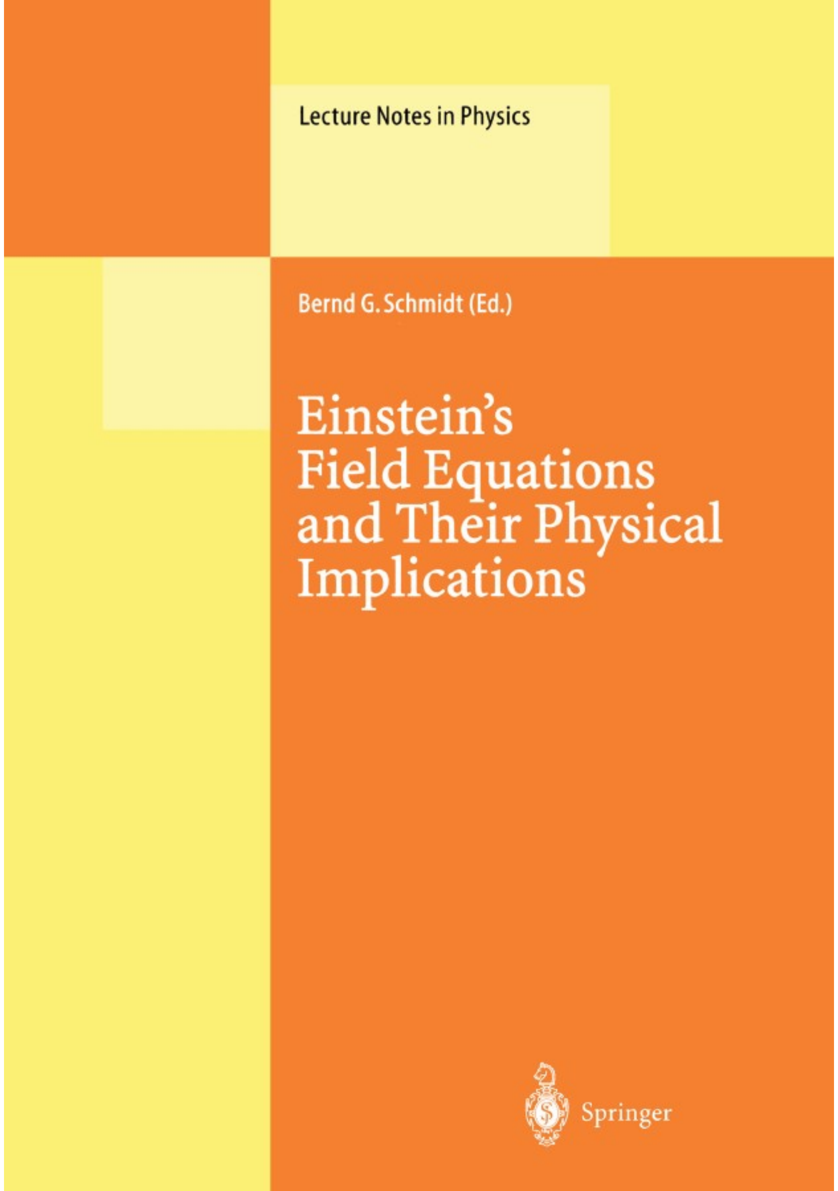
Importance of Being Exact

Some notes on J. Bičák's treatise

Solutions of Einstein's Equations:
Their Meaning and Uses



right: Jürgen Ehlers ... receives the Medal of the Charles University in Prague (09/2007)

left: J.Ehlers Festschrift (2000) where
J. Bičák prepares 120 pages on
Selected Solutions of Einstein's Field Equations:
Their Role in General Relativity and Astrophysics

Preface of the updated Review

It was just before the turn of millenium when I was asked by Bernd Schmidt to write a **chapter on exact solutions** to Einstein's equations for the Festschrift dedicated to Jürgen Ehlers' 70th birthday. The book \citep{BGS} appeared in 2000 and a few years later I was asked by Bernard Schutz, the founder of the **Living Reviews in Relativity**, to write a contribution on the solutions. I continued to devote some time to collecting new references and writing new sections of the article until about 2011. Its **length** and number of **references grew substantially** as compared with the original chapter.
However, in 2012 I and my colleagues spent considerable time by organizing the international conference "**Relativity and Gravitation**" in Prague to commemorate the 100 anniversary of Albert Einstein stay in Prague.
I was then involved in editing two volumes of the Proceedings \citep{aeiprg1,aeiprg2}, in various other activities and duties, and left the article unfinished. When **last year** the new editors asked me what is the state of the article and revealed an interest to publish it, I decided to update the article.

Selected Solutions of Einstein’s Field Equations: Their Role in General Relativity and Astrophysics

(2000)

....
On “Non-Kerr” Stationary Vacuum Solutions Used In Astrophysics
....

Vaidya ``Radiating'' Solution

....

The Boost-Rotation Symmetric Radiative Spacetimes
Uniformly Accelerated Sources And Their Fields--General Features
C-Metric
Rotating Charged Black Hole Falling In An Electric Field
The Newtonian Limit
Uniformly Accelerated Sources--A Miscellany
The Material Sources
Static Perfect Fluid Spheres
Static Cylinders
Static Cylindrically Symmetric Spacetimes With $\Lambda $
Rotating Cylinders
Cosmic Strings
Charged Matter Sources
Gödel's Universe
The Cosmological Models
Friedmann-Lemaitre-Robertson-Walker universes
Standard Flrw Models
Inflating Flrw Models
Spatially Homogeneous Cosmologies
Kantowski-Sachs Spacetimes
Bianchi Models
Inhomogeneous Cosmologies
Gowdy Solutions
Varia
Lemaitre-Tolman-Bondi models and other inhomogeneous spherical universes

Newly prepared version

In preparation: 2000-2024

380 → 800 references
330 → 720 kBytes
108 → 196 pages (approx. & without references)

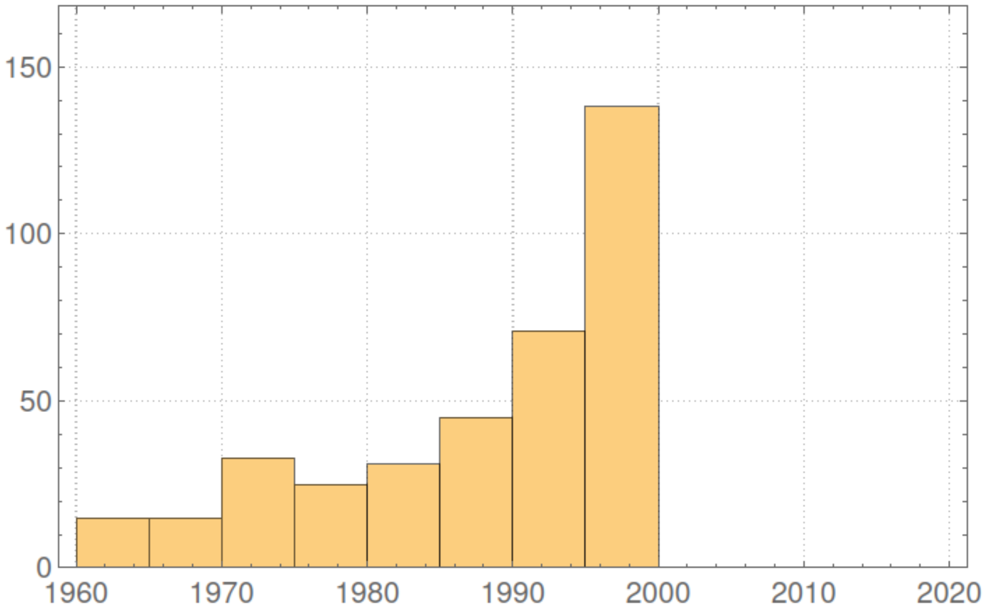
~ 90 new pages, 400 new references

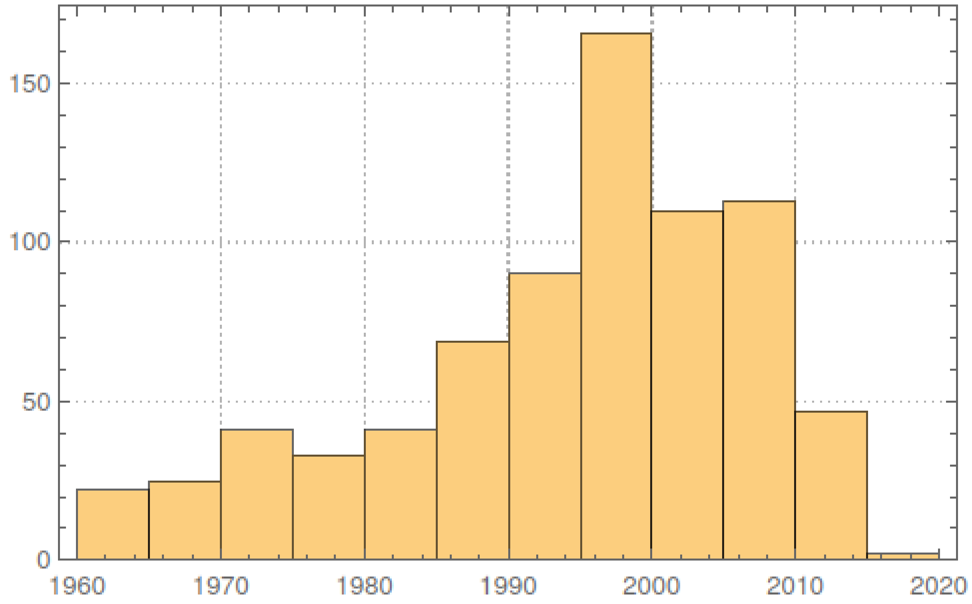
cf: Griffiths & Podolský ~ 1030 refs.
Stephani et al: ~ 2000 refs

cf.: Class Quantum Grav 40 years → ~10 000 papers

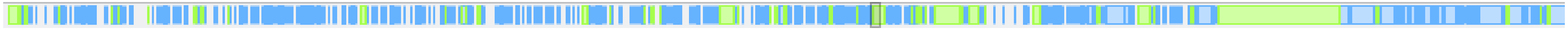
Newly prepared version

References





Text



New form of the C metric with cosmological constant

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The new form of the C metric proposed by Hong and Teo, in which the two structure functions are factorized, has proved useful in its analysis. In this paper, we extend this form to the case when a cosmological constant is present. The new form of this solution has two structure functions which are partially factorized; moreover, the roots of the structure functions are now regarded as fundamental parameters. This leads to a natural representation of the solution in terms of its so-called domain structure, in which the allowed coordinate range can be visualized as a "box" in a two-dimensional plot. The solution is then completely parametrized by the locations of the edges of this box, at least in the uncharged case. We also briefly analyze other possible domain structures—in the shape of a triangle and trapezoid—that might describe physically interesting space-times within the anti–de Sitter C metric.

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I. INTRODUCTION

The C metric is a static solution to the vacuum Einstein field equations, whose history dates back to 1918 when it was discovered by Levi-Cività [1]. It was subsequently rediscovered by various other authors in the early 1960s [2–4]; in particular, it was Ehlers and Kundt [4] who, in the process of classifying degenerate static vacuum solutions, gave it the "C" designation that it is known by today. However, its interpretation remained obscure until 1970, when Kinnersley and Walker [5] showed that the C metric actually describes a Schwarzschild black hole undergoing uniform acceleration. It was also these two authors who introduced the well-known form of the C metric that would remain the *de facto* standard form for the next three decades or so.

To see how Kinnersley and Walker obtained their form of the C metric, we need to start with the slightly more general form used by Ehlers and Kundt [4]:

$$ds^2 = \frac{1}{(x-y)^2}\left[F(y)dt^2 - \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2\right], \tag{1}$$

where the structure functions $G(x)$ and $F(y)$ are cubic polynomials in x and y, respectively, satisfying the condition

$$F(x) = G(x). \tag{2}$$

Thus, the two polynomials share the same coefficients. It would appear that this solution has four parameters, which can be taken to be the coefficients of $G(x)$, say. However, two of them are actually unphysical, and can be gauged away by a suitable coordinate transformation. Kinnersley and Walker considered the following affine coordinate transformation:

$$x' = Ac_0x + c_1, \qquad y' = Ac_0y + c_1,$$
$$t' = c_0t, \qquad \phi' = c_0\phi, \tag{3}$$

under which the metric (1) gains an overall factor but otherwise retains the same general form:

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[F(y)dt^2 - \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2\right]. \tag{4}$$

Note that the structure functions $G(x)$ and $F(y)$ are still cubic polynomials satisfying (2), although with new coefficients depending on A, c_0, and c_1. Kinnersley and Walker then used the coordinate freedom in (3) to set $G(x)$ to be

$$G(x) = 1 - x^2 - 2mAx^3. \tag{5}$$

In particular, the linear coefficient has been set to zero. The parameters m and A are related to the mass and acceleration of the black hole, respectively. In the limit $A \to 0$, the usual Schwarzschild metric with mass parameter m can be recovered from this form of the C metric. On the other hand, in the limit $m \to 0$, the usual Rindler space metric with acceleration parameter A can be recovered.

A major disadvantage of the Kinnersley-Walker form of the C metric is that the roots of the structure function (5) are cumbersome to write down in terms of the parameters m and A. Nevertheless, knowledge of these roots is important, since they encode the locations of the axes and horizons in the space-time. Almost any study of the geometrical properties of the space-time will involve these roots and would be very complicated as a result. Even if the roots were not explicitly expressed in terms of m and A, one would need to have a handle on their dependence on these parameters.

In 2003, Hong and Teo [6] proposed a new form of the C metric that would alleviate this difficulty. Instead of using the coordinate freedom in (3) to set the linear coefficient of $G(x)$ to zero, they used this freedom to set it to the value $2mA$. As a result, $G(x)$ can be put in the factorized form:

$$G(x) = (1 - x^2)(1 + 2mAx). \tag{6}$$

In this form, the roots of the structure functions are obvious to read off: the two axes of the space-time are located at $x = \pm 1$, while the acceleration and black-hole horizons are located at $y = -1, -\frac{1}{2mA}$, respectively. These simple expressions lead to potentially drastic simplifications when analyzing the properties of the C metric, as demonstrated in Ref. [6].

The new form (6) is related to the previous one (5) by a coordinate transformation and redefinition of parameters. In particular, m and A still retain their interpretations as the mass and acceleration parameters of the black hole, respectively. Again, the Schwarzschild metric can be recovered in the limit $A \to 0$, while the Rindler space metric can be recovered in the limit $m \to 0$. However, we emphasize that in the general case $m, A \neq 0$, the parameters appearing in (6) are inequivalent to those appearing in (5).

The C metric can be straightforwardly extended to include charge, by adding a quartic term to the structure functions. In the Kinnersley-Walker form, the metric is still given by (4), but the structure function (5) is generalized to

$$G(x) = 1 - x^2 - 2mAx^3 - q^2A^2x^4, \tag{7}$$

where q is the charge parameter of the black hole. Being a quartic polynomial, the roots of $G(x)$ are now even more cumbersome to write down than in the vacuum case. Fortunately, the factorized form (6) can be extended to the charged case. It was shown in Ref. [6] that, by a coordinate transformation and redefinition of parameters, (7) can be written as

$$G(x) = (1 - x^2)(1 + r_+Ax)(1 + r_-Ax), \tag{8}$$

where $r_\pm = m \pm \sqrt{m^2 - q^2}$ are the locations of the horizons in the usual form of the Reissner-Nordström metric. In this form, the roots of $G(x)$ are trivial to read off: the two axes of the space-time are again located at $x = \pm 1$, while the acceleration and two black-hole horizons are located at $y = -1, -\frac{1}{r_\pm A}$, respectively.

The (charged) C metric can also be extended to include rotation. In this case, the metric (4) has to be replaced by a more complicated stationary form—not reproduced here—which nevertheless still depends on two structure functions $G(x)$ and $F(y)$ satisfying (2). In the Kinnersley-Walker form, $G(x)$ is given by

$$G(x) = 1 - x^2 - 2mAx^3 - (a^2 + q^2)A^2x^4, \tag{9}$$

where a is the rotation parameter of the black hole. In Ref. [7], Hong and Teo showed that $G(x)$ can again be written in the factorized form (8), but with $r_\pm = m \pm \sqrt{m^2 - a^2 - q^2}$. The latter are just the locations of the horizons in the Boyer-Lindquist form of the Kerr-Newman metric. However, as Hong and Teo pointed out, one key difference in this case is that this new form of the rotating C metric is *not* related to the traditional form (9) by a coordinate transformation. It turns out that the traditional form of the rotating C metric possesses so-called Dirac-Misner singularities along the axes, while the new form does not. To avoid such singularities, the structure functions necessarily take the factorized form (8).

A natural question at this stage is whether this new form of the (static, charged) C metric can be extended to include a cosmological constant Λ. The C metric with cosmological constant is traditionally written in the form (4), with the structure functions

$$G(x) = 1 - x^2 - 2mAx^3 - q^2A^2x^4,$$
$$F(y) = \left(1 - \frac{1}{\ell^2A^2}\right) - y^2 - 2mAy^3 - q^2A^2y^4, \tag{10}$$

where $\ell^2 \equiv -3/\Lambda$. Note that $G(x)$ has exactly the same form as in (7), but that $F(x)$ now differs from $G(x)$ by a constant term:

$$F(x) = G(x) - \frac{1}{\ell^2A^2}. \tag{11}$$

This implies that there is no simple relation between the roots of $G(x)$ and those of $F(y)$. In particular, a factorized form for $G(x)$ does not lead to one for $F(y)$, or vice versa. In Ref. [7], a tentative proposal was made to write $G(x)$ in the factorized form (8), at the expense of leaving $F(y)$ unfactorized. However, an unsatisfactory consequence is that the $r_\pm$ appearing in $G(x)$ have no relation to the locations of the horizons of the Kerr-Newman-de Sitter/anti–de Sitter black hole. This is perhaps not unexpected, since the locations of the horizons are encoded by the roots of $F(y)$, which as mentioned are now not the same as those of $G(x)$.

In this paper, we would like to find a new form of the C metric with cosmological constant that retains the nice features of the factorized form of Ref. [6]. To this end, recall that two of the roots of $G(x)$ are physically significant, in that they represent the two axes in the space-time. The coordinate range for x lies between these two roots. On the other hand, two of the roots of $F(y)$ are physically significant, in that they represent the acceleration and (outer) black-hole horizons. The coordinate range for y lies between these two roots. It is therefore natural to take these two roots of $G(x)$ and two roots of $F(y)$ as

Examples

Introduction

“The physicist is always interested in the special case; he is never interested in the general case. He is talking about something; he is not talking abstractly about anything. He wants to discuss the gravity law in three dimensions; he never wants the arbitrary force case in n dimensions. So a certain amount of reducing is necessary, because the mathematicians have prepared these things for a wide range of problems. This is very useful, and later on it always turns out that the poor physicist has to come back and say, ‘Excuse me, when you wanted to tell me about four dimensions. . . ’ ” Of course, this is Feynman, and from 1965. . .

However, physicists are still rightly impressed by special explicit formulae. Explicit solutions enable us to discriminate more easily between a ``physical'' and ``pathological'' feature.

Where are there singularities?

What is their character?

How do test particles and fields behave in given background spacetimes?

What are their global structures?

Is a solution stable and, in some sense, generic?

Clearly, such questions have been asked not only within general relativity.

By studying a *special* explicit solution one acquires an intuition which, in turn, stimulates further questions relevant to more *general* situations. Consider, for example, charged black holes as described by the Reissner-Nordström solution. We have learned that in their interior a Cauchy horizon exists and that the singularities are timelike. The singularities can be seen by, and thus exert an influence on, an observer travelling in their neighborhood. However, will this violation of the (strong) cosmic censorship persist when the black hole is perturbed by weak ("linear") or even strong ("nonlinear") perturbations? We shall see that, remarkably, this question can also be studied by explicit exact special model solutions. Still more surprisingly, perhaps, a similar question can be addressed and analyzed by means of explicit solutions describing completely diverse situations — the collisions of plane waves. Such collisions may develop Cauchy horizons and subsequent timelike singularities. The theory of black holes and the theory of colliding waves have intriguing structural similarities which, first of all, stem from the circumstance that in both cases there exist two symmetries, i.e. two Killing fields. What, however, about more general situations? This is a natural question inspired by the explicit solutions. Then "the poor physicists have to come back" to a mathematician, or today alternatively, to a numerical relativist, and hope that somehow they will firmly learn whether the cosmic censorship is the "truth", or that it has been a very inspirational, but in general false conjecture. However, even *after* the formulation of a conjecture about a general situation inspired by particular exact solutions, *newly* discovered exact solutions can play an important role in verifying, clarifying, modifying, or ruling out the conjecture. And also "old" solutions may turn out to act as asymptotic states of general classes of models, and so become still more significant.

16 pages of Introduction:

1.1 A Word on the Role of Explicit Solutions in Other Parts of Physics and Astrophysics
1.2 Einstein's Field Equations
1.3 "Just So" Notes on the Simplest Solutions: The **Minkowski, de Sitter, and Anti-de Sitter Spacetimes**
1.4 On the Interpretation and Characterization of Metrics
1.5 The Choice of Solutions

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Examples

Reissner-Nordstrröm

For the Ricci tensor type [(11,2)] the group G_3 on V_2 does not imply the existence of a G_4: the Vaidya metric (Table 15.1)

$$\mathrm{d}s^2 = r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) - 2\mathrm{d}u\mathrm{d}r - (1 - 2m(u)/r)\mathrm{d}u^2, \qquad (15.20)$$

$m(u)$ being an arbitrary function of the null coordinate u, has G_3 on V_2 as the maximal group of motions (unless $m = \mathrm{const}$).

15.4.4 Spherically- and plane-symmetric fields

The spherically-symmetric Einstein–Maxwell field with $\Lambda = 0$ is the Reissner–Nordström solution

$$\mathrm{d}s^2 = r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) + (1 - 2m/r + e^2/r^2)^{-1}\mathrm{d}r^2 - (1 - 2m/r + e^2/r^2)\mathrm{d}t^2, \qquad (15.21)$$

which describes the exterior field of a spherically-symmetric charged body (its form in isotropic coordinates can be found e.g. in Prasanna (1968)). For $e = 0$, we obtain the Schwarzschild solution (15.19). We give it here in various other coordinate systems which are frequently used:

Isotropic coordinates:

$$\mathrm{d}s^2 = [1 + m/2\bar{r}]^4[\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2] - [1 - m/2\bar{r}]^2\mathrm{d}t^2/[1 + m/2\bar{r}]^2,$$
$$r = \bar{r}[1 + m/2\bar{r}]^2 \qquad (15.22)$$

(for isotropic coordinates covering also $r < 2m$, see Buchdahl 1985).

Eddington–Finkelstein coordinates (Eddington 1924, Finkelstein 1958):

$$\mathrm{d}s^2 = r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) - 2\mathrm{d}u\,\mathrm{d}r - (1 - 2m/r)\mathrm{d}u^2,$$
$$u = t - \int(1 - 2m/r)^{-1}\mathrm{d}r = t + 2m\ln(r - 2m). \qquad (15.23)$$

Kruskal–Szekeres coordinates (Kruskal 1960, Szekeres 1960):

$$\mathrm{d}s^2 = r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) - 32m^3r^{-1}\mathrm{e}^{-r/2m}\mathrm{d}u\,\mathrm{d}v,$$
$$u = -(r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\mathrm{e}^{-t/4m}, \quad v = (r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\mathrm{e}^{t/4m}. \qquad (15.24)$$

Lemaitre–Novikov coordinates:

$$\mathrm{d}s^2 = Y^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) + [1 - \varepsilon f^2(r)]^{-1}(Y'\mathrm{d}r)^2 - \mathrm{d}\tau^2,$$
$$\dot{Y}^2 - 2m/Y = -\varepsilon f^2(r) \qquad (15.25)$$

($\varepsilon = 0$: Lemaître (1933); $\varepsilon = 1$, $f^2 = (1 + r^2)^{-1}$: Novikov (1963)).

3 The Reissner-Nordström Solution

This spherically symmetric solution of the Einstein-Maxwell equations was derived independently[10] by H. Reissner in 1916, H. Weyl in 1917, and G. Nordström in 1918. It represents a spacetime with no matter sources except for a radial electric field, the energy of which has to be included on the right-hand side of the Einstein equations. Since Birkhoff's theorem, mentioned in connection with the Schwarzschild solution in Section 2.2, can be generalized to the electrovacuum case, the Reissner-Nordström solution is the unique spherical electrovacuum solution. Similarly to the Schwarzschild solution, it thus describes the exterior gravitational and electromagnetic fields of an arbitrary – static, oscillating, collapsing or expanding – spherically symmetric, charged body of mass M and charge Q. The metric reads

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (9)$$

the electromagnetic field in these spherical coordinates is described by the "classical" expressions for the time component of the electromagnetic potential and the (only non-zero) component of the electromagnetic field tensor:

$$A_t = -\frac{Q}{r}, \quad F_{tr} = -F_{rt} = -\frac{Q}{r^2}. \qquad (10)$$

A number of authors have discussed spherically symmetric, static charged dust configurations producing a Reissner-Nordström metric outside, some of them with a hope to construct a "classical model" of a charged elementary particle (see Stephani et al. (2003) for references). The main influence the metric has exerted on the developments of general relativity, and more recently in supersymmetric and superstring theories (see Section 3.2), is however in its analytically extended electrovacuum form when it represents charged, spherical black holes.

3.1 Reissner-Nordström black holes and the question of cosmic censorship

The analytic extensions have qualitatively different character in three cases, depending on the relationship between the mass M and the charge Q. In the

[10] In the literature one finds the solution to be repeatedly connected only with the names of Reissner and Nordström, except for the "exact-solutions-book" Stephani et al. (2003): there in four places the solution is called as everywhere else, but in one place (p. 257) it is referred to as the "Reissner-Weyl-solutions". An enlightening discussion on p. 209 in Stephani et al. (2003) shows that the solution belongs to a more general "Weyl's electrovacuum class" of electrostatic solutions discovered by Weyl (in 1917) which follow from an Ansatz that there is a functional relationship between the gravitational and electrostatic potentials. As will be noticed also in the case of cylindrical waves in Section 10, if "too many" solutions are given in one paper, the name of the author is not likely to survive in the name of an important subclass...

Examples

A page with 27 references:

transitive, so Killing orbits admit orthogonal surfaces (Berger et al., 1995; Carot et al., 1999; Mena Marugán, 2000). The orthogonal transitivity thus excludes the possibility of a global rotation. In other words *in vacuo* there can be in fact no smooth 'rotating cylindrical waves'. With a material source present as in the case of the rigidly rotating dust cylinder (Bonnor, 1980b), for example, the spacetime can, of course, be regular everywhere with a non-vanishing angular momentum per unit length. Bondi (1994) studied general changes in time of such systems which can lead to radiation. As he noticed, the conservation of angular momentum occurs even if gravitational waves are emitted by the cylinder since the cylindrical symmetry of the waves precludes their carrying angular momentum.

The metric containing a second degree of freedom was discovered by Jürgen Ehlers (working in the group of Pascual Jordan), who used a trick similar to Beck's on the generalized (stationary) Weyl metrics, and independently by Kompaneets (see the discussion in Stachel (1966)). In the literature (e.g. Piran et al. (1986); d'Inverno (1997)) one refers to the Jordan-Ehlers-Kompaneets form of the metric:

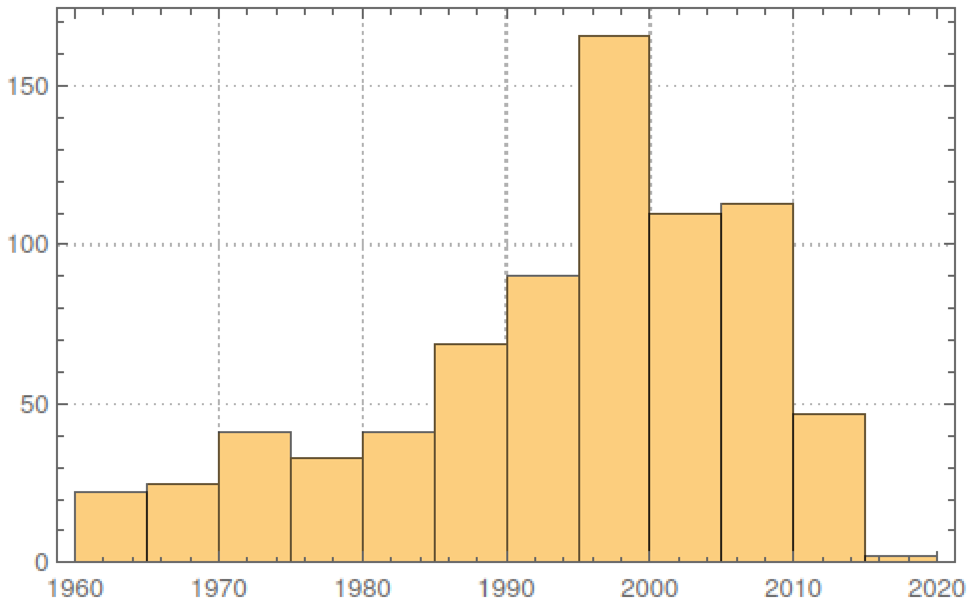
$$ds^2 = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + e^{2\psi}\left(dz + \omega\, d\varphi\right)^2 + \rho^2 e^{-2\psi} d\varphi^2. \qquad (84)$$

Here, the additional function $\omega(t,\rho)$ represents the second polarization.

Despite the fact that cylindrically symmetric waves cannot describe exactly the radiation from bounded sources, both the Einstein-Rosen waves and their generalization (84) have played an important role in clarifying a number of complicated issues, such as the energy loss due to gravitational waves (Thorne, 1965). An extensive literature exists on cylindrical waves interacting nonlinearly with cosmic strings (see Garriga and Verdaguer (1987); Xanthopoulos (1987, 1986, 1987); Economou and Tsoubelis (1988); Dagotto et al. (1990); Manojlovic and Mena Marugán (2001), and the monograph Anderson (2002)). Other applications of radiative cylindrical metrics involve the asymptotic structure of radiative spacetimes (Stachel, 1966), the dispersion of waves (Chandrasekhar and Ferrari, 1987), testing the quasilocal mass-energy (Tod, 1990), testing codes in numerical relativity (d'Inverno, 1997), investigation of the cosmic censorship (Berger et al., 1995), and quantum gravity in a simplified but field theoretically interesting context of midisuperspaces (Kuchař, 1971; Ashtekar and Pierri, 1996; Korotkin and Samtleben, 1998).

As mentioned above, the vacuum metrics (82) considered by Mashhoon et al. (2000) as "rotating gravitational waves" cannot have a regular axis since this requires the Killing orbits to admit orthogonal surfaces. However, the axis can represent a rotating cosmic string. One can investigate dragging of inertial frames by combined effects of waves interacing with a rotating cosmic string (Bičák et al., 2008). Assuming the 'rotation parameter' ω in (82) is small so that the terms in $\mathcal{O}(\omega^2)$ can be neglected, the inspection of vacuum field equations following from the Anzatz (82) then reveals that one can choose $W = \rho$ and field equations for ψ and γ are the same as for Einstein-Rosen waves. The rotational perturbation ω is determined by evolution equation $\left(\rho^3 e^{-2\gamma}\omega'\right)^{\cdot} = 0$

Review’s horizon



New form of the *C* metric with cosmological constant

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The new form of the *C* metric proposed by Hong and Teo, in which the two structure functions are factorized, has proved useful in its analysis. In this paper, we extend this form to the case when a cosmological constant is present. The new form of this solution has two structure functions which are partially factorized; moreover, the roots of the structure functions are now regarded as fundamental parameters. This leads to a natural representation of the solution in terms of its so-called domain structure, in which the allowed coordinate range can be visualized as a "box" in a two-dimensional plot. The solution is then completely parametrized by the locations of the edges of this box, at least in the uncharged case. We also briefly analyze other possible domain structures—in the shape of a triangle and trapezoid—that might describe physically interesting space-times within the anti–de Sitter *C* metric.

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I. INTRODUCTION

The *C* metric is a static solution to the vacuum Einstein field equations, whose history dates back to 1918 when it was discovered by Levi-Cività [1]. It was subsequently rediscovered by various other authors in the early 1960s [2–4]; in particular, it was Ehlers and Kundt [4] who, in the process of classifying degenerate static vacuum solutions, gave it the "*C*" designation that it is known by today. However, its interpretation remained obscure until 1970, when Kinnersley and Walker [5] showed that the *C* metric actually describes a Schwarzschild black hole undergoing uniform acceleration. It was also these two authors who introduced the well-known form of the *C* metric that would remain the *de facto* standard form for the next three decades or so.

To see how Kinnersley and Walker obtained their form of the *C* metric, we need to start with the slightly more general form used by Ehlers and Kundt [4]:

$$ds^2 = \frac{1}{(x-y)^2}\left[F(y)dt^2 - \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2\right], \tag{1}$$

where the structure functions $G(x)$ and $F(y)$ are cubic polynomials in x and y, respectively, satisfying the condition

$$F(x) = G(x). \tag{2}$$

Thus, the two polynomials share the same coefficients. It would appear that this solution has four parameters, which can be taken to be the coefficients of $G(x)$, say. However, two of them are actually unphysical, and can be gauged away by a suitable coordinate transformation. Kinnersley and Walker considered the following affine coordinate transformation:

$$x' = Ac_0x + c_1, \qquad y' = Ac_0y + c_1,$$
$$t' = c_0t, \qquad \phi' = c_0\phi, \tag{3}$$

under which the metric (1) gains an overall factor but otherwise retains the same general form:

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[F(y)dt^2 - \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)d\phi^2\right]. \tag{4}$$

Note that the structure functions $G(x)$ and $F(y)$ are still cubic polynomials satisfying (2), although with new coefficients depending on A, c_0, and c_1. Kinnersley and Walker then used the coordinate freedom in (3) to set $G(x)$ to be

$$G(x) = 1 - x^2 - 2mAx^3. \tag{5}$$

In particular, the linear coefficient has been set to zero. The parameters m and A are related to the mass and acceleration of the black hole, respectively. In the limit $A \to 0$, the usual Schwarzschild metric with mass parameter m can be recovered from this form of the *C* metric. On the other hand, in the limit $m \to 0$, the usual Rindler space metric with acceleration parameter A can be recovered.

A major disadvantage of the Kinnersley-Walker form of the *C* metric is that the roots of the structure function (5) are cumbersome to write down in terms of the parameters m and A. Nevertheless, knowledge of these roots is important, since they encode the locations of the axes and horizons in the space-time. Almost any study of the geometrical properties of the space-time will involve these roots and would be very complicated as a result. Even if the roots were not explicitly expressed in terms of m and A, one would need to have a handle on their dependence on these parameters.

In 2003, Hong and Teo [6] proposed a new form of the *C* metric that would alleviate this difficulty. Instead of using the coordinate freedom in (3) to set the linear coefficient of $G(x)$ to zero, they used this freedom to set it to the value $2mA$. As a result, $G(x)$ can be put in the factorized form:

$$G(x) = (1-x^2)(1+2mAx). \tag{6}$$

In this form, the roots of the structure functions are obvious to read off: the two axes of the space-time are located at $x = \pm 1$, while the acceleration and black-hole horizons are located at $y = -1, -\frac{1}{2mA}$, respectively. These simple expressions lead to potentially drastic simplifications when analyzing the properties of the *C* metric, as demonstrated in Ref. [6].

The new form (6) is related to the previous one (5) by a coordinate transformation and redefinition of parameters. In particular, m and A still retain their interpretations as the mass and acceleration parameters of the black hole, respectively. Again, the Schwarzschild metric can be recovered in the limit $A \to 0$, while the Rindler space metric can be recovered in the limit $m \to 0$. However, we emphasize that in the general case $m, A \neq 0$, the parameters appearing in (6) are inequivalent to those appearing in (5).

The *C* metric can be straightforwardly extended to include charge, by adding a quartic term to the structure functions. In the Kinnersley-Walker form, the metric is still given by (4), but the structure function (5) is generalized to

$$G(x) = 1 - x^2 - 2mAx^3 - q^2A^2x^4, \tag{7}$$

where q is the charge parameter of the black hole. Being a quartic polynomial, the roots of $G(x)$ are now even more cumbersome to write down than in the vacuum case. Fortunately, the factorized form (6) can be extended to the charged case. It was shown in Ref. [6] that, by a coordinate transformation and redefinition of parameters, (7) can be written as

$$G(x) = (1-x^2)(1+r_+Ax)(1+r_-Ax), \tag{8}$$

where $r_\pm = m \pm \sqrt{m^2 - q^2}$ are the locations of the horizons in the usual form of the Reissner-Nordström metric. In this form, the roots of $G(x)$ are trivial to read off: the two axes of the space-time are again located at $x = \pm 1$, while the acceleration and two black-hole horizons are located at $y = -1, -\frac{1}{r_\pm A}$, respectively.

The (charged) *C* metric can also be extended to include rotation. In this case, the metric (4) has to be replaced by a more complicated stationary form—not reproduced here—which nevertheless still depends on two structure functions $G(x)$ and $F(y)$ satisfying (2). In the Kinnersley-Walker form, $G(x)$ is given by

$$G(x) = 1 - x^2 - 2mAx^3 - (a^2+q^2)A^2x^4, \tag{9}$$

where a is the rotation parameter of the black hole. In Ref. [7], Hong and Teo showed that $G(x)$ can again be written in the factorized form (8), but with $r_\pm = m \pm \sqrt{m^2 - a^2 - q^2}$. The latter are just the locations of the horizons in the Boyer-Lindquist form of the Kerr-Newman metric. However, as Hong and Teo pointed out, one key difference in this case is that this new form of the rotating *C* metric is *not* related to the traditional form (9) by a coordinate transformation. It turns out that the traditional form of the rotating *C* metric possesses so-called Dirac-Misner singularities along the axes, while the new form does not. To avoid such singularities, the structure functions necessarily take the factorized form (8).

A natural question at this stage is whether this new form of the (static, charged) *C* metric can be extended to include a cosmological constant Λ. The *C* metric with cosmological constant is traditionally written in the form (4), with the structure functions

$$G(x) = 1 - x^2 - 2mAx^3 - q^2A^2x^4,$$
$$F(y) = \left(1 - \frac{1}{\ell^2A^2}\right) - y^2 - 2mAy^3 - q^2A^2y^4, \tag{10}$$

where $\ell^2 \equiv -3/\Lambda$. Note that $G(x)$ has exactly the same form as in (7), but that $F(x)$ now differs from $G(x)$ by a constant term:

$$F(x) = G(x) - \frac{1}{\ell^2A^2}. \tag{11}$$

This implies that there is no simple relation between the roots of $G(x)$ and those of $F(y)$. In particular, a factorized form for $G(x)$ does not lead to one for $F(y)$, or vice versa. In Ref. [7], a tentative proposal was made to write $G(x)$ in the factorized form (8), at the expense of leaving $F(y)$ unfactorized. However, an unsatisfactory consequence is that the $r_\pm$ appearing in $G(x)$ have no relation to the locations of the horizons of the Kerr-Newman-de Sitter/anti–de Sitter black hole. This is perhaps not unexpected, since the locations of the horizons are encoded by the roots of $F(y)$, which as mentioned are now not the same as those of $G(x)$.

In this paper, we would like to find a new form of the *C* metric with cosmological constant that retains the nice features of the factorized form of Ref. [6]. To this end, recall that two of the roots of $G(x)$ are physically significant, in that they represent the two axes in the space-time. The coordinate range for x lies between these two roots. On the other hand, two of the roots of $F(y)$ are physically significant, in that they represent the acceleration and (outer) black-hole horizons. The coordinate range for y lies between these two roots. It is therefore natural to take these two roots of $G(x)$ and two roots of $F(y)$ as

Conclusions

Classical and Quantum Gravity Exact Solutions Policy:

In the field of classical relativity, the discovery of a new exact solution does not justify publication simply for its own sake.

Justification for publishing a new solution would be provided by showing for example that

- it has an interesting physical application or
- unusual geometrical properties, or
- that it illustrates an important mathematical point.

The onus is on the author to provide convincing evidence that the solution is in fact new.

Conclusion: Jiří Bičák’s Review is great resource for such considerations.

Supplementary conclusion:

When asked to give a seminar, do not reply:

> “I do no science these days, I only do some editing.”

It won’t help.

Thank you !