Post-Newtonian expansions of extreme mass ratio inspirals of spinning bodies into Schwarzschild black holes

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VS and Vojtěch Witzany, arXiv:2406.14291

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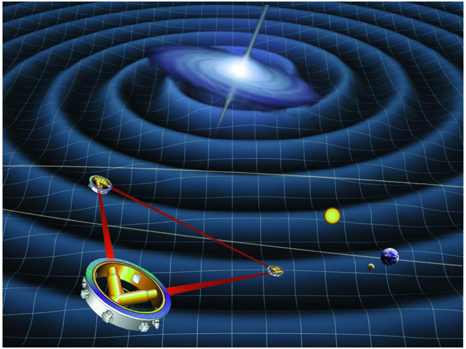
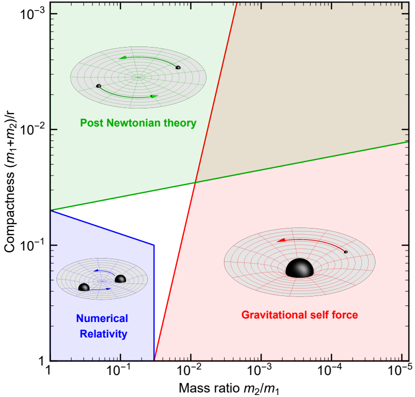
Motivation

- Spin of the secondary body needed for the postadiabatic (PA) term
- Lower accuracy of PA term
- Post-Newtonian (PN) approximation of the linear-in-spin part of the fluxes
- Eccentric orbits in Schwarzschild spacetime



Modelling of binary systems

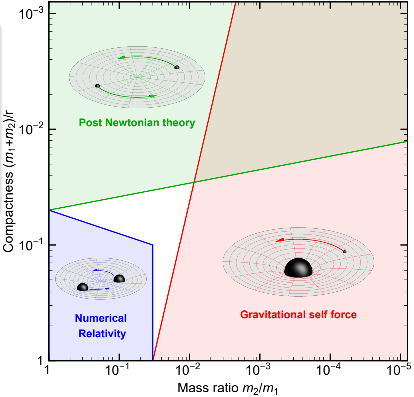


[LISA whitepaper, arXiv:2311.01300]

Modelling of binary systems

Gravitational self-force

- Expansion in the mass ratio
- Valid in strong and weak field
- Particles moving on perturbed geodesics in the background spacetime



[LISA whitepaper, arXiv:2311.01300]

Modelling of binary systems

Gravitational self-force

- Expansion in the mass ratio
- Valid in strong and weak field
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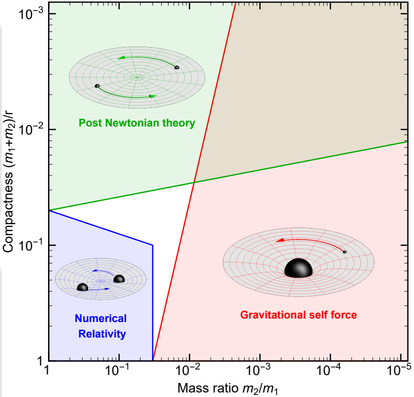
Post-Newtonian (PN) theory

- Expansion in the velocity v^2/c^2
- Valid for comparable masses and high mass ratio
- Quasi-Keplerian parametrization of the motion



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Gravitational self-force

- Black hole perturbation theory: spacetime expanded in the mass ratio as

$$g_{\mu\nu}^{\text{exact}} = g_{\mu\nu} + q h_{\mu\nu}^{(1)} + q^2 h_{\mu\nu}^{(2)} + \mathcal{O}(q^3)$$

- $h_{\mu\nu}^{(n)}$ calculated from expanded Einstein equations

$$\Box \bar{h}_{\mu\nu}^{(1)} + 2 R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} \bar{h}_{\alpha\beta}^{(1)} = -16\pi T_{\mu\nu}^{(1)}$$

- Self-force and self-torque:

$$\frac{D^2 z^\mu}{\mathrm{d}\tau^2} = f_{\text{s-f}}^{\mu}[h, u, S] + f_{\text{s-c}}^{\mu}[u, S] , \qquad \frac{D S^{\mu\nu}}{\mathrm{d}\tau} = \tau_{\text{s-t}}^{\mu\nu}[h, u, S]$$

Gravitational self-force

- Two-timescale expansion

$$\Phi_{\mathrm{GW}} = \underbrace{q^{-1}\Phi_0(qt)}_{\langle\text{diss. 1SF}\rangle} + \underbrace{q^{-\frac{1}{2}}\Phi_{1/2}(qt)}_{\text{resonances}} + \underbrace{\Phi_1(qt)}_{\text{cons. 1SF, diss. 2SF, diss. spin}} + \mathcal{O}(q)$$

- Flux balance laws: dissipative part calculated from the asymptotic GW fluxes (Teukolsky equation)
- Effects of the secondary spin in the postadiabatic term
- Lower accuracy needed for the postadiabatic term: PN expansion [Burke et al., arXiv:2310.08927]

Spinning particle in Schwarzschild spacetime

- Stress-energy tensor $T^{\mu\nu} = \int \mathrm{d}\tau \left(P^{(\mu} u^{\nu)} \frac{\delta^4}{\sqrt{-g}} - \nabla_\alpha \left(S^{\alpha(\mu} u^{\nu)} \frac{\delta^4}{\sqrt{-g}} \right) \right)$
- Linearized Mathisson-Papapetrou-Dixon equations

$$\mu \frac{D^2 z^\mu}{\mathrm{d}\tau^2} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau} S^{\rho\sigma} \qquad \frac{DS^{\mu\nu}}{\mathrm{d}\tau} = 0,$$

- Tulczyjew-Dixon SSC $S^{\mu\nu} P_\nu = 0$
- Constants of motion:

- $\mu = \sqrt{-P^\mu P_\mu}$, $P^\mu = \mu u^\mu + \mathcal{O}(s^2)$
- $S = \sqrt{S^{\mu\nu} S_{\mu\nu}/2} = s\mu$, $s/M \lesssim q \ll 1$
- $s_\parallel = Y_{\mu\nu} u^\mu S^\nu / \sqrt{K_{\mu\nu} u^\mu u^\nu}$
- $E = -\xi^\mu_{(t)} P_\mu + \xi^{(t)}_{\mu;\nu} S^{\mu\nu}/2$
- $J_i = \xi^\mu_{(i)} P_\mu - \xi^{(i)}_{\mu;\nu} S^{\mu\nu}/2$
- $J^2 = J_x^2 + J_y^2 + J_z^2$

Bound orbits

- Integrable system, analytic solution exists (Witzany and Piovano, PRL **132**, 171401 (2024))
- Rotation of the coordinate system such that $J = J_z \Rightarrow$ nearly equatorial orbits
- Bound orbits: parametrization with semi-latus rectum p and eccentricity e

$$r_1 = \frac{p}{1-e} \qquad r_2 = \frac{p}{1+e}$$

- Constants of motion: $E(p, e, s_{\parallel})$, $J_z(p, e, s_{\parallel})$
- Orbital frequencies wrt. Mino time $\lambda = \int r^{-2} \mathrm{d}\tau$: $\Upsilon^t(p, e, s_{\parallel})$, $\Upsilon^r(p, e, s_{\parallel})$, $\Upsilon^\phi(p, e, s_{\parallel})$, $\Upsilon^\psi(p, e)$

Bound orbits

- Motion parametrized with the radial phase $q^r = \Upsilon^r \lambda$ and precession phase $q^\psi = \Upsilon^\psi \lambda$

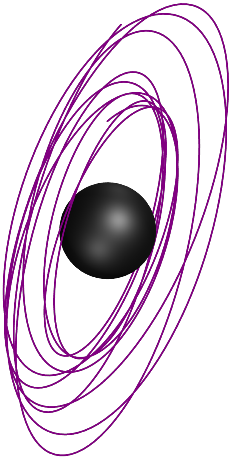
$$t(p, e, s_\parallel, q^r) = \frac{\Upsilon^t}{\Upsilon^r} q^r + \Delta t(q^r)$$

$$r(p, e, s_\parallel, q^r) = \frac{r_3(r_1 - r_2)\mathrm{sn}^2(\frac{K(k)}{\pi} q^r, k) - r_2(r_1 - r_3)}{(r_1 - r_2)\mathrm{sn}^2(\frac{K(k)}{\pi} q^r, k) - (r_1 - r_3)}$$

$$\theta(p, e, s_\perp, q^r, q^\psi) = \pi/2 + s_\perp \delta\theta(p, e, q^r, q^\psi)$$

$$\phi(p, e, s_\parallel, q^r) = \frac{\Upsilon^\phi}{\Upsilon^r} q^r$$

- Linearization in the secondary spin: $f(p, e, s_\parallel) = f^{(\mathrm{g})}(p, e) + s_\parallel \delta f(p, e)$



PN expansion of the trajectory

- Expansion in the "velocity" $v = \sqrt{M/p}$ and eccentricity e
- Constants of motion E, J_z (rational functions, square roots)
- Frequencies $\Upsilon_{t,r,\phi}$ (Elliptic integrals $K(k)$, $E(k)$, $\Pi(n,k)$, where $k = \mathcal{O}(ev^2)$)
- Trajectory $\Delta t(q^r)$, $r(q^r)$ (Finite Fourier series for each order in e)
- Expansion to v^9 and e^{10} beyond the leading order
- Linearization in $s_\parallel$: $f = f_{(\mathrm{g})} + s_\parallel \delta f$

$$\delta\Omega_r = \frac{3}{2}\left(1-e^2\right)^{5/2}\left(v^6 + \left(5e^2 + 4\sqrt{1-e^2} + 1\right)v^8 + \ldots + \mathcal{O}\left(v^{16}\right)\right)$$

$$\delta\Omega_\phi = -\frac{3}{2}\left(1-e^2\right)^{3/2}\left(\left(1+e^2\right)v^6 + \left(5e^4 + 4\left(1-e^2\right)^{3/2} + 6e^2 + 4\right)v^8 + \ldots + \mathcal{O}\left(v^{16}\right)\right)$$

Teukolsky equation

- GW fluxes calculated using the Teukolsky equation in the frequency domain

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}\psi_{lm\omega}}{\mathrm{d}r}\right)+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)\psi_{lm\omega}=\mathcal{T}_{lm\omega}$$

- Asymptotic behavior:

$$\psi_{lm\omega}(r)=\begin{cases} C^{+}_{lm\omega}r^3e^{i\omega r} & r\to\infty \\ C^{-}_{lm\omega}r(r-2M)e^{-i\omega r^*} & r\to 2M \end{cases}$$

- Discrete spectrum of frequencies $\omega_{mnj}=m\Omega_\phi+n\Omega_r+j\Omega_\psi$

$$C^{\pm}_{lmnj}=\frac{1}{2\pi\Upsilon^t}\int \mathrm{d}q^r\mathrm{d}q^\psi I^{\pm}_{lmnj}(q^r,q^\psi)e^{i\omega_{mnj}t(q^r)-im\phi(q^r)}$$

Gravitational-wave fluxes

- Waveform

$$h \equiv h_+ - ih_\times = \frac{2}{r} \sum_{lmnj} \frac{C^+_{lmnj}}{\omega^2_{mnj}} Y_{lm}(\theta) e^{-i\omega_{mnj} t + im\phi}$$

- Energy and angular momentum fluxes to infinity

$$\mathcal{F}^E = \sum_{lmnj} \frac{\left| C^+_{lmnj} \right|^2}{4\pi\omega^2_{mnj}}, \qquad \mathcal{F}^{J_z} = \sum_{lmnj} m \frac{\left| C^+_{lmnj} \right|^2}{4\pi\omega^3_{mnj}}$$

PN expansion of the fluxes

- Linearization of the amplitudes: $C^{+}_{lmn} = C^{(\mathrm{g})+}_{lmn} + s_{\parallel} \delta C^{+}_{lmn}$
- Fluxes independent of $s_{\perp}$ to linear order
- Expansion of the radial function $R^{\mathrm{in}}_{lm\omega}$ [Tagoshi and Sasaki, PTP **92**, 745–771 (1994)]
- Substitution of PN- and e-expanded trajectory
- Finite sums over l, m, n
  - $\mathcal{F}_{l,m,n} = \mathcal{F}_{l,-m,-n}$: only modes with positive frequency $-m < n \leq 5$
  - $2 \leq l \leq 5$
  - $-l \leq m \leq l$
- Linear-in-spin part of the fluxes to 3.5PN NLO (5PN)
- Horizon fluxes are of higher PN order

Results

$$\delta\mathcal{F}^E = \mathcal{F}^E_{\mathrm{N}}\left(1-e^2\right)^{3/2}\left(\delta f_3 v^3 + \delta f_5 v^5 + \delta f_6 v^6 + \delta f_7 v^7 + \delta f_8 v^8 \right.$$
$$\left. + \left(\delta f_9 + \delta f_9^{\log v}\left(\gamma - \frac{35\pi^2}{107} + \log v\right)\right) v^9 + \delta f_{10} v^{10} + \mathcal{O}\left(v^{11}\right)\right),$$

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- δf_3, δf_5, δf_7 $\delta f_9^{\log(v)}$ in closed form, other as series in eccentricity

$$\delta f_5 = \frac{1731}{112} + \frac{15399e^2}{112} + \frac{11811e^4}{224} - \frac{81743e^6}{896} - \frac{139613e^8}{14336} + 6(1-e^2)^{3/2}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

$$\delta f_i(e) = \sum_{j=0}^{5} f_{ij} e^{2j} + \mathcal{O}\left(e^{12}\right)$$

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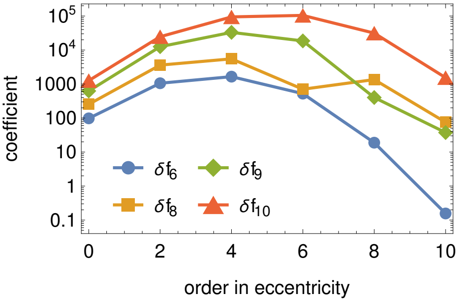

Comparison with PN theory

- Fluxes from eccentric binaries with aligned spins to 3PN and e^8 [Henry and Khalil, PRD **108**, 104016 (2023)]
- Functions of $x = (M\Omega_\phi)^{2/3}$ and e_t
- Quasi-Keplerian parametrization with harmonic coordinates $t_H = t$, $r_H = r - M$
- Relation between e_t and e (linear-in-spin part)

$$\delta(e_t^2/e^2) = \frac{2x^{3/2}}{\sqrt{1-e^2}} + \frac{6(e^2 - 2 + 2\sqrt{1-e^2})x^{5/2}}{(1-e^2)^{3/2}} + \mathcal{O}\left(x^{7/2}\right)$$

- Agreement between the energy and angular momentum fluxes from PN theory and from black hole perturbation theory expanded in v

Evolution of orbital parameter

- Parallel spin is conserved: $\langle \dot{s}_{\parallel} \rangle = 0$
- From the fluxes $\mathcal{F}^E = -\,\mathrm{d}E/\mathrm{d}t$, $\mathcal{F}^{J_z} = -\,\mathrm{d}J_z/\mathrm{d}t$ the evolution of the orbital parameters p, e

$$\begin{pmatrix} \frac{\mathrm{d}p}{\mathrm{d}t} \\ \frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = -\mathbb{J}^{-1} \begin{pmatrix} \mathcal{F}^E \\ \mathcal{F}^{J_z} \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} \frac{\partial E}{\partial p} & \frac{\partial E}{\partial e} \\ \frac{\partial J_z}{\partial p} & \frac{\partial J_z}{\partial e} \end{pmatrix}$$

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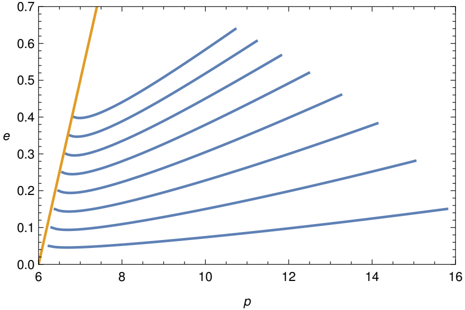
- Linearization in the secondary spin:

$$\begin{pmatrix} \frac{\mathrm{d}p}{\mathrm{d}t} \\ \frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = -\mathbb{J}^{-1}_{(\mathrm{g})} \begin{pmatrix} \mathcal{F}^E_{(\mathrm{g})} \\ \mathcal{F}^{J_z}_{(\mathrm{g})} \end{pmatrix} + s_{\parallel} \left(\mathbb{J}^{-1}_{(\mathrm{g})} \delta\mathbb{J} \mathbb{J}^{-1}_{(\mathrm{g})} \begin{pmatrix} \mathcal{F}^E_{(\mathrm{g})} \\ \mathcal{F}^{J_z}_{(\mathrm{g})} \end{pmatrix} - \mathbb{J}^{-1}_{(\mathrm{g})} \begin{pmatrix} \delta\mathcal{F}^E \\ \delta\mathcal{F}^{J_z} \end{pmatrix} \right)$$

- Hybrid model

$$\begin{pmatrix} \frac{\mathrm{d}p}{\mathrm{d}t} \\ \frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \dot{p}^{\mathrm{num}}_{(\mathrm{g})} \\ \dot{e}^{\mathrm{num}}_{(\mathrm{g})} \end{pmatrix} - s_{\parallel} \left(\mathbb{J}^{-1}_{(\mathrm{g})} \delta\mathbb{J} \begin{pmatrix} \dot{p}^{\mathrm{num}}_{(\mathrm{g})} \\ \dot{e}^{\mathrm{num}}_{(\mathrm{g})} \end{pmatrix} + \mathbb{J}^{-1}_{(\mathrm{g})} \begin{pmatrix} \delta\mathcal{F}^E_{\mathrm{PN}} \\ \delta\mathcal{F}^{J_z}_{\mathrm{PN}} \end{pmatrix} \right)$$

- Masses $M = 10^6 M_\odot$, $\mu = 10^2 M_\odot$, spin $s = 10^{-4} M$ (maximally spinning Kerr)
- Inspirals starting with $\omega_{20} = 2\Omega_\phi = 1\,\mathrm{mHz}$, ending near LSO at different e
- Inspiral waveform:



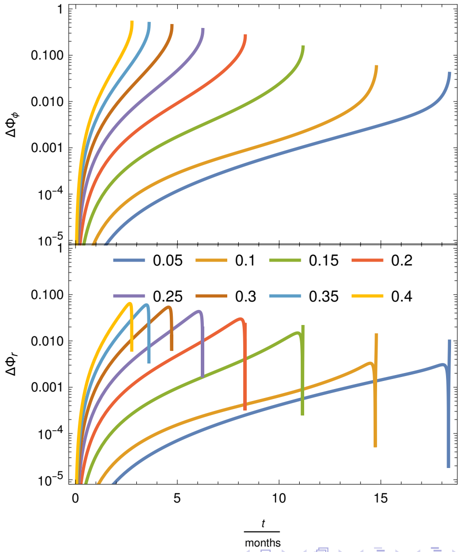
$$h = \frac{1}{r} \sum_{lmn} A_{lmn}(t) Y_{lm}(\theta) e^{-i\Phi_{mn}(t) + im\phi}$$

$$A_{lmn}(t) = \frac{2C^{+}_{lmn}(p(t), e(t))}{\omega^2_{mn}(p(t), e(t))}, \quad \Phi_{mn} = \int_0^t \omega_{mn}(p(t'), e(t'))\mathrm{d}t'$$

Phase shifts

- Comaprison of 2 models
  - Hybrid model
  - Fully relativistic model
- Phase shifts

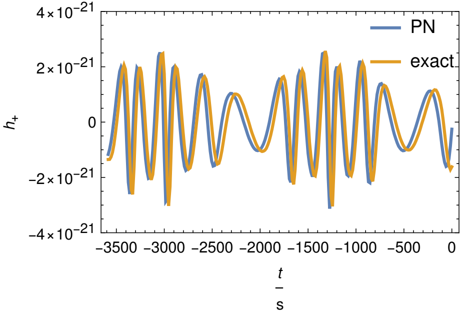
$$\Delta\Phi_i = \left|\Phi_i^{\mathrm{num}} - \Phi_i^{\mathrm{PN}}\right|$$

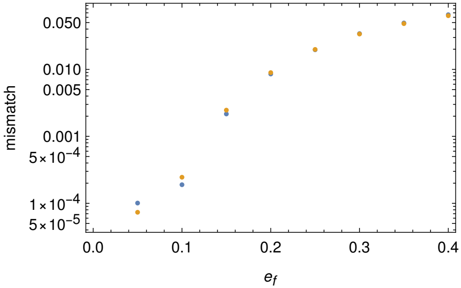


Mismatch

- Waveforms calculated using FastEMRIWaveforms [Katz et al., PRD **104**, 064047 (2021); Chua et al., PRL **126**, 051102 (2021)]
- Mismatch between 2 waveforms:

$$\mathcal{M} = 1 - \frac{\langle h_1, h_2 \rangle}{\sqrt{\langle h_1, h_1 \rangle \langle h_2, h_2 \rangle}}$$





Summary

- GW templates from EMRIs for LISA
- Spin of the secondary in the postadiabatic term
- Expansion of the spin contribution to the energy and angular momentum fluxes in PN series
- Confirmation of PN theory results with BHPT to 3PN and extension to 5PN
- Inspirals with exact adiabatic, PN postadiabatic term
- Good accuracy for low eccentricities
- Future work: extension to Kerr, calibration of EOB models, SuperKludge ...

Thank you

Relative differences between numerical and PN fluxes