Adiabatic inspirals into a black hole surrounded by matter

calculated using a modified Teukolsky equation

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We aim for a fully relativistic, self-consistent treatment

$\Rightarrow$ calculation of GW fluxes and EMRIs using a modified Teukolsky equation

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Perturbations in type D spacetimes

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Teukolsky equation for $\Psi_4^{(1)}$

$$\mathcal{O}^{(0)}\Psi^{(1)}_4 = \mathcal{T}^{(1)}$$

$O^{(0)}$, $\mathfrak{T}^{(0)ab}$ are constructed from $g^{(0)}_{\mu\nu}$, $\qquad \mathcal{T}^{(1)} = \mathfrak{T}^{(0)ab} T^{(1)}_{ab}$

- $\Psi^{(1)}_4 \rightarrow$ GW fluxes and waveforms

Extending the Teukolsky equation

Past extensions

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Our approach

- Inspired by the work of Li et al. (2023):
- Type D: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \mathcal{O}(\varepsilon^2)$

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$$g_{\mu\nu} = \underbrace{g^{(0,0)}_{\mu\nu} + \zeta h^{(1,0)}_{\mu\nu}}_{\textit{background}} + \underbrace{\varepsilon \left(h^{(0,1)}_{\mu\nu} + \zeta h^{(1,1)}_{\mu\nu} \right)}_{\textit{radiation}} + \ldots$$

- $g^{(0,0)}_{\mu\nu}$ = Type D black-hole spacetime, $h^{(1,0)}_{\mu\nu}$ =matter perturbation, $h^{(0,1)}_{\mu\nu}$ =radiation on type D background, $h^{(1,1)}_{\mu\nu}$ = the unknown of our problem

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- Using the same Bianchi and Ricci identities in NP formalism we obtain the modified Teukolsky equation for the variable $\Psi_4^{(1,1)}$

The underlining formalism

NP formalism

- NP tetrad $e_a^\mu = \{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ satisfying $l^\mu n_\mu = -1, \quad m^\mu \bar{m}_\mu = 1$

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- derivatives þ, þ$'$, ð, ð$'$
- more compact form of expressions

The unperturbed NP equations

Operators and Identities

$$E_3 \equiv \bar{\delta} + 3\alpha + \bar{\beta} + \pi - \bar{\tau}\,,$$

$$E_4 \equiv \Delta + \mu + \bar{\mu} + 3\gamma - \bar{\gamma}\,,$$

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- Expanding in Two parameters $\{\varepsilon, \zeta\}$ gives us the $(1, 1)$ modified Teukolsky equation

The resulting equation

- Instead of $(1,1)$ Einstein equations

$$\delta G_{\mu\nu}(h^{(1,1)}) + \delta^2 G_{\mu\nu}(h^{(1,0)}, h^{(0,1)}) = \kappa T^{(1,1)}_{\mu\nu}$$

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we have

Modified Teukolsky equation for $\Psi_4^{(1,1)}$

$$\mathcal{O}^{(0,0)}\Psi_4^{(1,1)} + \mathcal{G}^{(1,1)} = \mathcal{T}^{(1,1)}$$

$$\mathcal{G}^{(1,1)} = \mathcal{O}^{(1,0)}\Psi_4^{(0,1)} + \mathcal{O}^{(0,1)}\Psi_4^{(1,0)} + \mathcal{K}^{(1,1)}\big(\Psi_3^{(0,1)}, \Psi_3^{(1,0)}\big)$$

- $\mathcal{O}^{(0,0)} = \hat{O}$ is the original Teukolsky operator
- $\mathcal{O}^{(1,0)} = \mathcal{O}^{(1,0)}(\Psi_2^{(1,0)}, \Psi_3^{(1,0)}, T_{ab}^{(1,0)})$
- $K^{(1,1)} = K^{(1,1)}(\Psi_3^{(1,0)}, \Psi_3^{(0,1)}, T_{ab}^{(1,0)}, T_{ab}^{(0,1)})$,
  using gauge freedom one can impose $\Psi_3^{(1,0)} = 0 = \Psi_3^{(0,1)} \Rightarrow K^{(1,1)} = 0$
- $\mathcal{T}^{(1,1)} = \mathcal{T}^{(1,1)}(T_{ab}^{(1,0)}, T_{ab}^{(0,1)}, T_{ab}^{(1,1)})$
- $\mathcal{O}^{(0,1)}$, $K^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ require reconstruction of $h_{\mu\nu}^{(0,1)}$

Modified Teukolsky equation: a detailed look

- The original operator reads

$$\mathcal{O}^{(0,0)} = \mathcal{E}_4^{(0,0)} F_4^{(0,0)} - \mathcal{E}_3^{(0,0)} F_3^{(0,0)} - 3\Psi_2^{(0,0)}$$

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- and its perturbation

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\mathcal{O}^{(1,0)}_{\text{D-vac}} &= \mathcal{E}_4^{(1,0)} F_4^{(0,0)} + \mathcal{E}_4^{(0,0)} F_4^{(1,0)} - \mathcal{E}_3^{(1,0)} F_3^{(0,0)} - \mathcal{E}_3^{(0,0)} F_3^{(1,0)} - 3\Psi_2^{(1,0)} \\
O^{(1,0)}_{\text{corr}} &= \left(\Psi_2^{(0,0)}\right)^{-1} \left(G_3^{(0,0)}(\Psi_3^{(1,0)}) F_3^{(0,0)} - G_4^{(0,0)}(\Psi_3^{(1,0)}) F_4^{(0,0)} + S_5^{(1,0)} F_3^{(0,0)} - S_6^{(1,0)} F_4^{(0,0)} \right)
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- Some operators in terms of spin coefficients and NP derivatives

$$\mathcal{E}_3 = \delta^* + 3\alpha + \beta^* + 4\pi - \tau^*, \qquad F_3 = \delta + 4\beta - 3\tau, \qquad G_3 = D + 2\epsilon - 2\rho$$

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- $S_5^{(1,0)} = S_5^{(1,0)}(T_{ab}^{(1,0)})$
- We can then substitute for the NP quantities and find the source for the modified Teukolsky equation

$$\hat{O}^{(0,0)} \Psi_4^{(1,1)} = \mathcal{S}^{(1,1)}$$

EMRI and modified Teukolsky equation: general strategy

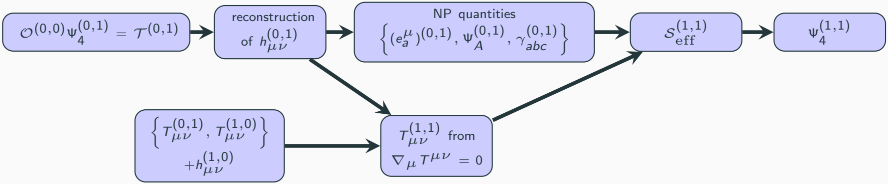
Modified Teukolsky equation $\Leftrightarrow$ construction of the effective source

$$\mathcal{O}^{(0,0)}\Psi_4^{(1,1)} = \mathcal{S}_{\text{eff}}^{(1,1)}$$

- $\mathcal{S}_{\text{eff}}^{(1,1)} = \mathcal{T}^{(1,1)} - \mathcal{O}^{(1,0)}\Psi_4^{(0,1)} - \mathcal{O}^{(0,1)}\Psi_4^{(1,0)} - \mathcal{K}^{(1,1)}(\Psi_3^{(0,1)}, \Psi_3^{(1,0)})$
- 2 parts of stress-energy

$$T^{\mu\nu} = T^{\mu\nu}_{\text{particle}} + T^{\mu\nu}_{\text{matter}}$$

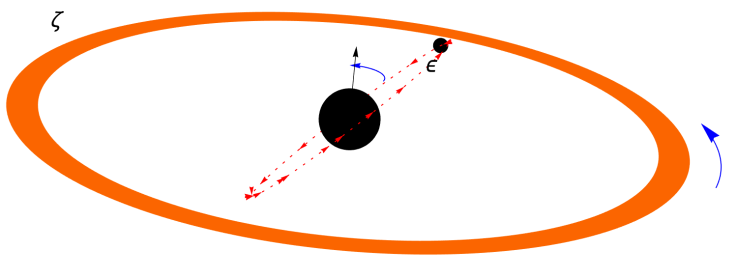
$$T^{\mu\nu}_{\text{particle}} = \varepsilon T^{(0,1)\mu\nu} + \varepsilon\zeta T^{(1,1)\mu\nu}_{\text{particle}}$$
$$T^{\mu\nu}_{\text{matter}} = \zeta T^{(1,0)\mu\nu} + \varepsilon\zeta T^{(1,1)\mu\nu}_{\text{matter}}$$



Application: blackhole surrounded by a ring

The ideal model

- Kerr black hole surrounded by a rotating disc



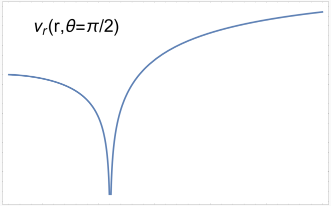
Our simplified model

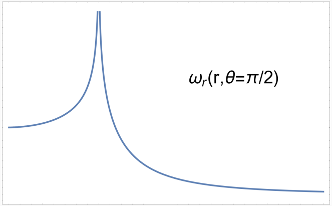
- $g^{(0,0)}_{\mu\nu}$: Kerr $\rightarrow$ Schwarzschild (or linearised Kerr)
- Disc $\rightarrow$ ring located at $r = r_s$
- The perturbed Schwarzschild metric can be written as

$$
\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)(1 + 2\nu_r)\mathrm{d}t^2 + \frac{1 + 2\xi_r - 2\nu_r}{1 - 2M/r}\mathrm{d}r^2 \\
+(1 - 2\nu_r)r^2\left[(1 + 2\xi_r)\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right] - 2\omega r^2\sin^2\theta\mathrm{d}t\mathrm{d}\phi
$$

$$
\text{with } \omega = \omega_K + \omega_r, \qquad \omega_K = \frac{Ma}{r^3}
$$

- The ring perturbation is described by functions ν_r and ω_r





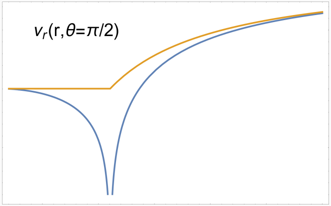
Our simplified model

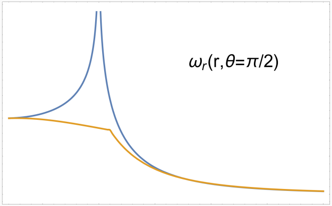
- $g^{(0,0)}_{\mu\nu}$: Kerr $\rightarrow$ Schwarzschild (or linearised Kerr)
- Disc $\rightarrow$ ring located at $r = r_s$
- The perturbed Schwarzschild metric can be written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)(1 + 2\nu_r)\mathrm{d}t^2 + \frac{1 + 2\xi_r - 2\nu_r}{1 - 2M/r}\mathrm{d}r^2$$
$$+(1 - 2\nu_r)r^2\left[(1 + 2\xi_r)\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right] - 2\omega r^2 \sin^2\theta\mathrm{d}t\mathrm{d}\phi$$

with $\omega = \omega_K + \omega_r, \qquad \omega_K = \frac{Ma}{r^3}$

- We can simplify the spacetime further by adopting a pole-dipole approximation





The pole-dipole approximation

- The background metric is $g_{\mu\nu}^{BG} = g_{\mu\nu}^{(0,0)} + \zeta h_{\mu\nu}^{(1,0)}$

The pole-dipole approximation of the ring

The metric $g_{\mu\nu}^{BG}$ is equivalent to two linearised Kerr spacetimes matched at the sphere $r = r_s$

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The metric $g^{BG}_{\mu\nu}$ is equivalent to two linearised Kerr spacetimes matched at the sphere $r = r_s$

- The spacetime is type D vacuum everywhere but on the matter shell $r = r_s$

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- Then outside of the particle and $r = r_s$ we have two Teukolsky equations in linearised Kerr spacetimes

$$\mathcal{O}_\pm \left(\Psi_4^{(0,1)} + \zeta \Psi_4^{(1,1)} \right) = 0$$

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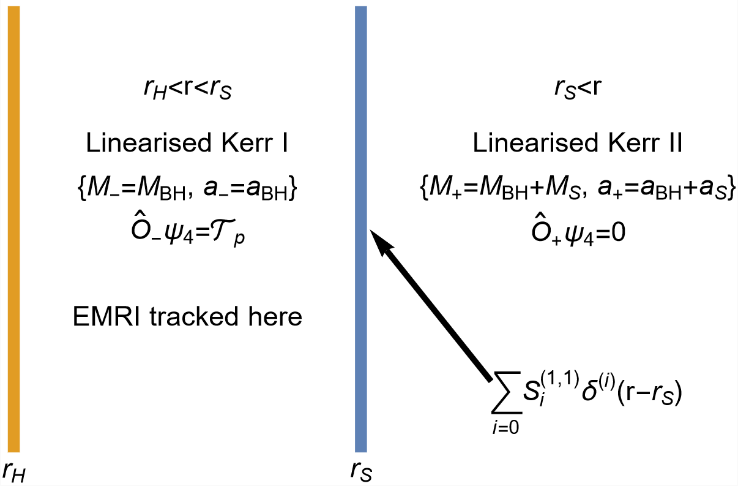
- On the sphere $r = r_s$ the source can be written as

$$\mathcal{S}_{r_\mathrm{S}}^{(1,1)} = \sum_{i=0}^{3} \mathcal{S}_{r_\mathrm{S}(i)}^{(1,1)} \delta^{(i)}(r - r_\mathrm{S})$$


$r_H < r < r_S$
Linearised Kerr I
$\{M_- = M_{BH},\ a_- = a_{BH}\}$
$\hat{O}_- \psi_4 = \mathcal{T}_p$
EMRI tracked here
$r_S < r$
Linearised Kerr II
$\{M_+ = M_{BH} + M_S,\ a_+ = a_{BH} + a_S\}$
$\hat{O}_+ \psi_4 = 0$
$\sum_{i=0} S_i^{(1,1)} \delta^{(i)}(r - r_S)$
r_H
r_S

Our strategy

Separating the solution into two parts

$$\Psi_4^{(1,1)} = \Psi_{4(sm)}^{(1,1)} + \Psi_{4(r_{\mathrm{S}})}^{(1,1)}$$

- By imposing the matching conditions $\Psi_{4+}^{(1,1)}\Big|_{r=r_s} = \Psi_{4-}^{(1,1)}\Big|_{r=r_s}$ and $\left(\partial_\mu \Psi_{4+}^{(1,1)}\right)\Big|_{r=r_s} = \left(\partial_\mu \Psi_{4-}^{(1,1)}\right)\Big|_{r=r_s}$

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- we get the smooth part

$$\Psi_{4(sm)}^{(1,1)} = \begin{cases} \Psi_{4-}^{(1,1)} & r \leq r_s \\ \Psi_{4+}^{(1,1)} & r > r_s \end{cases}$$

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- The other part is sourced by the terms living on the matter shell $r = r_s$

$$\hat{O}^{(0,0)}\Psi_4^{(1,1)\delta} = \mathcal{S}_{r_\mathrm{S}}^{(1,1)} = \sum_{i=0}^{3} \mathcal{S}_{r_\mathrm{S}(i)}^{(1,1)}\delta^{(i)}(r - r_\mathrm{S}).$$

$\Psi^{(1,1)}_{4(sm)}$: matching $\Psi^{(\pm)}_4$

Mode-decomposition

$$\Psi_4^{(-)} = \frac{1}{r^4} \sum_{lm\omega^{(in)}} R^{(-)}_{lm\omega^{(in)}}(r) Y_{lm}(\theta) e^{\mathrm{i}(m\phi^{(in)} - \omega^{(in)} t^{(in)})}$$

$$\Psi_4^{(+)} = \frac{1}{(r - ia\cos\theta)^4} \sum_{lm\omega} R^{(+)}_{lm\omega}(r) S_{lm}(\theta) e^{\mathrm{i}(m\phi - \omega t)}$$

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- Expanding $R_l^{(\pm)}(M + m_s, a_s) \approx R_{(\mathrm{Schw})\mathrm{l}} + m_s \partial_m R_l^{(\pm)} + a_s \partial_a R_l^{(\pm)}$
- Slow rotation: $S_{lm}(a\omega, \theta) \approx Y_{lm}(\theta) + a\omega[b^-_{lm} Y_{l-1m}(\theta) + b^+_{lm} Y_{l+1m}(\theta)]$

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New radial functions

$$\partial_a \mathfrak{R}_l^{(-)}(r) = \partial_a R_l^{(-)}(r)$$

$$\partial_a \mathfrak{R}_l^{(+)}(r) = \partial_a R_l^{(+)}(r) + \sum_{j=l-1}^{j=l+1} d_j(r) R_{(\mathrm{Schw})j}(r).$$

The source for $\Psi^{(1,1)}_{4(r_{\mathrm{S}})}$

- The source living on the $r = r_{\mathrm{S}}$ $\mathcal{S}^{(1,1)}_{r_{\mathrm{S}}} = \sum_{i=0}^{3} \mathcal{S}^{(1,1)}_{r_{\mathrm{S}}(i)} \delta^{(i)}(r - r_{\mathrm{S}})$.

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$$\begin{aligned}\mathcal{S}^{(1,1)}_{r_{\mathrm{S}}} &= \mathfrak{T}^{(0,1)ab} T^{(1,0)}_{ab} + \mathfrak{T}^{(0,0)ab} T^{(1,1)}_{ab(\mathrm{matter})} \\ &\quad - \left[\left(\mathcal{O}^{(1,0)(r_{\mathrm{S}})}_{\mathrm{D\text{-}vac}} + \mathcal{O}^{(1,0)}_{\mathrm{corr}} \right) \Psi^{(0,1)}_4 + \mathcal{K}^{(1,1)}(\Psi^{(0,1)}_3, \Psi^{(1,0)}_3) \right].\end{aligned}$$

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Metric reconstruction using the ORG Hertz potential

- $h^{(0,1)}_{\mu\nu} = \left(\mathscr{S}^{(0,0)\dagger}_4 \Psi^{(0,1)}_{\mathrm{Hz}} \right)_{\mu\nu}$
- where $\Psi^{(0,1)}_{\mathrm{Hz}}$ satisfies

$$\mathcal{O}^{(0,0)\dagger} \left(\Psi^{(0,1)}_{\mathrm{Hz}} \right) = 0, \quad \frac{1}{2} (\text{þ}')^4 \overline{\Psi^{(0,1)}_{\mathrm{Hz}}} = \Psi^{(0,1)}_4$$

- $\Psi^{(0,1)}_{\mathrm{Hz}}$ and $\Psi^{(0,1)}_4$ satisfy Teukolsky equation with spin weight ± 2

$$C^{\mathrm{Hz}}_{lmnk} = (-1)^{l+m+k} \frac{2}{\omega^4_{mnk}} {}_{-2}C^{(\mathrm{H})}_{(\mathrm{Schw})lmnk}$$

Constructing the source

- $\mathcal{S}^{(1,1)}_{r_\mathrm{S}} = -\left(\mathcal{O}^{(1,0)(r_\mathrm{S})}_{\text{D-vac}} + \mathcal{O}^{(1,0)}_{\mathrm{corr}}\right)\Psi^{(0,1)}_4 + \mathcal{S}^{(1,1)}_{\mathrm{reconst}}$

Constructing the source

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- The part needing metric reconstruction is

$$\mathcal{S}^{(1,1)}_{\mathrm{reconst}} = \Sigma^{(1,0)}\overline{\Psi^{(0,1)}_{\mathrm{Hz}}} + \mathfrak{T}^{(1,1)}_{\mathrm{dynamical}}$$

- The last term requires solving equations for matter $\mathfrak{T}^{(1,1)}_{\mathrm{dynamical}} \Leftrightarrow T^{(1,1)}_{\mu\nu}$

- The desired result is the decomposition of the form

$$\mathcal{S}^{(1,1)}_{r_\mathrm{S}} = \sum_{lm\omega}\mathcal{R}\left[\mathcal{S}^{(1,1)}_{r_\mathrm{S}}\right]_{lm\omega}{}_{-2}Y_{lm}(\theta)e^{\mathrm{i}(m\phi-\omega t)}$$

Constructing the source

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- The radial part can be written as

$$\mathcal{R}\left[\mathcal{S}^{(1,1)}_{r_\mathrm{S}}\right]_{lm\omega} = \sum_{i=0}^{3} \mathcal{R}\left[\mathcal{S}^{(1,1)}\right]_{(i)lm\omega}(r)\delta^{(i)}(r - r_s)$$

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- So that we can get ODEs for $\Psi^{(1,1)}_{4(r_\mathrm{S})}$

$${}_{-2}\mathfrak{D}_{lm\omega}R^{(1,1)}_{(r_\mathrm{S})lm\omega}(r) = \sum_{i=0}^{3}\mathcal{S}^{(1,1)}_{(i)lm\omega}(r)\delta^{(i)}(r-r_s)$$

An example of a source term

$$\mathfrak{T}^{(0,1)ab} T^{(1,0)}_{ab}$$

where

$$\mathfrak{T}^{(0,1)ab} = \mathcal{E}_4^{(0,0)} \mathcal{S}_4^{(0,1)ab} + \mathcal{E}_4^{(0,1)} \mathcal{S}_4^{(0,0)ab} - \mathcal{E}_3^{(0,0)} \mathcal{S}_3^{(0,1)ab} - \mathcal{E}_3^{(0,1)} \mathcal{S}_3^{(0,0)ab}$$

Expressed using the metric perturbation

$$\frac{1}{2}\Big(3\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \Psi_2 - 3\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \bar\Psi_2 - 2\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \rho\,\rho' + 3\, h^{\{0,1\}}_{12}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'^2 + 3\, h^{\{0,1\}}_{34}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'^2 + 6\, h^{\{0,1\}}_{44}\, \Phi_{11}^{\{1,0\}}\, \rho'^2 - 6\, h^{\{0,1\}}_{14}\, \bar\Phi_{12}^{\{1,0\}}\, \rho'^2 - 14\, h^{\{0,1\}}_{12}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'\,\bar\rho' + 2\, h^{\{0,1\}}_{34}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'\,\bar\rho' - 6\, h^{\{0,1\}}_{44}\, \Phi_{11}^{\{1,0\}}\, \rho'\,\bar\rho' +$$
$$6\, h^{\{0,1\}}_{14}\, \bar\Phi_{12}^{\{1,0\}}\, \rho'\,\bar\rho' - 5\, h^{\{0,1\}}_{12}\, \bar\Phi_{02}^{\{1,0\}}\, \bar\rho'^2 - 5\, h^{\{0,1\}}_{34}\, \bar\Phi_{02}^{\{1,0\}}\, \bar\rho'^2 + 10\, h^{\{0,1\}}_{14}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'^2 + 3\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \rho'\,\bar\rho - h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \bar\rho'\,\bar\rho - 14\, h^{\{0,1\}}_{44}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\tau - 6\, h^{\{0,1\}}_{23}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'\,\tau' - 6\, h^{\{0,1\}}_{44}\, \Phi_{12}^{\{1,0\}}\, \rho'\,\tau' + 6\, h^{\{0,1\}}_{14}\, \Phi_{22}^{\{1,0\}}\, \rho'\,\tau' -$$
$$12\, h^{\{0,1\}}_{23}\, \bar\Phi_{02}^{\{1,0\}}\, \bar\rho'\,\tau' + 6\, h^{\{0,1\}}_{44}\, \Phi_{12}^{\{1,0\}}\, \bar\rho'\,\tau' + 22\, h^{\{0,1\}}_{12}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\tau' - 2\, h^{\{0,1\}}_{34}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\tau' - 10\, h^{\{0,1\}}_{14}\, \Phi_{22}^{\{1,0\}}\, \bar\rho'\,\tau' + 6\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \tau\,\tau' + 6\, h^{\{0,1\}}_{23}\, \bar\Phi_{12}^{\{1,0\}}\, \tau'^2 - 3\, h^{\{0,1\}}_{12}\, \Phi_{22}^{\{1,0\}}\, \tau'^2 - 3\, h^{\{0,1\}}_{34}\, \Phi_{22}^{\{1,0\}}\, \tau'^2 -$$
$$6\, h^{\{0,1\}}_{44}\, \bar\Phi_{12}^{\{1,0\}}\, \rho'\,\bar\tau' + 4\, h^{\{0,1\}}_{44}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\bar\tau' - 4\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \tau'\,\bar\tau' - 6\, h^{\{0,1\}}_{23}\, \bar\Phi_{02}^{\{1,0\}}\, \rho'\,\bar\tau - 2\, h^{\{0,1\}}_{44}\, \Phi_{12}^{\{1,0\}}\, \rho'\,\bar\tau + 10\, h^{\{0,1\}}_{12}\, \bar\Phi_{12}^{\{1,0\}}\, \rho'\,\bar\tau - 14\, h^{\{0,1\}}_{34}\, \bar\Phi_{12}^{\{1,0\}}\, \rho'\,\bar\tau - 4\, h^{\{0,1\}}_{14}\, \Phi_{22}^{\{1,0\}}\, \rho'\,\bar\tau + 10\, h^{\{0,1\}}_{23}\, \bar\Phi_{02}^{\{1,0\}}\, \bar\rho'\,\bar\tau +$$
$$2\, h^{\{0,1\}}_{44}\, \Phi_{12}^{\{1,0\}}\, \bar\rho'\,\bar\tau - 8\, h^{\{0,1\}}_{12}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\bar\tau + 4\, h^{\{0,1\}}_{34}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\,\bar\tau + 8\, h^{\{0,1\}}_{44}\, \Phi_{22}^{\{1,0\}}\, \tau\,\bar\tau + 10\, h^{\{0,1\}}_{23}\, \bar\Phi_{12}^{\{1,0\}}\, \tau'\,\bar\tau - 2\, h^{\{0,1\}}_{12}\, \Phi_{22}^{\{1,0\}}\, \tau'\,\bar\tau + 6\, h^{\{0,1\}}_{34}\, \Phi_{22}^{\{1,0\}}\, \tau'\,\bar\tau - 10\, h^{\{0,1\}}_{23}\, \bar\Phi_{12}^{\{1,0\}}\, \bar\tau^2 + 5\, h^{\{0,1\}}_{12}\, \Phi_{22}^{\{1,0\}}\, \bar\tau^2 + 5\, h^{\{0,1\}}_{34}\, \Phi_{22}^{\{1,0\}}\, \bar\tau^2 +$$
$$\bar\Phi_{02}^{\{1,0\}}\, \rho'\, (\text{þ}\, h^{\{0,1\}}_{22}) - 2\, \bar\Phi_{02}^{\{1,0\}}\, \bar\rho'\, (\text{þ}\, h^{\{0,1\}}_{22}) + 3\, \bar\Phi_{12}^{\{1,0\}}\, \tau'\, (\text{þ}\, h^{\{0,1\}}_{22}) + 3\, \bar\Phi_{12}^{\{1,0\}}\, \bar\tau\, (\text{þ}\, h^{\{0,1\}}_{22}) - 5\, \bar\Phi_{12}^{\{1,0\}}\, \rho'\, (\text{þ}\, h^{\{0,1\}}_{24}) - 6\, \bar\Phi_{12}^{\{1,0\}}\, \bar\rho'\, (\text{þ}\, h^{\{0,1\}}_{24}) - 3\, \Phi_{22}^{\{1,0\}}\, \tau'\, (\text{þ}\, h^{\{0,1\}}_{24}) + 2\, \Phi_{22}^{\{1,0\}}\, \bar\tau\, (\text{þ}\, h^{\{0,1\}}_{24}) + 4\, \Phi_{22}^{\{1,0\}}\, \rho'\, (\text{þ}\, h^{\{0,1\}}_{44}) + \ldots$$

$$
\begin{aligned}
&\frac{1}{4}\\
&\Big(\Phi_{22}^{\{1,0\}}\Big(12\rho'^2(\rho'-\bar\rho')(\text{þ}\bar\Psi_{HZ}^{\{0,1\}})+12\rho'^2(\text{þ}\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-28\rho'\bar\rho'(\text{þ}\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-2\rho'(\text{þ}\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-4\bar\rho'(\text{þ}\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2(\text{þ}\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+12\Psi_2\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+43\bar\Psi_2\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 3\Psi_2\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\Psi_2\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+10\rho\rho'\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-18\rho\bar\rho'^2(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+12\rho'^2\bar\rho(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+24\rho'\bar\rho'\bar\rho(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\rho'^2\bar\rho(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-40\rho'\tau\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 8\bar\rho'\tau\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+16\bar\rho'\tau'\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-34\rho'\tau\bar\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+3\bar\rho'\tau\bar\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-38\rho'\bar\tau'\bar\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+22\bar\rho'\bar\tau'\bar\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+6\Psi_2(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+14\bar\Psi_2(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 4\rho\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+6\rho'\bar\rho(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+18\bar\rho'\bar\rho(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-28\tau\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-16\tau'\bar\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-32\tau\bar\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-12\bar\tau'\bar\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-8\rho(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 8\bar\rho(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-18\tau'(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-16\bar\tau(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+6(\text{þ}'\text{þ}'\eth\eth'\bar\Psi_{HZ}^{\{0,1\}})+8\tau(\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})+6\bar\tau'(\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-4\rho'\tau'(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-8\bar\rho'\tau'(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 4\rho'\bar\tau(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+16\bar\rho'\bar\tau(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+6(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth\bar\tau)-12\bar\rho'(\text{þ}'\eth\eth'\bar\Psi_{HZ}^{\{0,1\}})-12\bar\rho'\tau(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})+12\rho'\bar\tau'(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-24\bar\rho'\bar\tau'(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})+15(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth'\bar\tau')+\\
&\quad 12\rho'^2\tau'(\eth\bar\Psi_{HZ}^{\{0,1\}})-28\rho'\bar\rho'\tau'(\eth\bar\Psi_{HZ}^{\{0,1\}})-12\rho'^2\bar\tau(\eth\bar\Psi_{HZ}^{\{0,1\}})+16\rho'\bar\rho'\bar\tau(\eth\bar\Psi_{HZ}^{\{0,1\}})+24\bar\rho'^2\bar\tau(\eth\bar\Psi_{HZ}^{\{0,1\}})+6\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\tau)+62\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\tau)+16(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\tau)-\\
&\quad 24\rho'^2\tau(\eth'\bar\Psi_{HZ}^{\{0,1\}})+12\rho'\bar\rho'\tau(\eth'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\rho'^2\tau(\eth'\bar\Psi_{HZ}^{\{0,1\}})+12\rho'^2\bar\tau'(\eth'\bar\Psi_{HZ}^{\{0,1\}})-12\rho'\bar\rho'\bar\tau'(\eth'\bar\Psi_{HZ}^{\{0,1\}})+7\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\tau)-4(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\tau)+2\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\bar\tau')-\\
&\quad 48\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\bar\tau')+8(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\bar\tau')+\\
&\quad \bar\Psi_{HZ}^{\{0,1\}}\big(39\bar\Psi_2\rho'^2+9\Psi_2\rho'\bar\rho'-9\bar\Psi_2\rho'\bar\rho'+18\rho\rho'^2\bar\rho'+9\Psi_2\bar\rho'^2-18\rho\bar\rho'^3+48\rho'^2\bar\rho'\bar\rho-15\rho'\bar\rho'\tau\bar\tau+38\bar\rho'^2\tau\bar\tau-66\rho'^2\bar\tau'\bar\tau+42\bar\rho'^2\bar\tau'\bar\tau+9(\text{þ}'\text{þ}'\eth'\bar\tau')+42\bar\rho'(\text{þ}'\eth\bar\tau)+9\bar\rho'(\text{þ}'\eth'\tau)+9\rho'(\text{þ}'\eth'\bar\tau')-\\
&\qquad 45\bar\rho'(\text{þ}'\eth'\bar\tau')+24\rho'\bar\rho'(\eth\bar\tau)-18\bar\rho'^2(\eth\bar\tau)+9\rho'\bar\rho'(\eth'\tau)-18\bar\rho'^2(\eth'\tau)+18\rho'^2(\eth'\bar\tau')-18\rho'\bar\rho'(\eth'\bar\tau')+18\bar\rho'^2(\eth'\bar\tau')\big)\Big)+\\
&\ 2\Big(2\rho'^2(\text{þ}\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\Phi_{22}^{\{1,0\}})-2\bar\rho'^2(\text{þ}\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\Phi_{22}^{\{1,0\}})+2\rho'(\text{þ}\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\Phi_{22}^{\{1,0\}})+(\text{þ}\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\Phi_{22}^{\{1,0\}})-12\Phi_{11}^{\{1,0\}}\rho'^2\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\Phi_{12}^{\{1,0\}}\rho'^2\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 32\bar\Phi_{12}^{\{1,0\}}\rho'\bar\rho'\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\Phi_{12}^{\{1,0\}}\bar\rho'^2\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+20\bar\Phi_{12}^{\{1,0\}}\rho'\bar\rho'\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-4\bar\Phi_{12}^{\{1,0\}}\bar\rho'^2\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-2\rho'\bar\rho'(\text{þ}\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\rho'^2(\text{þ}\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 4\rho'\bar\rho'(\text{þ}'\Phi_{11}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-20\rho'\tau(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+12\bar\rho'\tau(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\rho'\bar\tau'(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\Psi_2(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+4\bar\Psi_2(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 2\rho'\bar\rho(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+8\bar\rho'\bar\rho(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-6\tau\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-8\tau'\bar\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-2\tau\bar\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-2\bar\tau'\bar\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 4\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\text{þ}'\bar\Phi_{12}^{\{1,0\}})-4\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\text{þ}'\bar\Phi_{12}^{\{1,0\}})-6\Phi_{11}^{\{1,0\}}\rho'^2(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-6\Phi_{11}^{\{1,0\}}\rho'\bar\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\Phi_{12}^{\{1,0\}}\rho'\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+42\bar\Phi_{12}^{\{1,0\}}\bar\rho'\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 18\bar\Phi_{12}^{\{1,0\}}\rho'\bar\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+6\bar\Phi_{12}^{\{1,0\}}\bar\rho'\bar\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\rho'(\text{þ}\Phi_{22}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\bar\rho'(\text{þ}\Phi_{22}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\rho'(\text{þ}'\Phi_{11}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\rho'(\text{þ}'\Phi_{11}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 22\tau(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-12\bar\tau'(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-2\rho(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\rho(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-6\Phi_{11}^{\{1,0\}}\rho'(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-6\bar\Phi_{12}^{\{1,0\}}\tau(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 4\bar\Phi_{12}^{\{1,0\}}\bar\tau'(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2(\text{þ}'\Phi_{11}^{\{1,0\}})(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\Phi_{11}^{\{1,0\}}(\text{þ}'\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-4\bar\Phi_{12}^{\{1,0\}}(\text{þ}'\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+16\bar\Phi_{12}^{\{1,0\}}\rho'(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+18\bar\Phi_{12}^{\{1,0\}}\bar\rho'(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 14(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+4\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth\bar\Phi_{12}^{\{1,0\}})+2(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth\bar\Phi_{12}^{\{1,0\}})+8\bar\Phi_{12}^{\{1,0\}}\rho'\bar\rho'(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+8\bar\Phi_{12}^{\{1,0\}}\bar\rho'^2(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+8\rho'(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 6\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-2\bar\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})-4(\text{þ}'\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})+2(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\eth\eth'\bar\Psi_{HZ}^{\{0,1\}})+4\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth'\Phi_{22}^{\{1,0\}})+4\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth'\Phi_{22}^{\{1,0\}})+\\
&\quad 4(\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})(\text{þ}'\eth'\Phi_{22}^{\{1,0\}})+2\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-10\rho'\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\Phi_{12}^{\{1,0\}})-4\bar\rho'^2(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\Phi_{12}^{\{1,0\}})-5\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\Phi_{12}^{\{1,0\}})-\\
&\quad \bar\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\Phi_{12}^{\{1,0\}})+(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\Phi_{12}^{\{1,0\}})+8\bar\rho'\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\Phi_{22}^{\{1,0\}})-10\bar\rho'\bar\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\Phi_{22}^{\{1,0\}})+4\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\Phi_{22}^{\{1,0\}})-(\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})(\eth\Phi_{22}^{\{1,0\}})+\\
&\quad 4\bar\Phi_{12}^{\{1,0\}}\bar\rho'^3(\eth\bar\Psi_{HZ}^{\{0,1\}})+8\rho'\bar\rho'(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})+4\bar\rho'^2(\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})+2\rho'\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})-8\bar\rho'\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})-2\rho'\bar\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 8\bar\rho'\bar\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})-4\bar\rho'(\text{þ}'\text{þ}'\bar\Phi_{12}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})+4\bar\rho'(\text{þ}'\eth'\Phi_{22}^{\{1,0\}})(\eth\bar\Psi_{HZ}^{\{0,1\}})+2(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\bar\tau)-4\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\eth'\Phi_{22}^{\{1,0\}})-2(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth\eth'\Phi_{22}^{\{1,0\}})-\\
&\quad 4\rho'\bar\rho'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{12}^{\{1,0\}})-2\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{12}^{\{1,0\}})-2\bar\rho'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{12}^{\{1,0\}})-2(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{12}^{\{1,0\}})+22\rho'\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})-4\bar\rho'\tau(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})-\\
&\quad 4\bar\rho'\bar\tau'(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})+20\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})+8\bar\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})+11(\text{þ}'\text{þ}'\eth\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})-2\bar\rho'^2(\eth\bar\Psi_{HZ}^{\{0,1\}})(\eth'\Phi_{22}^{\{1,0\}})-2\rho'\tau(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth'\bar\Psi_{HZ}^{\{0,1\}})+\\
&\quad 2\rho'\bar\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth'\bar\Psi_{HZ}^{\{0,1\}})-4\bar\rho'\bar\tau'(\text{þ}'\Phi_{22}^{\{1,0\}})(\eth'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\rho'^2(\eth\Phi_{22}^{\{1,0\}})(\eth'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\quad 2\Phi_{12}^{\{1,0\}}\big(16\bar\Psi_{HZ}^{\{0,1\}}\bar\rho'^2(\rho'+2\bar\rho')\bar\tau-2\bar\rho'(3\rho'\tau'+\bar\rho'\tau'-6\rho'\bar\tau-16\bar\rho'\bar\tau)(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-3\rho'\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-5\bar\rho'\tau'(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\rho'\bar\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+15\bar\rho'\bar\tau(\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})-\\
&\qquad 4\tau'(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+2\bar\tau(\text{þ}'\text{þ}'\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})+\text{þ}'\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}}+\rho'(\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-2\bar\rho'(\text{þ}'\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-2\bar\rho'^2(\text{þ}'\eth'\bar\Psi_{HZ}^{\{0,1\}})-2\rho'\bar\rho'^2(\eth'\bar\Psi_{HZ}^{\{0,1\}})\big)+4(\text{þ}'\Phi_{22}^{\{1,0\}})(\text{þ}'\bar\Psi_{HZ}^{\{0,1\}})(\eth'\bar\tau')+\\
&\quad \bar\Psi_{HZ}^{\{0,1\}}\big(-16\bar\rho'^2\bar\tau(\eth\Phi_{22}^{\{1,0\}})+(\text{þ}'\Phi_{22}^{\{1,0\}})(5\bar\Psi_2\rho'+3\Psi_2\bar\rho'+8\bar\Psi_2\bar\rho'+6\rho\rho'\bar\rho'-6\rho\bar\rho'^2+8\rho'\bar\rho'\bar\rho+8\bar\rho'^2\bar\rho+3\bar\rho'\tau\bar\tau-14\rho'\bar\tau'\bar\tau+10\bar\rho'\bar\tau'\bar\tau+3(\text{þ}'\eth'\bar\tau')+14\bar\rho'(\eth\bar\tau)+3\bar\rho'(\eth'\tau)+6\rho'(\eth'\bar\tau')-12\bar\rho'(\eth'\bar\tau'))\big)\Big)\Big);
\end{aligned}
$$

Decomposing into spherical harmonics

$$= \frac{1}{4} (-1)^{2-m} Y_{-2,1,m}$$

$$\Big(\Phi_{22}^{(1,0)} \Big(-16 \rho'^2 (\text{þ}\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 6 \rho' (\text{þ}\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 2 (\text{þ}\,\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 12 c^{-}_{-2,1} c^{+}_{-3,1} \xi^{-12} \rho' (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 9 \Psi_2 \rho' (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 51 \bar{\Psi}_2 \rho' (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 36 \rho \rho'^2 (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 24 c^{-}_{-2,1} c^{+}_{-3,1} \xi^{-1} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) +$$

$$6 \bar{R}^{(0,1)}_{2,1,-m} \big(\rho'^2 (3 \Psi_2 + 5 \bar{\Psi}_2 + 8 \rho \rho') + 2 c^{-}_{-2,1} c^{+}_{-3,1} (-2 \xi^{-1} \rho' (\text{þ}' \xi^{-1}) + (\text{þ}' \xi^{-1})^2 + \xi^{-1} (\text{þ}'\,\text{þ}' \xi^{-1}))\big) + 6 c^{-}_{-2,1} c^{+}_{-3,1} \xi^{-12} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 6 \Psi_2 (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 14 \bar{\Psi}_2 (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 28 \rho \rho' (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m})\Big) -$$

$$2 \Big(4 c^{+}_{-2,1+1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho'^3 C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} - 4 c^{-}_{-2,-1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,-1+1,-m} \rho'^3 C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} - 4 c^{-}_{-2,1} \xi^{-1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1,-m} \rho'^3 C^{(-2,1,m)}_{1,1,0,-3,1,m} - 4 c^{-}_{-2,1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho'^3 C^{(-2,1,m)}_{1,1,0,-3,1+1,m} +$$

$$16 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho'^2 C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) - 4 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,-1+1,-m} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}' \xi^{-1}) - 4 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1,-m} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}' \xi^{-1}) -$$

$$4 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}' \xi^{-1}) + 12 c^{+}_{-2,1+1} \xi^{-1} \bar{R}^{(0,1)}_{2,1+1,-m} \rho'^2 C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) + 8 c^{+}_{-2,1+1} \bar{R}^{(0,1)}_{2,1+1,-m} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) -$$

$$3 \Psi_2 \bar{R}^{(0,1)}_{2,1,-m} \rho' (\text{þ}' \Phi_{22}^{(1,0)}) - 13 \bar{\Psi}_2 \bar{R}^{(0,1)}_{2,1,-m} \rho' (\text{þ}' \Phi_{22}^{(1,0)}) - 16 \bar{R}^{(0,1)}_{2,1,-m} \rho \rho'^2 (\text{þ}' \Phi_{22}^{(1,0)}) - 2 \rho' (\text{þ}\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) (\text{þ}' \Phi_{22}^{(1,0)}) - (\text{þ}\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) (\text{þ}' \Phi_{22}^{(1,0)}) - 4 c^{-}_{-2,1} c^{+}_{-3,1} \xi^{-1} \bar{R}^{(0,1)}_{2,1,-m} (\text{þ}' \xi^{-1}) (\text{þ}' \Phi_{22}^{(1,0)}) +$$

$$4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 14 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 4 c^{-}_{-2,-1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) +$$

$$4 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 4 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + 4 c^{+}_{-1,1} \xi^{-1} \rho' C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) +$$

$$12 \Phi_{11}^{(1,0)} \rho'^3 (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 14 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 4 c^{-}_{-2,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) +$$

$$4 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 4 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 4 \rho'^2 (\text{þ}' \Phi_{11}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 4 c^{+}_{-1,1} \xi^{-1} \rho' C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) -$$

$$2 c^{-}_{-2,1} c^{+}_{-3,1} \xi^{-12} (\text{þ}' \Phi_{22}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 2 \Psi_2 (\text{þ}' \Phi_{22}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 4 \bar{\Psi}_2 (\text{þ}' \Phi_{22}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 10 \rho \rho' (\text{þ}' \Phi_{22}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 16 c^{+}_{-2,1+1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 14 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 4 c^{-}_{-2,1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho'^2 C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$68 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 4 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 4 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$8 c^{+}_{-2,1+1} \xi^{-1} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 4 c^{+}_{-1,1} \xi^{-1} \rho' C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 28 c^{+}_{-2,1+1} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$34 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}'\,\text{þ}' \xi^{-1}) - 2 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,-1+1,-m} \rho' C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}'\,\text{þ}' \xi^{-1}) - 2 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1,-m} \rho' C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}'\,\text{þ}' \xi^{-1}) -$$

$$2 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} \rho' C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}'\,\text{þ}' \xi^{-1}) - 14 c^{+}_{-2,1+1} \bar{R}^{(0,1)}_{2,1+1,-m} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \xi^{-1}) + 6 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) (\text{þ}'\,\text{þ}' \xi^{-1}) +$$

$$6 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) (\text{þ}'\,\text{þ}' \xi^{-1}) - 12 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) (\text{þ}'\,\text{þ}' \xi^{-1}) + 6 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) (\text{þ}'\,\text{þ}' \xi^{-1}) -$$

$$4 c^{+}_{-2,1+1} \xi^{-1} \bar{R}^{(0,1)}_{2,1+1,-m} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) - 4 c^{+}_{-2,1+1} \bar{R}^{(0,1)}_{2,1+1,-m} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) - 4 c^{+}_{-2,1+1} \xi^{-1} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) +$$

$$4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 6 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 2 c^{-}_{-2,-1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) +$$

$$2 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + 6 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + 2 c^{+}_{-1,1} \xi^{-1} C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) +$$

$$12 \Phi_{11}^{(1,0)} \rho'^2 (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 6 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 2 c^{-}_{-2,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) +$$

$$2 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 6 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 4 \rho' (\text{þ}' \Phi_{11}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 2 c^{+}_{-1,1} \xi^{-1} C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) +$$

$$34 c^{+}_{-2,1+1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 4 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 6 c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) -$$

$$2 c^{-}_{-2,1+1} \xi^{-1} R\Phi_{12}^{(1,0)} \rho' C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 12 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 2 c^{+}_{-1,1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$6 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 14 c^{+}_{-2,1+1} \xi^{-1} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 2 c^{+}_{-1,1} \xi^{-1} C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) -$$

$$4 c^{+}_{-2,1+1} R\bar{\Phi}_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1}) + 2 c^{-}_{-2,-1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,-1+1,-m} C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1}) + 2 c^{-}_{-2,1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1,-m} C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1}) +$$

$$2 c^{-}_{-2,1+1} R\Phi_{12}^{(1,0)} \bar{R}^{(0,1)}_{2,1+1,-m} C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1}) + 2 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,-1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) +$$

$$2 c^{-}_{-2,-1+1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,-1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + 2 c^{+}_{-2,-1+1} C^{(-2,1,m)}_{-1,1,0,-1,-1+1,m} \big(4 \xi^{-1} \rho' (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 14 (\text{þ}' \xi^{-1}) (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) -$$

$$2 \xi^{-1} (\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) + \bar{R}^{(0,1)}_{2,-1+1,-m} \big((\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (6 \xi^{-1} \rho'^2 + 4 \rho' (\text{þ}' \xi^{-1}) - 7 (\text{þ}'\,\text{þ}' \xi^{-1})) - 2 (\xi^{-1} \rho' + \text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)})\big) - 7 \xi^{-1} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + R\bar{\Phi}_{12}^{(1,0)}$$

$$\big((\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) (8 \xi^{-1} \rho'^2 + 34 \rho' (\text{þ}' \xi^{-1}) - 6 (\text{þ}'\,\text{þ}' \xi^{-1})) + 17 \xi^{-1} \rho' (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) - 6 (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m}) + \bar{R}^{(0,1)}_{2,-1+1,-m} (2 \xi^{-1} \rho'^3 + 8 \rho'^2 (\text{þ}' \xi^{-1}) + 17 \rho' (\text{þ}'\,\text{þ}' \xi^{-1}) - 2 (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1})) - 2 \xi^{-1} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,-1+1,-m})\big)\big) +$$

$$6 \Phi_{11}^{(1,0)} \rho' (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 2 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 2 c^{-}_{-2,1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) -$$

$$2 (\text{þ}' \Phi_{11}^{(1,0)}) (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + 2 c^{+}_{-2,1} C^{(-2,1,m)}_{-1,1,0,-1,1,m} \big(4 \xi^{-1} \rho' (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 14 (\text{þ}' \xi^{-1}) (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) -$$

$$2 \xi^{-1} (\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) + \bar{R}^{(0,1)}_{2,1,-m} \big((\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (6 \xi^{-1} \rho'^2 + 4 \rho' (\text{þ}' \xi^{-1}) - 7 (\text{þ}'\,\text{þ}' \xi^{-1})) - 2 (\xi^{-1} \rho' + \text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' R\bar{\Phi}_{12}^{(1,0)})\big) - 7 \xi^{-1} (\text{þ}' R\bar{\Phi}_{12}^{(1,0)}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) +$$

$$R\bar{\Phi}_{12}^{(1,0)} \big((\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) (8 \xi^{-1} \rho'^2 + 34 \rho' (\text{þ}' \xi^{-1}) - 6 (\text{þ}'\,\text{þ}' \xi^{-1})) + 17 \xi^{-1} \rho' (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) - 6 (\text{þ}' \xi^{-1}) (\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m}) + \bar{R}^{(0,1)}_{2,1,-m} (2 \xi^{-1} \rho'^3 + 8 \rho'^2 (\text{þ}' \xi^{-1}) + 17 \rho' (\text{þ}'\,\text{þ}' \xi^{-1}) - 2 (\text{þ}'\,\text{þ}'\,\text{þ}' \xi^{-1})) - 2 \xi^{-1} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m})\big)\big) -$$

$$4 c^{+}_{-2,1+1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{-1,1,0,-1,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + 2 c^{-}_{1,1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) + c^{+}_{-1,1} \xi^{-1} R\bar{\Phi}_{12}^{(1,0)} C^{(-2,1,m)}_{0,1,0,-2,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) +$$

$$2 c^{-}_{-2,1+1} \xi^{-1} R\Phi_{12}^{(1,0)} C^{(-2,1,m)}_{1,1,0,-3,1+1,m} (\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1+1,-m}) - 2 \Phi_{11}^{(1,0)} (\text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}' \bar{R}^{(0,1)}_{2,1,-m})\Big)\Big)$$

And the resulting radial expression in coordinates

$$\frac{1}{64\,(2M-r_s)\,r_s^4\,r^3}\,(-1)^{-m}\,\overline{\mathrm{RHz}}_{-1+l,-m,-\omega}[r]$$

$$\Big(-512\,i\,a_s\sqrt{-2-l+l^2}\,M^6\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+512\,i\,a_s\sqrt{-2-l+l^2}\,M^5\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-128\,i\,a_s\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+2112\,i\,a_s\sqrt{-2-l+l^2}\,M^5\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]$$

$$r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+704\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^5\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-704\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^5\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-2112\,i\,a_s\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$704\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+704\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+528\,i\,a_s\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$176\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-176\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-1184\,i\,a_s\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$656\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+656\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^4\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-2240\,a_s\sqrt{-2-l+l^2}\,M^5\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$1184\,i\,a_s\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+656\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-656\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$2240\,a_s\sqrt{-2-l+l^2}\,M^4\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-296\,i\,a_s\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-164\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$164\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-560\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+176\,i\,a_s\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$152\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-152\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^3\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+592\,a_s\sqrt{-2-l+l^2}\,M^4\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$96\,a_s\,l\sqrt{-2-l+l^2}\,M^4\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-96\,a_s\,l^2\sqrt{-2-l+l^2}\,M^4\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-176\,i\,a_s\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$152\,i\,a_s\,l\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+152\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M^2\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-592\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$96\,a_s\,l\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+96\,a_s\,l^2\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+44\,i\,a_s\sqrt{-2-l+l^2}\,M\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$38\,i\,a_s\,l\sqrt{-2-l+l^2}\,M\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-38\,i\,a_s\,l^2\sqrt{-2-l+l^2}\,M\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+148\,a_s\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$24\,a_s\,l\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-24\,a_s\,l^2\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+80\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$48\,a_s\,l\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+48\,a_s\,l^2\sqrt{-2-l+l^2}\,M^3\,\omega\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-448\,i\,a_s\sqrt{-2-l+l^2}\,M^4\,\omega^2\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$80\,a_s\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+48\,a_s\,l\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-48\,a_s\,l^2\sqrt{-2-l+l^2}\,M^2\,\omega\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$448\,i\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega^2\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+20\,a_s\sqrt{-2-l+l^2}\,M\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-12\,a_s\,l\sqrt{-2-l+l^2}\,M\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$12\,a_s\,l^2\sqrt{-2-l+l^2}\,M\,\omega\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-112\,i\,a_s\sqrt{-2-l+l^2}\,M^2\,\omega^2\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^4\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+272\,i\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega^2\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^5\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-$$

$$272\,i\,a_s\sqrt{-2-l+l^2}\,M^2\,\omega^2\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^5\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+68\,i\,a_s\sqrt{-2-l+l^2}\,M\,\omega^2\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^5\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-64\,a_s\sqrt{-2-l+l^2}\,M^3\,\omega^3\sqrt{6\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^6\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+$$

$$64\,a_s\sqrt{-2-l+l^2}\,M^2\,\omega^3\sqrt{6\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^6\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}-16\,a_s\sqrt{-2-l+l^2}\,M\,\omega^3\sqrt{6\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^6\,C^{\{-2,l,m\}}_{-1,1,0,-1,-1+l,m}+1024\,i\,a_s\,M^6\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-$$

$$1024\,i\,a_s\,M^5\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+256\,i\,a_s\,M^4\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-1920\,i\,a_s\,M^5\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-256\,i\,a_s\,l\,M^5\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+$$

$$256\,i\,a_s\,l^2 M^5\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+1920\,i\,a_s\,M^4\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+256\,i\,a_s\,l\,M^4\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-256\,i\,a_s\,l^2 M^4\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-$$

$$480\,i\,a_s\,M^3\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-64\,i\,a_s\,l\,M^3\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+64\,i\,a_s\,l^2 M^3\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+1024\,i\,a_s\,M^4\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+$$

$$320\,i\,a_s\,l\,M^4\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-320\,i\,a_s\,l^2 M^4\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+960\,a_s\,M^5\,\omega\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-1024\,i\,a_s\,M^3\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-$$

$$320\,i\,a_s\,l\,M^3\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+320\,i\,a_s\,l^2 M^3\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-960\,a_s\,M^4\,\omega\sqrt{3\pi}\,r_s\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+256\,i\,a_s\,M^2\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+$$

$$80\,i\,a_s\,l\,M^2\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-80\,i\,a_s\,l^2 M^2\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+240\,a_s\,M^3\,\omega\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^2\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-192\,i\,a_s\,M^3\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-$$

$$96\,i\,a_s\,l\,M^3\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+96\,i\,a_s\,l^2 M^3\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-704\,a_s\,M^4\,\omega\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-144\,a_s\,l\,M^4\,\omega\sqrt{3\pi}\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+$$

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$$24\,i\,a_s\,l\,M\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+24\,i\,a_s\,l^2 M\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-176\,a_s\,M^2\,\omega\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}-36\,a_s\,l\,M^2\,\omega\sqrt{3\pi}\,r_s^2\,\mathrm{DiracDelta}[-r_s+r]\,r^3\,C^{\{-2,l,m\}}_{0,1,0,-2,-1+l,m}+$$

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- The ring absorbs gravitation waves $\Rightarrow$ it oscillates:

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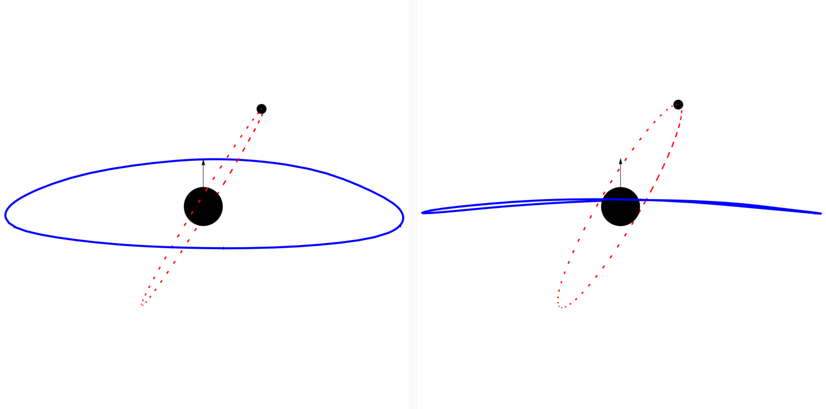
The ring oscillations



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References

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