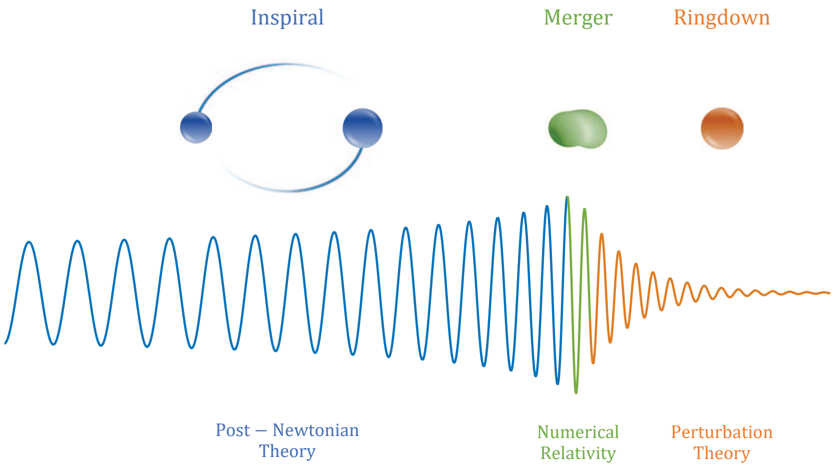
Post-Newtonian methods applied to scalar-tensor theory

Scope


Inspiral
Merger
Ringdown
Post – Newtonian Theory
Numerical Relativity
Perturbation Theory

Generalized Fierz-Pauli-Brans-Dicke theory

Action defined in Jordan frame :

$$\boxed{S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\phi R - \underbrace{\frac{\omega(\phi)}{\phi}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}_{\text{kinetic term but no potential}}\right] + \underbrace{S_\mathrm{m}[g_{\alpha\beta},\mathfrak{m}]}_{\text{no coupling to }\phi}}$$

For the post-Newtonian setup, better to work in Einstein frame. Define

$$\varphi = \frac{\phi}{\phi_0} \qquad \text{and} \qquad \tilde{g}_{\mu\nu} = \frac{\phi}{\phi_0}g_{\mu\nu} \qquad \text{where} \qquad \phi \underset{r\to\infty}{\longrightarrow} \phi_0$$

The action in Einstein frame then reads

$$S = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{3+2\omega(\phi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right] + S_\mathrm{m}[\varphi^{-1}\tilde{g}_{\alpha\beta},\mathfrak{m}]$$

Equivalence to DEF gravity

Our Einstein frame action

$$S = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\Big[\tilde{R} - \frac{3+2\omega(\phi_0\varphi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\Big] + S_{\mathrm{m}}[\varphi^{-1}\tilde{g}_{\alpha\beta}, \mathfrak{m}]$$

is equivalent to Damour & Esposito-Farèse (DEF) gravity [gr-qc/9602056]:

$$S_{\mathrm{DEF}} = \frac{c^3}{16\pi G_*}\int \mathrm{d}^4x\sqrt{-g_*}\left[R_* - 2g_*^{\alpha\beta}\partial_\alpha\varphi_*\partial_\beta\varphi_*\right] + \mathcal{S}_{\mathrm{m}}\left[\mathcal{A}(\varphi_*)g^*_{\alpha\beta}, \mathfrak{m}\right]$$

where $G_* = G/\phi_0$, $g^*_{\mu\nu} = \tilde{g}_{\mu\nu}$ and $\varphi_* = \mathcal{T}(\varphi)$, where

$$\mathcal{T}(x) = \frac{1}{2}\int^x \mathrm{d}y\sqrt{\frac{3+2\omega(\phi_0 y)}{2y^2}} \qquad \text{and} \qquad \mathcal{A}(\varphi_*) = \frac{1}{\mathcal{T}^{-1}(\varphi_*)}$$

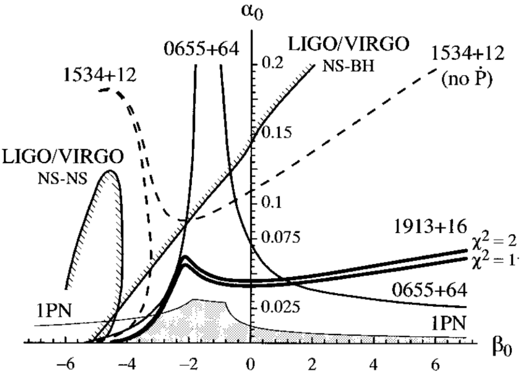
Freedom of choice in $\omega(\phi) \Leftrightarrow \mathcal{T}(\varphi) \Leftrightarrow \mathcal{A}(\varphi_*)$

Constraints of DEF gravity

We recall the DEF gravity is written

$$S_{\rm DEF} = \frac{c^3}{16\pi G_*}\int \mathrm{d}^4x\sqrt{-g_*}\left[R_* - 2g_*^{\alpha\beta}\partial_\alpha\varphi_*\partial_\beta\varphi_*\right] + \mathcal{S}_{\rm m}\left[\mathcal{A}(\varphi_*)g^*_{\alpha\beta}, \mathfrak{m}\right]$$

which can be described by the parameters $\alpha_0 = \frac{\partial\mathcal{A}(\varphi_*)}{\partial\varphi_*}$ and $\beta_0 = \frac{\partial^2\mathcal{A}(\varphi_*)}{\partial\varphi_*^2}$



[gr-qc/9803031]

LIGO-Virgo experiments cannot constrain better than binary pulsars, but how about ET and LISA ?

Field equations

The field equations should be expressed using the Landau & Lifschitz formulation. Perturbation of (conformal inverse) metric around Minkowski:

$$\boxed{h^{\mu\nu} = \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu} - \eta^{\mu\nu}}$$

[At linear level, this is equivalent the "trace reversed metric"]

Perturb (normalized) scalar field around background value:

$$\varphi = 1 + \psi$$

Restriction to harmonic gauge $\partial_\mu h^{\mu\nu} = 0$, the field equations read:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4 \phi_0}\left[\varphi(-g)T^{\mu\nu} + \frac{c^4\phi_0}{16\pi G}\Lambda^{\mu\nu}[h,\psi]\right]$$

$$\Box \psi = \frac{8\pi G}{c^4\phi_0}\left[\frac{\varphi\sqrt{-g}}{[3+2\omega(\phi)]}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right) + \frac{c^4\phi_0}{8\pi G}\Lambda_s[h,\psi]\right]$$

where the non-linear couplings are described by $\Lambda^{\mu\nu}[h,\psi]$ and $\Lambda_s[h,\psi]$

Near-zone and exterior vacuum zone

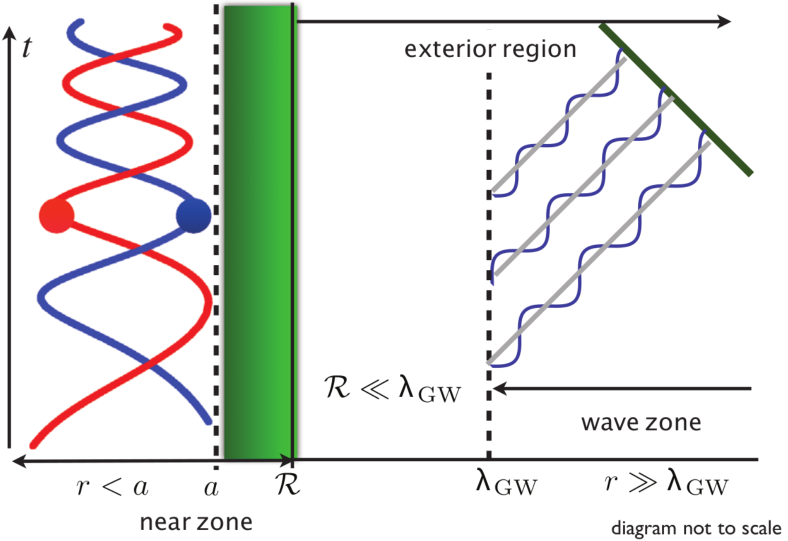


Figure from [1410.7832]

Linearized metric in exterior vacuum

N.B. I will focus on the scalar field for pedagogy

In the exterior vacuum zone, we formally perform a multipolar post-Minkowskian expansion $\psi = G\psi_1 + G^2\psi_2 + ...$

At linear level, the scalar field equation reads $\Box\psi_1 = 0$, so we can express it as a multipolar expansion [Thorne 1980]:

$$\psi_1 = -\frac{2}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{I}^s_L\right]$$

The "source moments" can be matched to a near-zone, post-Newtonian ($v \ll c$) computation involving the matter, such that they can be expressed as functions of the phase space variable of the compact binary system

$$\mathrm{I}^s_L[\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2]$$

Multipolar moments: an example

For example, we have [2201.10924]

$$I_i^s = -\frac{m_1(1-2s_1)y_1^i}{\phi_0(3+\omega_0)} - \frac{m_2(1-2s_2)y_2^i}{\phi_0(3+\omega_0)} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

where various ST parameters come from

$$\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega_0' + \ldots$$

and [for $A \in \{1,2\}$] :

$$m_A(\psi) = m_A\left(1 + s_A\psi + \ldots\right)$$

Note that the weak equivalence principle is broken so the inertial mass of a star (seen as a point-particle) can depend on the local value of the scalar field, hence the need to introduce sensitivities, e.g.

$$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}$$

This allows to account for spontaneous scalarized stars !

MPM metric in exterior vacuum

Now that the linear metric is entirely determined, we go back to the MPM expansion: $\psi = G\psi_1 + G^2\psi_2 + ...$ and inject it into our full vacuum field equation

$$\Box\psi = \Lambda_s[h, \psi]$$

where $\Lambda_s[h, \psi]$ is at least quadratic in the fields. Thus, we contruct the MPM metric by iterating:

$$\Box\psi_n = \Lambda_s^{(n)}[h_1, ..., h_{n-1}; \psi_1, ..., \psi_{n-1}]$$

This generates nonlocal effects such as tail, the quadratic memory, etc. !

Radiative moments

Once the MPM metric constructed, we can discard all subdominant terms in the $rt \to \infty$ limit. We thus recover an (asymptotically) multipolar structure:

$$\psi \sim \frac{1}{r} \sum \hat{n}_L \mathcal{U}^s_L$$

We recover the tail terms of GR , but also find new ST tail terms and a new ST memory term:

$$\mathcal{U}_{ij} = \overset{(2)}{\mathrm{I}}_{ij} + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}_{ij}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] + \frac{G(3+2\omega_0)}{3c^3} \int_0^{+\infty} \mathrm{d}\tau \left[\overset{(2)}{\mathrm{I}}{}^s_{\langle i} \overset{(2)}{\mathrm{I}}{}^s_{j\rangle} \right](u-\tau) + (\mathrm{inst}) + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$$\mathcal{U}^s_i = \overset{(1)}{\mathrm{I}}{}^s_i + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(3)}{\mathrm{I}}{}^s_i(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] + (\mathrm{inst}) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Fluxes at infinity

Now that we know that asymptotic structure of the scalar waves [idem for GWs] at $\mathcal{I}^+$, we can deduce the fluxes of energy and angular momentum that they carry [2401.06844]:

$$\begin{aligned}\mathcal{F}^s &= \frac{c^3 R^2 (3+2\omega_0)\phi_0}{16\pi G}\int \mathrm{d}^2\Omega \dot{\psi}^2 \\ &= \sum_{\ell=0}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}\ell!(2\ell+1)!!}\dot{\mathcal{U}}^s_L \dot{\mathcal{U}}^s_L\end{aligned}$$

$$\begin{aligned}\mathcal{G}^s_i &= \frac{c^3 R^3 (3+2\omega_0)\phi_0}{16\pi G}\int \mathrm{d}^2\Omega \dot{\psi}\epsilon_{iab} n_a \partial_b \psi \\ &= \sum_{\ell=1}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}(\ell-1)!(2\ell+1)!!}\epsilon_{iab}\mathcal{U}^s_{aL-1}\dot{\mathcal{U}}^s_{bL-1}\end{aligned}$$

where we have used $\psi \sim \frac{1}{r}\sum \hat{n}_L \mathcal{U}^s_L(t-r/c)$.

Quasicircular orbits

Why are we interested in the fluxes ? Consider the case of a quasicircular orbit. First, the angular momentum flux is related to the energy flux by $\mathcal{F} = \omega \mathcal{G}$, so we only consider the energy balance law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} - \mathcal{F}^s$$

In the COM frame, the only dynamical variables are $r = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$, $\boldsymbol{n} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$.

The fluxes depend on them only through $r \approx (Gm/\omega^2)^{2/3}$, $v^2 \approx (Gm\omega)^{2/3}$ and $\boldsymbol{n} \cdot \boldsymbol{v} \approx 0$, where ω is the orbital frequency.

Thus, the energy balance equation reduces to an equation of the type:

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = f(\omega)}$$

This immediately yields the phase and frequency evolution !

The phase at 1.5PN for quasi-circular orbits

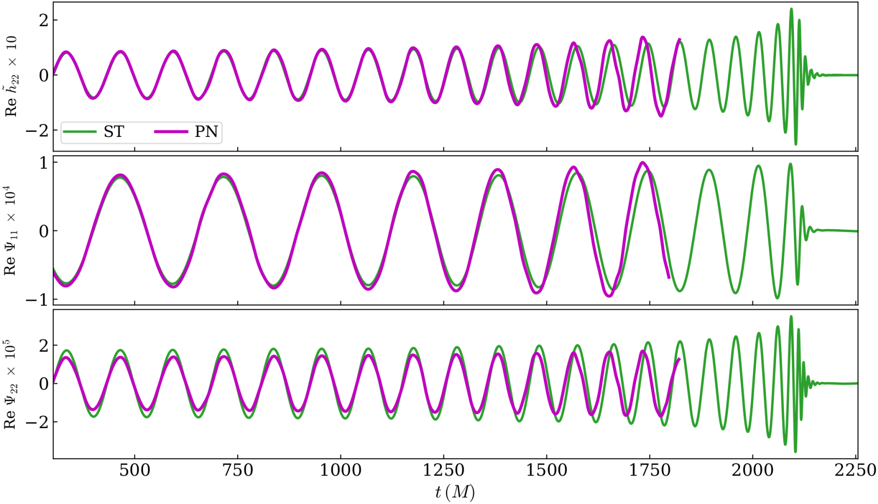
In [2201.10924], we found:

$$
\begin{aligned}
\phi_{\text{circ}} = -\frac{1}{4\zeta\mathcal{S}_-^2\nu x^{1/2}}\Bigg[&x^{-1} \\
&+\frac{3}{2}+8\bar{\beta}_+-2\bar{\gamma}-12\bar{\beta}_+\bar{\gamma}^{-1}-\frac{72}{5}\zeta^{-1}\mathcal{S}_-^{-2} \\
&-6\zeta^{-1}\bar{\gamma}\mathcal{S}_-^{-2}-12\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ \\
&+\delta\Big[-8\bar{\beta}_-+12\bar{\beta}_-\bar{\gamma}^{-1}+12\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+\Big]+\frac{7}{2}\nu \\
&+3\pi x^{1/2}\log(x)\left(1+\frac{\bar{\gamma}}{2}\right) \\
&+x\Bigg\{\text{complicated expression}\Bigg\} \\
&+\frac{\pi x^{3/2}}{1-\zeta}\Bigg\{\text{complicated expression}\Bigg\}\Bigg].
\end{aligned}
$$

This is the main observable in a GW !

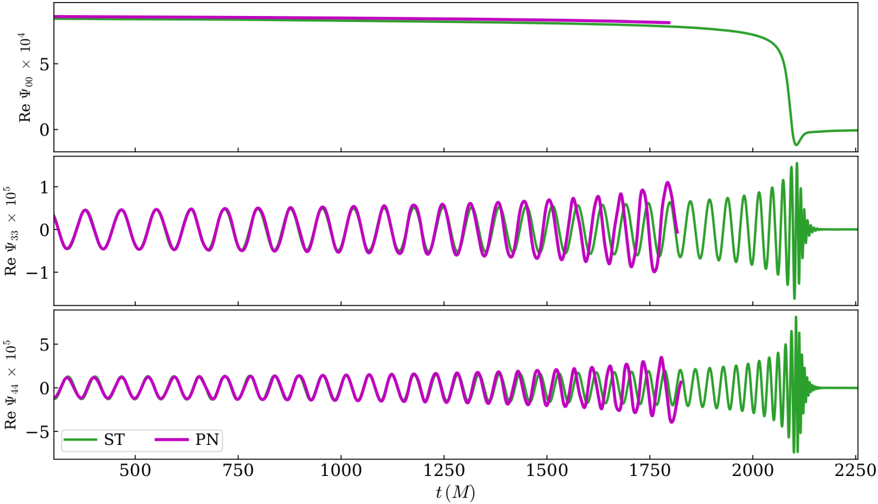
Comparison to NR

Numerical simulation [2304.11836] found good agreement with our results:



Comparison to NR (cont'd)

even for the DC memory effect !



(a) Scalar modes

The quasi-Keplerian representation for alternative theories of gravity

The Kepler solution

We study the two-body problem in the context of Newtonian gravity, described by the relative acceleration [in the COM frame]:

$$a^i = a_1^i - a_2^i = -\frac{G_{12} m n^i}{r^2}$$

where $G_{12} = G$ in general relativity, but in ST theory reads

$$G_{12} = \frac{G}{\phi_0}\left(1 + \frac{(1-2s_1)(1-2s_2)}{3+2\omega_0}\right)$$

In the bound case, we know that the orbit is an ellipse:

$$r = \frac{a(1-e^2)}{1 + e\cos(\phi - \phi_{\text{peri}})}$$

where a is the semimajor axis and e the eccentricity ($e < 1$ for bound orbits), given in terms of the energy ($E < 0$) and angular momentum J:

$$a = -\frac{Gm}{2E} \qquad \text{and} \qquad e = \sqrt{1 + \frac{2EJ}{G^2 m^2}}$$

The Kepler solution

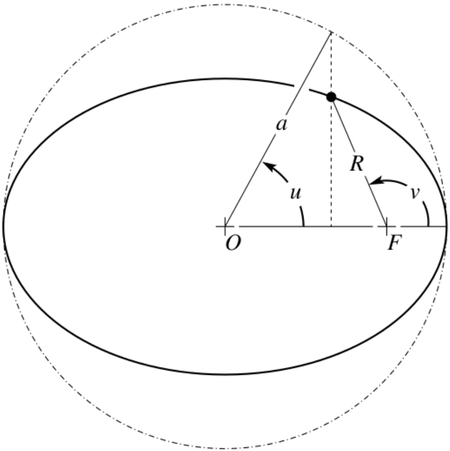
To describe the time evolution, it is however more practical to use the following set of three equations

$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v(u)$$

where we have introduced

- the *eccentric anomaly* u, which acts as an affine parameter
- the *true anomaly* $v(u) \equiv 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$
- the *mean motion* $n \equiv 2\pi/P$, where P is a time period
- the *mean anomaly* $\ell = n(t - t_0)$, which increases linearly with time and goes from 0 to 2π over one orbit

The Kepler solution



$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v$$
$$v = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$$

Figure from [gr-qc/0407049]

The quasi-Keplerian solution at 1PN order

What happens if we now want to solve the equations of motion for the 1PN acceleration ? $a^i = -\frac{G_{12} m n^i}{r^2} + \frac{1}{c^2}\,(\text{many terms})^i$

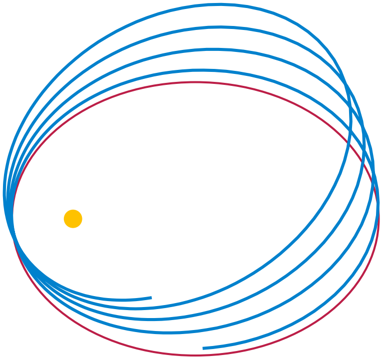
Damour & Deruelle [Ann.IHP.Phys.Th. 43, 1 (1985), p.107] showed that the equations of motion then reads

$$\begin{aligned} r &= a_r(1 - e_r \cos u) \\ \phi - \phi_0 &= K v \\ n(t - t_0) &= u - e_t \sin(u) \\ v(u) &= 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right] \end{aligned}$$

which is the same equation as before, except:

- there are now three eccentricities e_r, e_t, e_ϕ
- pericenter precession appears via the factor $K = 1 + k$ (with $k \ll 1$)
- a_r and n acquire post-Newtonian corrections

Doubly periodic structure of QK motion



The time between two periastrons is the *radial period* denote P, so the mean motion $n = 2\pi/P$ is the *radial frequency*.

The time for the angular coordinate ϕ to go from 0 to 2π is P/K, so $\omega = nK$ is the *angular frequency*

Thus, $K = 1 + k$ with $k \ll 1$ is a measure of the pericenter precession

(1)

The quasi-Keplerian solution at 2PN order

Damour & Schäfer [Nuovo Cim.B 101 (1988) 127] showed that the QK parametrization reads at 2PN

$$\begin{aligned}
r &= a_r(1 - e_r \cos u) \\
\phi - \phi_0 &= K\Big[v + f_\phi \sin(2v) + g_\phi \sin(3v)\Big] \\
n(t - t_0) &= u - e_t \sin(u) + f_t \sin(v) + g_t(v - u) \\
v(u) &= 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right]
\end{aligned}$$

Here, the new parameters f_ϕ, g_ϕ, f_t and g_t are all of order $\mathcal{O}(1/c^4)$, while all other parameters acquire 2PN corrections.

But how do we determine the values of these parameters ?

Determining the QK parameters

Assume we are working in *some theory of gravity* [e.g. GR or ST theory], and that we have determined (in a PN sense):

$$E = f(r, \dot{r}, \dot{\phi}) \qquad \text{and} \qquad J = g(r, \dot{r}, \dot{\phi})$$

For many theories of gravity, we can invert this as

$$\dot{r}^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$
$$\dot{\phi} = \frac{F}{r^2} + \frac{I_1}{r^3} + \frac{I_2}{r^4} + \frac{I_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

where A, B, C and F are of order 1, but D_1 and D_2 are 1PN and the others 2PN. **All these parameters are functions of E and J.**

Determining the QK parameters (cont'd)

Solving these EOM [technical !] directly yields the 2PN QK representation, and one reads off the expressions of the PK parameters (a_r, e_t, g_t, ...) in terms of A, B, D_1, ... For example [2401.06844]:

$$a_r = -\frac{B}{A} + \frac{D_1}{2C} + \frac{2BD_1^2 - 2BCD_2 + 4B^2D_3 - ACD_3}{2C^3} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Expression of A, B, ... depends of the theory. For example, in ST theory:

$$B = \tilde{G}\alpha m \left\{1 + \varepsilon\left[3 + \bar{\gamma} - \frac{7}{2}\nu\right] + \varepsilon^2\left[\frac{9}{4} + \frac{3}{4}\bar{\gamma} + \nu\Big(-12 - \frac{15}{4}\bar{\gamma}\Big) + \frac{21}{4}\nu^2\right]\right\}$$

where $\varepsilon = -2E/(m\nu c^2) > 0$ and $\varepsilon = \mathcal{O}(1/c^2)$.

To get the GR results, replace $\tilde{G}\alpha \to G$ and $\bar{\gamma} \to 0$.

If (i) you have $E = f(r, \dot{r}, \dot{\phi})$ and $J = g(r, \dot{r}, \dot{\phi})$ for your favorite theory; (ii) it has some nice properties
$\Rightarrow$ use these results to immediately obtain the QK representation

Applications at relative 1PN order

Fluxes at Newtonian order

At Newtonian order [reminder: the leading order is -1PN], the flux is *instantaneous*, i.e. no tails or memory. The QK representation allows us to write the fluxes only in terms of the eccentric anomaly:

$$\mathcal{F} = f[r, \phi, \dot{r}, \dot{\phi}] = g[r, \phi] = h[u]$$

After some trigonometry, we find that the structure is in fact

$$\mathcal{F} = \sum_k \left[\frac{\alpha_k}{[1 - e_t \cos(u)]^k} + \frac{\beta_k \sin(u)}{[1 - e_t \cos(u)]^k} \right]$$

The orbit averaged flux reads:

$$\langle \mathcal{F} \rangle = \frac{1}{P} \int_0^P \mathrm{d}t \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\ell \, \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u \frac{\mathrm{d}\ell}{\mathrm{d}u} \mathcal{F}$$

where $\mathrm{d}\ell / \mathrm{d}u = 1 - e_t \cos(u)$. We can then use:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}u}{[1 - e_t \cos(u)]^n} = \frac{P_{n-1}(1/\sqrt{1 - e_t^2})}{(1 - e_t^2)^{n/2}}$$

Averaged fluxes at Newtonian order

I find at Newtonian (relative 1PN) order [2401.06844]

$$\langle \mathcal{F} \rangle = \frac{32c^5x^5\nu^2(1+\bar{\gamma}/2)}{5\tilde{G}\alpha} \cdot \frac{1+\frac{73}{24}e_t^2+\frac{37}{96}e_t^4}{(1-e_t^2)^{7/2}}$$

$$\langle \mathcal{G} \rangle = \frac{32c^2mx^{7/2}\nu^2(1+\bar{\gamma}/2)}{5} \cdot \frac{1+\frac{7}{8}e_t^2}{(1-e_t^2)^2}$$

$$\langle \mathcal{F}^s \rangle = \frac{c^5x^4\nu^2\zeta}{3\tilde{G}\alpha}\left[4\mathcal{S}_-^2\frac{1+e_t^2/2}{(1-e_t^2)^{5/2}} + \frac{x}{(1-e_t^2)^{7/2}}\left(\mathcal{C}_0+\mathcal{C}_2e_t^2+\mathcal{C}_4e_t^4\right)\right]$$

$$\langle \mathcal{G}^s \rangle = \frac{c^2mx^{7/2}\nu^2\zeta}{3}\left[\frac{4\mathcal{S}_-^2}{1-e_t^2} + \frac{x}{(1-e_t^2)^{7/2}}\left(\mathcal{D}_0+\mathcal{D}_2e_t^2\right)\right]$$

where we introduce the dimensionless PN parameter

$$x = \left(\frac{\tilde{G}\alpha m\omega}{c^3}\right)^{2/3}$$

Here, $\omega = nK$ is the *angular* frequency (and n is the *radial* frequency)

Orbital evolution at leading Newtonian order

At leading order, we consider: (i) a Keplerian orbit; (ii) the Newtonian E and J; (iii) the -1PN fluxes $\mathcal{F}^s$ and $\mathcal{G}^s$. The balance equations $\mathrm{d}E/\mathrm{d}t = -\mathcal{F}^s$ and $\mathrm{d}J/\mathrm{d}t = -\mathcal{G}^s$ can be rewritten as [2401.06844]

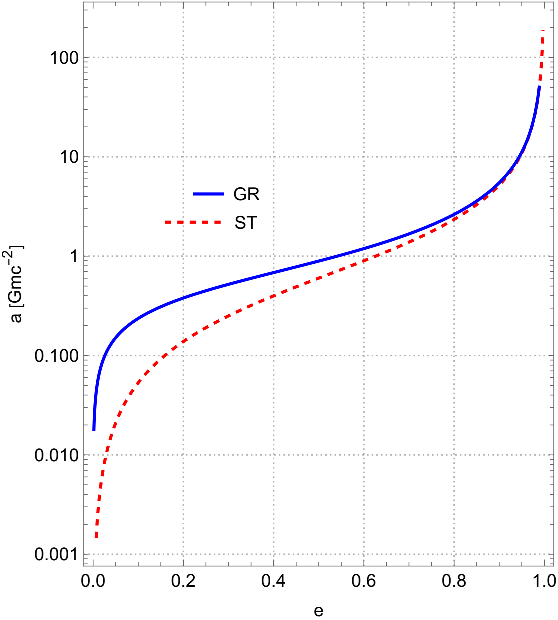
$$\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle = -\frac{8}{3}\,\frac{(\tilde{G}\alpha m)^2 \zeta \mathcal{S}_-^2 \nu}{c^3} \cdot \frac{1 + e^2/2}{a^2(1-e^2)^{5/2}}$$
$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = -\frac{(\tilde{G}\alpha m)^2 \zeta \mathcal{S}_-^2 \nu}{c^3} \cdot \frac{2e}{a^3(1-e^2)^{3/2}}.$$

Eliminating the time dependency and solving: $\boxed{a = \dfrac{c_0 e^{4/3}}{1-e^2}}$

In GR, the equivalent formula (due to Peters [PhysRev.136.B1224]) reads

$$a = \frac{c_0' e^{12/19}}{1-e^2}\left(1 + \frac{121}{304}e^2\right)^{870/2299}$$

The “Peters and Mathews” formula for ST theories



Evolution of orbital parameters at 1PN order

This time, we work at 1PN order in the QK parametrization and Newtonian order in the fluxes. THis time, we rewrite the balance equations $\mathrm{d}E/\mathrm{d}t = -\mathcal{F} - \mathcal{F}^s$ and $\mathrm{d}J/\mathrm{d}t = -\mathcal{G} - \mathcal{G}^s$ in terms of x and e_t [2401.06844]

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \frac{2c^3\zeta x^4\nu}{3\tilde{G}\alpha m}\left\{ \frac{4\mathcal{S}_-^2(1+\frac{1}{2}e_t^2)}{(1-e_t^2)^{5/2}} + \frac{x}{15(1-e_t^2)^{7/2}}\Big(\mathcal{C}_1 + e_t^2\mathcal{C}_2 + e_t^4\mathcal{C}_3\Big)\right\}$$

$$\left\langle \frac{\mathrm{d}e_t}{\mathrm{d}t} \right\rangle = -\frac{c^3\zeta x^3\nu}{\tilde{G}\alpha m}\left\{ \frac{2\mathcal{S}_-^2 e_t}{(1-e_t^2)^{3/2}} + \frac{x\, e_t}{15(1-e_t^2)^{5/2}}\Big(\mathcal{C}_4 + e_t^2\mathcal{C}_5\Big)\right\}$$

Since all other QK parameters (e.g. a_r, e_r, f_t, ...) can be expressed in terms of the pair (x, e_t), we have entirely characterized the motion!

GW waveform at Newtonian order

A common description of the GW waveform is to perform a mode decomposition.

$$h_+ - \mathrm{i}h_\times = \frac{2\tilde{G}(1-\zeta)m\nu x}{Rc^2}\sqrt{\frac{16\pi}{5}}\sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell} \hat{H}^{\ell m}e^{-\mathrm{i}m\phi}\,{}_{-2}Y^{\ell m}(\Theta,\Phi)$$

$$\psi = \frac{2\mathrm{i}\tilde{G}\zeta\sqrt{\alpha}\mathcal{S}_- m\nu\sqrt{x}}{Rc^2}\sqrt{\frac{8\pi}{3}}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} \hat{\Psi}^{\ell m}e^{-\mathrm{i}m\phi}\,Y^{\ell m}(\Theta,\Phi)$$

Complete mode decomposition at Newtonian (relative 1PN) order, e.g. [2401.06844]:

$$\hat{\Psi}^{11} = \frac{1-e_t^2-\mathrm{i}e_t\sqrt{1-e_t^2}\sin u}{\sqrt{1-e_t^2}(1-e_t\cos u)} + \mathcal{O}(x)$$

Inversing the Kepler equation $\ell = u - e_t\sin(u)$: numerically or using:

$$u = \ell + 2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin(n\ell)$$

Conclusion

Recapitulation

In this work, I have:

- solved the EOM for ST theories at 2PN order for an quasi-elliptic system with the 2PN quasi-Keplerian parametrization
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- solved the EOM for ST theories at 2PN order for an quasi-elliptic system with the 2PN quasi-Keplerian parametrization
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- computed the fluxes of energy and angular momentum up to Newtonian order [relative 1PN order]
- obtained the evolution equations for x and e_t at 1PN order
- computed the waveform modes at Newtonian [relative 1PN] order

Presentation is based on:

- Bernard, Blanchet & DT, JCAP 08, 008 (2022), arXiv:2201.10924
- DT, Phys. Rev. D 109, 104003 (2024), arXiv:2401.06844
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Recapitulation

In this work, I have:

- solved the EOM for ST theories at 2PN order for an quasi-elliptic system with the 2PN quasi-Keplerian parametrization
- extended the method in a way that greatly facilitates the same computation for other alternative theories of gravity
- computed the fluxes of energy and angular momentum up to 1.5PN order [relative 2.5PN order], including tail and memory contributions
- obtained the evolution equations for x and e_t at 2.5PN order
- computed the waveform modes at Newtonian [relative 1PN] order

Presentation is based on:

- Bernard, Blanchet & DT, JCAP 08, 008 (2022), arXiv:2201.10924
- DT, Phys. Rev. D 109, 104003 (2024), arXiv:2401.06844
- DT, *in preparation*

Prospects

Possible directions for future work

- full waveform at 1.5PN [relative 1.5PN] $\Rightarrow$ need to include some post-adiabatic contribution (i.e. not only the secular evolution of orbital parameters, but also their oscillations over an orbit)
- data analysis applications: can we use this to constraint the theory ? [most current test of alternative are done for agnostic deviations to GR]
- comparison with numerical relativity
- unbound (quasihyperbolic) systems
- including spins and/or precession
- slightly more complicated theories, e.g. scalar-Gauss-Bonnet
- study theories exhibiting screening, e.g. k-essence [is PN theory applicable ?]