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Záření v modelech s kosmologickou konstantou

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Diploma thesis

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Radiation in models with cosmological constant

Institute of Theoretical Physics

Supervisor: doc. RNDr. Jiří Podolský, CSc.

Branch of study: Physics–Theoretical Physics

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I declare that I have written my diploma thesis on my own and with exceptional use of quoted sources. I agree with lending of my work.

In Prague on 21st of April 2006 Hedvika Kadlecová

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Abstrakt: *V předložené práci studujeme asymptotickou směrovou strukturu konkrétních přesných řešení Einsteinových rovnic patřících do rozsáhlé rodiny černoděrových prostoročasů typu D s kosmologickou konstantou, nalezených Plebańskim a Demiańskim. Diskutujeme záření v případech prostorového a časového konformního nekonečna. Za účelem další interpretace nalézáme souvislost mezi strukturou zdrojů záření (hmotnost, náboj, NUT parametr, rotační parametr, zrychlení) a vlastnostmi záření, které je jimi generováno.*

Klíčová slova: *rodina řešení Plebańského–Demiańského typu D, asymptotická směrová zaření, konformní geometrie*

Title: *Radiation in models with cosmological constant*

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Abstract: *In this work we study the asymptotic directional structure of specific exact solutions belonging to the large family of black hole spacetimes of type D found by Plebański and Demiański. We discuss the radiation in the case of spacelike and timelike conformal infinity. With the aim of further interpretation, we have found the relation between the structure of the sources (the mass, the charge, the NUT parameter, the rotational parameter, the acceleration) and the properties of radiation generated by them.*

Keywords: *Plebański–Demiański family of solutions of type D, asymptotical directional structure of radiation, conformal geometry*

1 Introduction

Many rigorous theoretical studies have been devoted to investigation of gravitational waves within the full Einstein theory since 1950s. These are described in various review articles, such as [1, 2] and [3]. In many existing analyses the asymptotic flatness has been naturally assumed and the presence of a non-vanishing cosmological constant Λ was not usually considered. The importance of such studies rises due to the fact that the possible presence of a positive Λ has been indicated by recent observations. Moreover, the spacetimes with cosmological constant are now used in various branches of theoretical research, e.g. brane cosmologies, supergravity or string theories.

Krtouš and Podolský recently analyzed the asymptotic directional properties of electromagnetic and gravitational fields in spacetimes with a non-vanishing cosmological constant Λ. It had been known for a long time that the dominant (radiative) component of the fields in spacetimes with $\Lambda \neq 0$ depends on the direction along which a null geodesics approaches a given point at conformal infinity $\mathcal{I}$, contrary to the asymptotically flat spacetimes where the dominant term of radiation is unique.

First, they studied the radiation in the C-metric spacetime with Λ which represents accelerated black holes in de Sitter ($\Lambda > 0$) [4] and anti-de Sitter ($\Lambda < 0$) [5] universe. Their results were summarized and discussed in the topical review [3] where they presented the general asymptotic directional behaviour of any massless field of spin s. The fields of algebraic type D were investigated more explicitly in [6]. Furthermore, the generalization to higher dimensions was recently presented in [7].

The authors demonstrated in [3] that the directional structure of radiation has a universal character which is determined by the algebraic (Petrov) type of the field, namely by the number, degeneracy and specific orientation of the principal null directions with respect to $\mathcal{I}$. It covers all three possibilities $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$, corresponding to a spacelike, timelike or null conformal infinity.

The main intention of this diploma thesis is to apply the above general theory on particular exact model spacetimes of type D and to find the relation between the structure of the sources (namely the mass, the charge, the NUT parameter, the rotation and the acceleration of the corresponding black holes) and the properties of radiation which is generated by them, as observed at spacelike or timelike conformal infinity $\mathcal{I}$. As the specific models we use the Plebański–Demiański family of solutions, first presented in [8]. This class of solutions was studied recently in detail by Griffiths and Podolský in [9, 10, 11] and [12].

The text is organized as follows. In section 2 we review necessary facts about the family of spacetimes of type D found by Plebański and Demiański. In section 3 we first introduce geometrical concepts and objects and we set up the notation which will be followed throughout the text. Then we define tetrads and explicit expression which describes the behavior of the field at any conformal infinity. The relevant work [6] is more explicitly reviewed. In section 4 we apply the general theory introduced in section 3 to particular exact model spacetimes of type D described in section 2. First, we study the Kerr–Newman–de Sitter solution. Then we investigate the general Plebański–Demiański family of solutions, containing not only the mass and the charge of the black holes but also rotation, the NUT parameter and non-zero acceleration. We derive the corresponding radiation near de Sitter or anti-de Sitter-like conformal infinity. In the final subsection 9.5 we discuss the results, in particular the dependence of the amplitude of radiation on the parameters of the sources in various subcases. We hope that this can provide a deeper insight into the general theory of radiation.

2 The Plebański–Demiański class of solutions

The complete family of spacetimes of algebraic type D with an aligned electromagnetic field and a possibly non-zero cosmological constant Λ can be represented in a form of the Plebański–Demiański metric [8, 10] and those specific metrics which can be derived from it by certain limiting procedures (because some particular and well-known cases are not included explicitly in the original form of the Plebański–Demiański metric).

In [10] a new form of this metric was described which is more useful for physical interpretation and for identifying different subfamilies. The parameters employed in the new metric have clear physical interpretation and it is possible to classify the complete family in the way that clarifies their physical properties. The new metric explicitly contains two parameters α and ω which describe the acceleration of the sources and the twist of the principal null congruences. These solutions are characterized by two general quartic functions whose coefficients are related to- the physical parameters of the spacetime. The physical meaning of these coefficients was clarified and the relation between the Plebański–Demiański parameter n and the NUT parameter l was identified. These coefficients were traditionally misinterpreted in the general case.

We will study here the most important solutions from the Plebański–Demiański family. In particular, we wish to investigate the specific radiative properties of (possibly) accelerating black holes with charge and rotation in asymptotically Minkowski, de Sitter or anti–de Sitter universe.

First, we summarize basic facts about these solutions. In [10] the signature $(1,-1,-1,-1)$ of the metric has been used. We use here an opposite signature of the metric because we need to follow the notation of the review [3] for further study of the asymptotic behavior of these solutions. The metrics are thus changed as $\mathbf{g}_{ab} \to -\mathbf{g}_{ab}$. We will also make some other minor changes in the spacetimes forms presented in [9, 10, 11, 12].

2.1 Initial form of the metric

The original Plebański–Demiański metric [8] can be written in the form [10]

$$\begin{aligned}\mathbf{g}_{ab} = \frac{1}{(1-\alpha pr)^2}\Bigg[&\frac{-Q}{\omega^2p^2+r^2}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)^2+\frac{P}{\omega^2p^2+r^2}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)^2\\ &+\frac{\omega^2p^2+r^2}{P}\mathrm{d}p^2+\frac{\omega^2p^2+r^2}{Q}\,\mathrm{d}r^2\Bigg],\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}P(p) &= k+2\omega^{-1}np-\epsilon p^2+2\alpha mp^3-\left[\alpha^2(\omega^2k+e^2+g^2)+\omega^2\Lambda/3\right]p^4,\\ Q(r) &= (\omega^2k+e^2+g^2)-2mr+\epsilon r^2-2\alpha\omega^{-1}nr^3-(\alpha^2k+\Lambda/3)r^4,\end{aligned} \tag{2.2}$$

and m, n, e, g, Λ, ϵ, k, α and ω are arbitrary real parameters of which two can be chosen for convenience. Only the parameters Λ, e and g have their traditional physical interpretation in this metric. The large family of type D spacetimes given by the metric (9.1) admits (at least) two commuting Killing vectors ∂_σ and ∂_τ whose orbits are spacelike for $Q > 0$ and timelike for $Q < 0$. Surfaces on which $Q = 0$ are horizons, points at $P = 0$ are poles (axes).

Plebański and Demiański considered in their original work [8] that $\alpha = 1$ and $\omega = 1$. In [10, 11] it was shown that the physical interpretation of the parameters can be determined more easily when α and ω are retained as continuous parameters and ϵ and k are set to convenient values (without changing their signs).

It is thus appropriate to describe the full family of solutions in terms of seven continuous parameters m, n, e, g, Λ, α and ω and two auxiliary ones ϵ and k which can be set conveniently. Their meaning is as following:

- e and g are the electric and the magnetic charges
- Λ is the cosmological constant
- α is the acceleration of the sources
- ω is the twist parameter
- m is related to the mass of the source
- n is the Plebański and Demiański parameter.

In some particular cases ω is directly related to both the angular velocity of sources and the effects of the NUT parameter. The metric is flat when $m = n = 0$, $e = g = 0$ and $\Lambda = 0$ (see equation (9.14) below). The remaining parameters ϵ, k, α and ω may be non-zero even in this flat limit. The well-known solutions such as the Schwarzschild–de Sitter, the Reissner–Nordström, the Kerr metric, the NUT solution or the C-metric, and other type D spaces are also included: the simple transformation (9.16) leads to a form that explicitly includes all these well-known special cases, see [10, 11] for more details and section 9.2 for demonstration of this fact.

2.1.1 Natural null tetrad

In [10] the null tetrad was expressed in the coordinates (τ, σ, p, r) using the signature $(1, -1, -1, -1)$. For our later purposes, we need to have the null tetrad with opposite signature of the metric. Moreover, it is necessary to introduce the null tetrad both for $Q > 0$ and for $Q < 0$ because the sign of this function Q is different near $r \to \infty$, as

$$\begin{aligned} Q < 0 \quad &\text{if} \quad (\alpha^2 k + \Lambda/3) > 0, \\ Q > 0 \quad &\text{if} \quad (\alpha^2 k + \Lambda/3) < 0. \end{aligned} \tag{2.3}$$

The function Q thus may become negative under the square root near the conformal infinity where we wish to study radiative properties and therefore we need to consider also the case $Q < 0$. Concretely, for $\alpha = 0$ the sign of Q depends only on the sign of Λ. After some calculations outlined below we found the following null tetrads in the metric signature $(-1,1,1,1)$:

The null tetrad for $Q<0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{-Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{-Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{1}{\sqrt{-Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{-Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right),\\
\overline{\mathbf{m}}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right).
\end{aligned}
\tag{2.4}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{1-\alpha pr}\left(-\sqrt{\frac{-Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{-2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{-Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{-2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{1-\alpha pr}\left(i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{1-\alpha pr}\left(-i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right).
\end{aligned}
\tag{2.5}
$$

The null tetrad for $Q>0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{Q}}(r^2\partial_\tau-\omega\partial_\sigma)-\sqrt{Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right),\\
\overline{\mathbf{m}}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right).
\end{aligned}
\tag{2.6}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)+\sqrt{\frac{\omega^2p^2+r^2}{2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{1-\alpha pr}\left(i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{1-\alpha pr}\left(-i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right).
\end{aligned}
\tag{2.7}
$$

When deriving these null tetrads we began with the null tetrad presented in [10]. It was sufficient to transform only the vectors, the covariant one-forms were easily obtained by lowering the indeces, $\mathbf{k}_a = \mathbf{g}_{ab}\mathbf{k}^b$, etc., using the metric (9.1). We expected the changes to occur only in the signs of the two parts of each vector. We thus added coefficients in front of each part of the vector with possible values ± 1 in $\mathbf{k}_a, \mathbf{l}_a$, and also $\pm i$ in $\mathbf{m}_a, \overline{\mathbf{m}}_a$. The new vectors must satisfy the basic relations for the null tetrad, namely that the scalar products of the tetrad vectors all vanish, except for

$$\mathbf{k}^a\mathbf{l}_a = -1, \tag{2.8}$$

$$\mathbf{m}^a\overline{\mathbf{m}}_a = +1. \tag{2.9}$$

We thus obtained several conditions for the vector coefficients.

For the case when $Q < 0$, the condition (9.8) implies four possible combinations of the coefficients, two of them are equivalent as

$$(-\mathbf{k}^a)(-\mathbf{k}_a) = \mathbf{k}^a\,\mathbf{k}_a, \quad (-\mathbf{l}^a)(-\mathbf{l}_a) = \mathbf{l}^a\,\mathbf{l}_a, \quad \mathbf{k}^a(-\mathbf{l}_a) = (-\mathbf{k}^a)\,\mathbf{l}_a. \tag{2.10}$$

Each two pairs of equivalent vectors contain a combination of future oriented null vectors $\mathbf{k}^a, \mathbf{l}^a$ and a combination of past oriented vectors $\mathbf{k}^a, \mathbf{l}^a$ which differ only in the sign in front of the first part $(r^2\,\partial_\tau - \omega\partial_\sigma)$ of the vectors. The future/past orientation means that plus/minus stands in front of the second part ∂_r of the vector. The sign in front of the first part is thus eligible, of course, only ± 1. It corresponds to two possible orientations of the two null vectors aligned along the light cone. We have chosen the future orientation. The condition (9.9) implies two possible choices of vectors $\mathbf{m}_a, \overline{\mathbf{m}}_a$ which are completely independent of the condition (9.8), so we have chosen one orientation arbitrarily.

For the case $Q > 0$, the condition (9.8) again implies four combinations of the signs, of which two are equivalent but the two pairs contain one vector future oriented and one past oriented. We can not choose two vectors oriented in the same way as in the case $Q < 0$. We have rather chosen the vectors according to their first parts: we took the choice such that the first part of the vector $\mathbf{k}^a$ remains the same as in the case $Q < 0$. The vectors $\mathbf{m}_a, \overline{\mathbf{m}}_a$ were chosen the same as in the case $Q < 0$.

The above null tetrads $\{\mathbf{e}_A\} = \{\mathbf{m}^a, \overline{\mathbf{m}}^a, \mathbf{l}^a, \mathbf{k}^a\}$ given by (9.4) or (9.6) satisfy further relations (see also [13]). The vectors are related to the dual one-forms by the non-diagonal metric $\mathbf{g}_{AB}$ whose components are

$$\mathbf{g}_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad {\mathbf{g}_{AB}}^{-1} \equiv \mathbf{g}^{AB} = \mathbf{g}_{AB}. \tag{2.11}$$

In practice, the related one-forms to our vectors are obtained by the following algorithm:

1. Set up the basis of the vectors and the basis of the dual forms $\boldsymbol{\omega}^A$ as

$$\begin{aligned} \boldsymbol{e}_1 &= \mathbf{m}^a, & \boldsymbol{\omega}^1 &= \mathbf{m}_a, \\ \boldsymbol{e}_2 &= \overline{\mathbf{m}}^a, & \boldsymbol{\omega}^2 &= \overline{\mathbf{m}}_a, \\ \boldsymbol{e}_3 &= \mathbf{l}^a, & \boldsymbol{\omega}^3 &= \mathbf{l}_a, \\ \boldsymbol{e}_4 &= \mathbf{k}^a, & \boldsymbol{\omega}^4 &= \mathbf{k}_a. \end{aligned}$$

2. Insert all vectors into each one-form and find to which vector the one-form is associated, e.g. $\boldsymbol{\omega}^1(\mathbf{e}_1) = 1$. If -1 is obtained, change the sign of the given one-form.

3. Finally, confirm that the relations (9.8) and (9.9) are satisfied.

We have thus identified the one-forms associated to the vectors (9.4) and (9.6) as

$$\begin{aligned} \boldsymbol{\mu} &\equiv \boldsymbol{\omega}^1 = \mathbf{m}_a, \\ \overline{\boldsymbol{\mu}} &\equiv \boldsymbol{\omega}^2 = \overline{\mathbf{m}}_a, \\ \boldsymbol{\lambda} &\equiv -\boldsymbol{\omega}^4 = -\mathbf{k}_a, \\ \boldsymbol{\kappa} &\equiv -\boldsymbol{\omega}^3 = -\mathbf{l}_a, \end{aligned} \tag{2.12}$$

where we introduced a new (temporary) notation for the dual one-forms which satisfy

$$\begin{aligned} \boldsymbol{\mu}(\mathbf{e}_1) &= 1, \\ \overline{\boldsymbol{\mu}}(\mathbf{e}_2) &= 1, \\ \boldsymbol{\kappa}(\mathbf{e}_3) &= 1, \\ \boldsymbol{\lambda}(\mathbf{e}_4) &= 1. \end{aligned} \tag{2.13}$$

Consequently, $\{\mathbf{m}^a, \overline{\mathbf{m}}^a, \mathbf{l}^a, \mathbf{k}^a\}$ is dual to $\{\boldsymbol{\mu}, \overline{\boldsymbol{\mu}}, \boldsymbol{\kappa}, \boldsymbol{\lambda}\}$.

2.1.2 Weyl and Ricci tensors

It can be calculated (for example by using the Maple package *GR tensor*) that the only non-zero component of the Weyl in the tetrad (9.4) and (9.6) is given by

$$\Psi_2 = -(m+in)\left(\frac{1-\alpha pr}{r+i\omega p}\right)^3 + (e^2+g^2)\left(\frac{1-\alpha pr}{r+i\omega p}\right)^3 \frac{1+\alpha pr}{r-i\omega p}. \tag{2.14}$$

This confirms that the spacetimes studied are of algebraic type D and that the tetrad vectors $\mathbf{l}^a$ and $\mathbf{k}^a$ are aligned with the principal null directions of the Weyl tensor. The only non-zero component of the Ricci tensor is

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(1-\alpha pr)^4}{(\omega^2p^2+r^2)^2}. \tag{2.15}$$

These components of the curvature tensor indicate that there is a curvature singularity at $p = r = 0$. This singularity can be considered as the "source" of gravitational field. It is necessary that $P > 0$ to retain a Lorentzian signature in metrics. $P(p)$ is generally a quartic function, so the coordinate p must be restricted to a particular range between appropriate roots. If this range includes $p = 0$, it is necessary that $k > 0$. The condition $P > 0$ thus places restriction on the possible signs of the parameters ε and k. Note also, that there are non-singular solutions in the Plebański–Demiański family when $\omega \neq 0$.

2.2 More general form of the metric

The metric (9.1) does not include the type D non-singular NUT solution. It is necessary to introduce a specific shift in the coordinate p to cover such cases with the NUT parameter. This procedure is also essential to obtain the correct metric for accelerating and rotating black holes. We will start with the metric (9.1) and perform the coordinate transformation

$$p = \frac{l}{\omega} + \frac{a}{\omega}\tilde{p}, \qquad \tau = \tilde{t} - \frac{(l+a)^2}{a}\tilde{\phi}, \qquad \sigma = -\frac{\omega}{a}\tilde{\phi}, \tag{2.16}$$

where a and l are new arbitrary parameters. These solutions have two parameters k and ϵ which can be scaled to any convenient values. In addition, we have the further parameters a and l which can be chosen arbitrarily. In practice, it is convenient to choose a and l to satisfy certain conditions which simplify the form of the metric, and then to re-express n and ω in terms of these parameters. We will show that the parameter a corresponds to a Kerr-like rotation parameter and l corresponds to a NUT parameter. The properties of the solution depend on the character of the function P. It becomes $\tilde{P} = P\frac{\omega^2}{a^2}$ after transformation (9.16). This function is a quartic function and can have up to four distinct roots.

We will consider the case when $\tilde{P}$ has two roots $\tilde{p} = \pm 1$ and $\tilde{p}$ covers the range between these roots. The function $\tilde{P}$ has a simplified form

$$\tilde{P} = (1 - \tilde{p}^2)(a_0 - a_3\tilde{p} - a_4\tilde{p}^2). \tag{2.17}$$

Then the conditions specifying the two parameters ϵ and n in terms of a and l are

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}m - (a^2 + 3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right], \tag{2.18}$$

$$n = \frac{\omega^2 k l}{a^2 - l^2} - \alpha\frac{a^2 - l^2}{\omega}m + (a^2 - l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right]. \tag{2.19}$$

(see [9], [10], [11] for more details). The next equation defines the parameter k for any given value of a_0 as

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right) k = a_0 + 2\alpha\frac{l}{\omega}m - 3\alpha^2\frac{l^2}{\omega^2}(e^2 + g^2) - l^2\Lambda. \tag{2.20}$$

These constraints define the values of k, n and ϵ, the remaining scaling freedom is in the parameter ω which we can set to any convenient value with assumption that a and l do not both vanish. The remaining parameters are thus α, a and l in addition to m, e, g and Λ. We will concentrate on the physically most relevant case for which $\tilde{P}$ has two roots and $a_0 > 0$. The scaling freedom can be used to set $a_0 = 1$.

The coordinate $\tilde{p}$ covers the range between the roots $\tilde{p} = \pm 1$ and we put $\tilde{p} = \cos\vartheta$, where $\vartheta \in [0, \pi]$. The transformation (9.16) from the metric (9.1) to new general metric has thus the form

$$p = \frac{l}{\omega} + \frac{a}{\omega}\cos\vartheta, \qquad \tau = \tilde{t} - \frac{(l + a)^2}{a}\tilde{\phi}, \qquad \sigma = -\frac{\omega}{a}\tilde{\phi}. \tag{2.21}$$

Then the metrics (9.1) becomes

$$\begin{aligned}\mathbf{g}_{ab} = \frac{1}{\Omega^2}\Bigg[&-\frac{Q}{\rho^2}(\mathrm{d}\tilde{t} - (a\sin^2\vartheta + 4l\sin^2\tfrac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2 + \frac{\rho^2}{\hat{P}}\mathrm{d}\vartheta^2 + \frac{\rho^2}{Q}\,\mathrm{d}r^2 \\ &+ \frac{\hat{P}\sin^2\vartheta}{\rho^2}(\mathrm{d}\tilde{t} - (r^2 + (a + l)^2)\mathrm{d}\tilde{\phi})^2\Bigg],\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\Omega &= 1 - \frac{\alpha}{\omega}(l + a\cos\vartheta)r, \\ \rho^2 &= r^2 + (l + a\cos\vartheta)^2, \\ \hat{P} &\equiv \frac{\tilde{P}}{\sin^2\vartheta} = 1 - a_3\cos\vartheta - a_4\cos^2\vartheta, \\ Q &= (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha\omega^{-1}nr^3 - (\alpha^2 k + \Lambda/3)r^4,\end{aligned} \tag{2.23}$$

and

$$a_3 = 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k + e^2 + g^2) - 4\frac{\Lambda}{3}al, \tag{2.24}$$

$$a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + e^2 + g^2) - \frac{\Lambda}{3}a^2. \tag{2.25}$$

with ϵ, n and k given by (9.18)-(9.20).

This solution represents *the complete family of black hole spacetimes* and contains eight arbitrary parameters $m, e, g, a, l, \alpha, \Lambda$ and ω. The first seven of these parameters can be varied independently and ω can be set to any convenient value if a or l are not both zero. It was shown in [11] that the metric (9.22) represents for $\Lambda = 0$ accelerating and rotating charged black holes with a generally non-zero NUT parameter. The authors found subsequently in [12] that there is analytical extension of the metric which was expressed in terms of the Weyl–Lewis–Papapetrou metric and the boost-rotation form.

This more general form of metric (9.22) contains various special cases which can be obtained explicitly by setting certain parameters to zero. These will be reviewed below in subsections 9.2.1-9.2.3.

By performing the transformation (9.21) from the coordinates (τ, σ, p, r) to $(\tilde{t}, \tilde{\phi}, \vartheta, r)$, the null tetrads (9.4) and (9.5), (9.6) and (9.7), the Weyl tensor and the Ricci tensor will change their forms:

The vectors transform by "inverse" transformation of (9.21),

$$\begin{aligned} \partial_\sigma &= -\frac{a}{\omega}(\partial_{\tilde{\phi}} + \frac{(l+a)^2}{a}\partial_{\tilde{t}}), \\ \partial_p &= -\frac{\omega}{a}\frac{1}{\sin\vartheta}\partial_\vartheta, \\ \partial_\tau &= \partial_{\tilde{t}}, \end{aligned} \tag{2.26}$$

and the one-forms transform by "straight" differentiation of (9.21)

$$\begin{aligned} \mathrm{d}p &= -\frac{a}{\omega}\sin\vartheta \mathrm{d}\vartheta, \\ \mathrm{d}\sigma &= -\frac{\omega}{a}\mathrm{d}\tilde{\phi}, \\ \mathrm{d}\tau &= \mathrm{d}\tilde{t} - \frac{(l+a)^2}{a}\mathrm{d}\tilde{\phi}. \end{aligned} \tag{2.27}$$

When we substitute these relations into (9.4) and (9.5), we obtain *the null tetrad vectors for the case $Q < 0$:*

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{-Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{-Q}\,\partial_r\right), \\
\mathbf{l}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{1}{\sqrt{-Q}}[(a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{-Q}\,\partial_r\right), \\
\mathbf{m}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]) + \sqrt{\hat{P}}\,\partial_\vartheta\right), \\
\overline{\mathbf{m}}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right).
\end{aligned}
\tag{2.28}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Omega}\left(-\sqrt{\frac{-Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] - \frac{\rho}{\sqrt{-2Q}}\mathrm{d}r\right), \\
\mathbf{l}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{-Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] - \frac{\rho}{\sqrt{-2Q}}\mathrm{d}r\right), \\
\mathbf{m}_a &= \frac{1}{\Omega}\left(i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2+(l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right), \\
\overline{\mathbf{m}}_a &= \frac{1}{\Omega}\left(-i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2+(l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right).
\end{aligned}
\tag{2.29}
$$

When we substitute these relations into (9.6) and (9.7), we get *the null tetrad for* $Q > 0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{Q}\,\partial_r\right), \\
\mathbf{l}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] - \sqrt{Q}\,\partial_r\right), \\
\mathbf{m}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right), \\
\overline{\mathbf{m}}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right).
\end{aligned}
\tag{2.30}
$$

The corresponding one–forms are

$$\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2Q}}\mathrm{d}r\right), \\
\mathbf{l}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] - \frac{\rho}{\sqrt{2Q}}\mathrm{d}r\right), \\
\mathbf{m}_a &= \frac{1}{\Omega}\left(i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2 + (l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right), \\
\overline{\mathbf{m}}_a &= \frac{1}{\Omega}\left(-i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2 + (l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right).
\end{aligned} \tag{2.31}$$

The only non-zero component of the Weyl tensor in these tetrads is given by

$$\Psi_2 = \left[-(m+in) + (e^2+g^2)\left(\frac{1+\frac{\alpha}{\omega}r(l+a\cos\vartheta)}{r-i(l+a\cos\vartheta)}\right)\right]\left(\frac{1-\frac{\alpha}{\omega}r(l+a\cos\vartheta)}{r+i(l+a\cos\vartheta)}\right)^3. \tag{2.32}$$

The only non-zero component of the Ricci tensor is

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(1-\frac{\alpha}{\omega}r(l+a\cos\vartheta))^4}{(r^2+(l+a\cos\vartheta)^2)^2}. \tag{2.33}$$

2.2.1 Kerr–Newman–NUT–de Sitter spacetime ($\alpha = 0$)

We obtain this particular case when we set $\alpha = 0$. The constraint (9.20) becomes $\omega^2 k = (1-l^2\Lambda)(a^2-l^2)$, the relations (9.18) and (9.19) become

$$\epsilon = 1 - (\tfrac{1}{3}a^2 + 2l^2)\Lambda, \quad n = l + \tfrac{1}{3}(a^2 - 4l^2)l\Lambda. \tag{2.34}$$

The metric is the same as (9.22) with

$$\begin{aligned}
&\Omega = 1 \\
&\rho^2 = r^2 + (l+a\cos\vartheta)^2 \\
&\hat{P} = 1 + \tfrac{4}{3}\Lambda al\cos\vartheta + \tfrac{1}{3}\Lambda a^2\cos^2\vartheta \\
&Q = (a^2 - l^2 + e^2 + g^2) - 2mr + r^2 - \Lambda\left[(a^2-l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\right].
\end{aligned}$$

This solution represents a *non-accelerating black hole* with mass m, electric and magnetic charges e, g,
rotational parameter a and a NUT parameter l in Minkowski, de Sitter or anti–de Sitter background. It reduces to well-known forms when $l = 0$ or $a = 0$ or $\Lambda = 0$. However, it is necessary to distinguish the two cases in which $|a|$ is greater or less than $|l|$. When $a^2 \geq l^2$, $k \geq 0$, the metric has Kerr like ring singularity

at $r = 0$. This case represents a Kerr–Newman–de Sitter solution, it means a charged black hole with a small NUT parameter. This solution will be discussed properly in the next section. Alternatively, when $a^2 < l^2$, $k < 0$, the metric is singularity free. This case is best described as a charged NUT–de Sitter solution with a small Kerr-like rotation. Although these cases have identical metric forms, their singularity and global structures differ substantially.

2.2.2 Accelerating Kerr–Newman–de Sitter black holes ($l = 0$)

This is the case where the NUT parameter vanishes but α is arbitrary. Now the equation (9.20) implies that $\omega^2 k = a^2$. We use the remaining scaling freedom in ω to set $k = 1$ so then $\omega = a$ (the Kerr rotation parameter) and from the constraints (9.18)-(9.20) we obtain

$$\epsilon = 1 - \alpha^2(a^2 + e^2 + g^2) - \tfrac{1}{3}\Lambda a^2, \quad k = 1, \quad n = -\alpha a m.$$

Interestingly, while the NUT parameter vanishes, the Plebański–Demiański parameter n is not zero. The metric (9.22) becomes

$$\begin{aligned} \mathbf{g}_{ab} = \frac{1}{\Omega^2}\Bigg[& \frac{-Q}{\rho^2}(\mathrm{d}\tilde{t} - a\sin^2\vartheta \mathrm{d}\tilde{\phi})^2 + \frac{\hat{P}\sin^2\vartheta}{\rho^2}(a\mathrm{d}\tilde{t} - (r^2+a^2)\mathrm{d}\tilde{\phi})^2 \\ & + \frac{\rho^2}{\hat{P}}\mathrm{d}\,\vartheta^2 + \frac{\rho^2}{Q}\,\mathrm{d}r^2\Bigg], \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} \Omega &= 1 - \alpha r\cos\vartheta, \\ \rho^2 &= r^2 + a^2\cos^2\vartheta, \\ \hat{P} &= 1 - 2\alpha m\cos\vartheta + [\alpha^2(a^2+e^2+g^2) + \tfrac{1}{3}\Lambda a^2]\cos^2\vartheta, \\ Q &= (\omega^2 k + e^2 + g^2) - 2mr + r^2 - \frac{1}{3}\Lambda r^2(r^2+a^2). \end{aligned}$$

The only non-zero components of the curvature tensor are now given by

$$\Psi_2 = \left[-m(1 - i\alpha a) + (e^2+g^2)\frac{1+\alpha r\cos\vartheta}{r - ia\cos\vartheta}\right]\left(\frac{1-\alpha r\cos\vartheta}{r + ia\cos\vartheta}\right)^3, \tag{2.36}$$

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(1-\alpha r\cos\vartheta)^4}{(r^2 + a^2\cos^2\vartheta)^2}. \tag{2.37}$$

This metric (9.35) clearly exhibits the singularity and horizon structure of an *accelerating charged and rotating black hole* in Minkowski, de Sitter or anti-de Sitter background. It represents the spacetime from the singularity through the inner and outer black hole horizons and out to the acceleration horizon. Nevertheless, it does not cover the complete analytic extension inside the black hole horizon. In [12] it was shown for the case $\Lambda = 0$ that the complete spacetime contains two causally separated charged and rotating black holes which accelerate away from each other in opposite spatial directions. Special subcase of (9.35) is *the charged C-metric with a cosmological constant*. We get this when we put $a = 0$, and the metric (9.35) reduces to the simple diagonal form

$$\mathbf{g}_{ab} = \frac{1}{(1-\alpha r\cos\vartheta)^2}\left(-\frac{Q}{r^2}\mathrm{d}\tilde{t}^2 + \frac{r^2}{Q}\mathrm{d}r^2 + \hat{P}r^2\sin^2\vartheta\,\mathrm{d}\tilde{\phi}^2 + \frac{r^2}{\hat{P}}\mathrm{d}\vartheta^2\right), \tag{2.38}$$

where

$$\begin{aligned}\hat{P} &= 1 - 2\alpha m\cos\vartheta + \alpha^2(e^2+g^2)\cos^2\vartheta,\\ Q &= (e^2+g^2-2mr+r^2)(1-\alpha^2r^2) - \tfrac{1}{3}\Lambda r^4.\end{aligned}$$

For $\Lambda = 0$, it describes a pair of black holes of mass m and electric and magnetic charges e and g which accelerate towards infinity under the action of forces represented by a conical singularity, where α is the acceleration. The acceleration horizon is $r = \alpha^{-1}$. The location of other horizons depends on e, g, m and Λ.

2.2.3 Kerr–Newman–de Sitter spacetime ($\alpha = l = 0$)

This is obviously the $l = 0$ subcase of the spacetime discussed in subsection 9.2.1. It can be written in the standard form of the Kerr–Newman–de Sitter solution in Boyer–Lindquist coordinates by a simple rescaling transformation

$$\tilde{t} = t\,\Xi^{-1}, \quad \tilde{\phi} = \phi\,\Xi^{-1}, \tag{2.39}$$

where $\Xi = 1 + \frac{1}{3}\Lambda a^2$. This transformation leads us to

$$\begin{aligned}\mathbf{g}_{ab} = \frac{-\Delta_r}{\Xi^2\rho^2}\left[\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\phi\right]^2 &+ \frac{\Delta_\vartheta\sin^2\vartheta}{\Xi^2\rho^2}\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2\\ &+ \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2,\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\vartheta,\\ \Delta_r \equiv Q &= (r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2) - 2mr + (e^2+g^2),\\ \Delta_\vartheta \equiv \hat{P} &= 1 + \tfrac{1}{3}\Lambda a^2\cos^2\vartheta,\end{aligned} \tag{2.41}$$

In fact, there is no need to introduce the constant rescaling Ξ in t and ϕ. But it is included so that the metric has well-behaved axis at $\vartheta = 0$ and $\vartheta = \pi$ with $\phi \in [0, 2\pi)$.

Notice that there exists a direct transformation from the initial metric (9.1) to the metric (9.40): by inserting (9.39) into (9.16) we get

$$\begin{aligned}p &= \cos\vartheta,\\ \tau &= (t - a\phi)\,\Xi^{-1},\\ \sigma &= -\phi\,\Xi^{-1}.\end{aligned} \tag{2.42}$$

(Here we assume $\alpha = 0$, $l = 0$ and we set $a = \omega$, $\epsilon = 1 - \frac{1}{3}\Lambda a^2$, $k = 1$, $n = 0$).

Now we can directly rewrite the null tetrads (9.4) and (9.6), as well as the Weyl tensor and the Ricci tensor into coordinates of the Kerr–Newman–de Sitter solution (9.40). The vectors transform by inverse transformation of (9.42),

$$\begin{aligned} \partial_\sigma &= -\Xi(\partial_\phi + a\partial_t), \\ \partial_p &= -\frac{1}{\sin\vartheta}\partial_\vartheta, \\ \partial_\tau &= \Xi\,\partial_t, \end{aligned} \tag{2.43}$$

while the one-forms transform by

$$\begin{aligned} \mathrm{d}p &= -\sin\vartheta\,\mathrm{d}\vartheta, \\ \mathrm{d}\sigma &= -\Xi^{-1}\mathrm{d}\phi, \\ \mathrm{d}\tau &= \Xi^{-1}(\mathrm{d}t - a\mathrm{d}\phi). \end{aligned} \tag{2.44}$$

Substituting these relations into (9.4) and (9.5), we finally obtain *the null tetrad vectors for the case* $\Lambda > 0$*:*

The vectors are

$$\begin{aligned} \mathbf{k}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{-\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{-\Delta_r}\,\partial_r\right), \\ \mathbf{l}^a &= \frac{1}{\sqrt{2}\rho}\left(+\frac{\Xi}{\sqrt{-\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{-\Delta_r}\,\partial_r\right), \\ \mathbf{m}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta - i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right), \\ \overline{\mathbf{m}}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta + i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right), \end{aligned} \tag{2.45}$$

and the corresponding one-forms

$$\begin{aligned} \mathbf{k}_a &= -\frac{1}{\Xi\rho}\sqrt{\frac{-\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{-2\Delta_r}}\,\mathrm{d}r, \\ \mathbf{l}_a &= +\frac{1}{\Xi\rho}\sqrt{\frac{-\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{-2\Delta_r}}\,\mathrm{d}r, \\ \mathbf{m}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta + \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi), \\ \overline{\mathbf{m}}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta - \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi). \end{aligned} \tag{2.46}$$

Substituting these relations (9.43), (9.44) into (9.6) and (9.7), we analogously obtain *the null tetrad vectors for the case* $\Lambda < 0$*:*

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{\Delta_r}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] - \sqrt{\Delta_r}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta - i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right),\\
\overline{\mathbf{m}}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta + i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right),
\end{aligned}
\tag{2.47}
$$

and the corresponding one-forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Xi\rho}\sqrt{\frac{\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) + \frac{\rho}{\sqrt{2\Delta_r}}\,\mathrm{d}r,\\
\mathbf{l}_a &= \frac{1}{\Xi\rho}\sqrt{\frac{\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{2\Delta_r}}\,\mathrm{d}r,\\
\mathbf{m}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta + \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi),\\
\overline{\mathbf{m}}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta - \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi).
\end{aligned}
\tag{2.48}
$$

The only non-zero component of the Weyl tensor (9.14) is now

$$
\Psi_2 = -\frac{m}{(r+ia\cos\vartheta)^3} + \frac{e^2+g^2}{(r+ia\cos\vartheta)^3(r-ia\cos\vartheta)},
\tag{2.49}
$$

and the Ricci tensor (9.15) becomes

$$
\Phi_{11} = \frac{1}{2}\frac{e^2+g^2}{(r^2+a^2\cos^2\vartheta)}.
\tag{2.50}
$$

2.3 Alternative form of the general metric

For our later purposes it is also convenient to introduce a new coordinate q which is simply related to r by

$$
q = -\frac{1}{r}.
\tag{2.51}
$$

The metric (9.22) then becomes

$$
\begin{aligned}
\mathbf{g}_{ab} = \frac{1}{\mathbf{\Omega}^2}\Bigg[&\frac{-\mathcal{Q}}{\varrho^2}(\mathrm{d}\tilde{t} - (a\sin^2\vartheta + 4l\sin^2\frac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2 + \frac{\varrho^2}{\mathcal{P}}\mathrm{d}\vartheta^2 + \frac{\varrho^2}{\mathcal{Q}}\mathrm{d}q^2\\
&+ \frac{\mathcal{P}\sin^2\vartheta}{\varrho^2}(aq^2\mathrm{d}\tilde{t} - [1+(a+l)^2q^2]\mathrm{d}\tilde{\phi})^2\Bigg],
\end{aligned}
\tag{2.52}
$$

where

$$
\begin{aligned}
\mathbf{\Omega} \equiv & q\,\Omega = -(q + \frac{\alpha}{\omega}(l + a\cos\vartheta)), \\
\varrho^2 = & q^2\,\rho^2 = 1 + q^2(l + a\cos\vartheta)^2, \\
\mathcal{P}(\vartheta) \equiv & \hat{P} = 1 - a_3\cos\vartheta - a_4\cos^2\vartheta, \\
\mathcal{Q}(q) \equiv & q^4\,Q = -(\alpha^2 k + \Lambda/3) + 2\alpha\omega^{-1} n q + \epsilon q^2 + 2mq^3 + (\omega^2 k + e^2 + g^2)q^4,
\end{aligned} \tag{2.53}
$$

and the coefficients are as in (9.24), (9.25)

$$
a_3 = 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k + e^2 + g^2) - 4\frac{\Lambda}{3}al, \tag{2.54}
$$

$$
a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + e^2 + g^2) - \frac{\Lambda}{3}a^2. \tag{2.55}
$$

with ϵ, n and k again given by (9.18)-(9.20).

We obtain the null tetrads by performing the simple transformation (9.51) in (9.28) and (9.29), (9.30) and (9.31). Then the vectors and their dual forms are expressed in the coordinates $(\tilde{t}, \tilde{\phi}, \vartheta, q)$.

The null tetrad for $Q < 0$:

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1 + (l+a)^2 q^2]\partial_{\tilde{t}}) + \sqrt{-\mathcal{Q}}\,\partial_q\right), \\
\mathbf{l}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1 + (l+a)^2 q^2]\partial_{\tilde{t}}) + \sqrt{-\mathcal{Q}}\,\partial_q\right), \\
\mathbf{m}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-i}{\sqrt{\mathcal{P}}\sin\vartheta}((4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}) + \sqrt{\mathcal{P}}\,\partial_\vartheta\right), \\
\overline{\mathbf{m}}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{i}{\sqrt{\mathcal{P}}\sin\vartheta}((4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}) + \sqrt{\mathcal{P}}\,\partial_\vartheta\right).
\end{aligned} \tag{2.56}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\mathbf{\Omega}}\left(-\sqrt{\frac{-\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}) - \frac{\varrho}{\sqrt{-2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{l}_a &= \frac{1}{\mathbf{\Omega}}\left(+\sqrt{\frac{-\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}) - \frac{\varrho}{\sqrt{-2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{m}_a &= \frac{1}{\mathbf{\Omega}}\left(+i\sqrt{\frac{\mathcal{P}}{2}}\frac{\sin\vartheta}{\varrho}(aq^2\mathrm{d}\tilde{t} - [1 + (l+a)^2 q^2]\mathrm{d}\tilde{\phi}) + \frac{\varrho}{\sqrt{2\mathcal{P}}}\mathrm{d}\vartheta\right), \\
\overline{\mathbf{m}}_a &= \frac{1}{\mathbf{\Omega}}\left(-i\sqrt{\frac{\mathcal{P}}{2}}\frac{\sin\vartheta}{\varrho}(aq^2\mathrm{d}\tilde{t} - [1 + (l+a)^2 q^2]\mathrm{d}\tilde{\phi}) + \frac{\varrho}{\sqrt{2\mathcal{P}}}\mathrm{d}\vartheta\right).
\end{aligned} \tag{2.57}
$$

The null tetrad for $Q > 0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\boldsymbol{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2]q^2\partial_{\tilde{t}}) + \sqrt{\mathcal{Q}}\,\partial_q\right), \\
\mathbf{l}^a &= \frac{\boldsymbol{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) - \sqrt{\mathcal{Q}}\,\partial_q\right), \\
\mathbf{m}^a &= \frac{\boldsymbol{\Omega}}{\sqrt{2}\varrho}\left(\frac{-i}{\sqrt{\mathcal{P}}\sin\vartheta}((4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}) + \sqrt{\mathcal{P}}\,\partial_\vartheta\right), \\
\overline{\mathbf{m}}^a &= \frac{\boldsymbol{\Omega}}{\sqrt{2}\varrho}\left(\frac{i}{\sqrt{\mathcal{P}}\sin\vartheta}((4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}) + \sqrt{\mathcal{P}}\,\partial_\vartheta\right).
\end{aligned}
\tag{2.58}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\boldsymbol{\Omega}}\left(\sqrt{\frac{\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}) + \frac{\varrho}{\sqrt{2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{l}_a &= \frac{1}{\boldsymbol{\Omega}}\left(\sqrt{\frac{\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}) - \frac{\varrho}{\sqrt{2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{m}_a &= \frac{1}{\boldsymbol{\Omega}}\left(+i\sqrt{\frac{\mathcal{P}}{2}}\frac{\sin\vartheta}{\varrho}(aq^2\mathrm{d}\tilde{t} - [1+(l+a)^2q^2]\mathrm{d}\tilde{\phi}) + \frac{\varrho}{\sqrt{2\mathcal{P}}}\mathrm{d}\vartheta\right), \\
\overline{\mathbf{m}}_a &= \frac{1}{\boldsymbol{\Omega}}\left(-i\sqrt{\frac{\mathcal{P}}{2}}\frac{\sin\vartheta}{\varrho}(aq^2\mathrm{d}\tilde{t} - [1+(l+a)^2q^2]\mathrm{d}\tilde{\phi}) + \frac{\varrho}{\sqrt{2\mathcal{P}}}\mathrm{d}\vartheta\right).
\end{aligned}
\tag{2.59}
$$

The non-zero component of the Weyl in these tetrads is given by

$$
\Psi_2 = \left[(m+in) + (e^2+g^2)\left(\frac{q - \frac{\alpha}{\omega}(l+a\cos\vartheta)}{1+iq(l+a\cos\vartheta)}\right)\right]\left(\frac{q + \frac{\alpha}{\omega}(l+a\cos\vartheta)}{1-iq(l+a\cos\vartheta)}\right)^3.
\tag{2.60}
$$

The non-zero component of the Ricci tensor is

$$
\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(q + \frac{\alpha}{\omega}(l+a\cos\vartheta))^4}{(1+q^2(l+a\cos\vartheta)^2)^2}.
\tag{2.61}
$$

As we will demonstrate, this alternative form of the metric (2.52) is more useful for investigation of the directional structure of radiation near conformal infinity.

3 Asymptotic directional structure of radiation for fields of algebraic type D

In the next few sections we will review basic facts about the asymptotic properties of a physical spacetime and its conformally related spacetime which will be necessary for our further investigations.

3.1 Conformal infinity $\mathcal{I}$

As in [3] we will consider a manifold $\mathcal{M}$ with physical metric $\mathbf{g}$ which can be embedded into a larger conformal manifold $\tilde{\mathcal{M}}$ with conformal metric $\tilde{\mathbf{g}}$ through a conformal transformation

$$\tilde{\mathbf{g}}_{ab} = \Omega^2 \mathbf{g}_{ab}, \tag{3.1}$$

where Ω is a conformal factor. The spacetimes $(\mathcal{M}, \mathbf{g})$ and $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ have identical causal structures, which means that they have the same light cones. The conformal factor Ω is assumed to be positive in $\mathcal{M}$ and vanishes on the boundary of $\mathcal{M}$ in $\tilde{\mathcal{M}}$. This boundary is called *the conformal infinity* $\mathcal{I}$ and the condition for its localization is $\Omega = 0$. In the following, it is not necessary to require a global existence of $\mathcal{I}$. We assume that the conformal factor is smooth near $\mathcal{I}$ and it is sufficiently smooth along null geodesics approaching $\mathcal{I}$ in $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$.

The Weyl tensor remains unchanged under the conformal transformation,

$$\tilde{\mathbf{C}}^{d}_{abc} = \mathbf{C}_{abc}{}^{d}. \tag{3.2}$$

The character of $\mathcal{I}$ is given by the gradient $\mathrm{d}\Omega$ — it can be timelike, null, or spacelike. In [3] a normalized vector $\tilde{\mathbf{n}}$ was introduced (in conformal geometry) which is *normal to the conformal infinity* $\mathcal{I}$,

$$\tilde{\mathbf{n}}^a = \tilde{N}\tilde{\mathbf{g}}^{ab}\mathbf{d}_b\Omega, \qquad \tilde{\mathbf{g}}_{ab}\tilde{\mathbf{n}}^a\tilde{\mathbf{n}}^b = \sigma, \qquad \sigma = -1, 0, +1. \tag{3.3}$$

The function $\tilde{N} > 0$ is given for $\sigma = \pm 1$ by $\tilde{N} = |\tilde{\mathbf{g}}^{ab}\mathbf{d}_a\Omega\,\mathbf{d}_b\Omega|^{-1/2}$, for $\sigma = 0$ the function $\tilde{N}$ is chosen as an arbitrary constant on $\mathcal{I}$. The normalization parameter σ determines the character of the conformal infinity,

$$\sigma = \begin{cases} -1: & \mathcal{I} \text{ is } \textit{spacelike}, \\ 0: & \mathcal{I} \text{ is } \textit{null}, \\ +1: & \mathcal{I} \text{ is } \textit{timelike}. \end{cases} \tag{3.4}$$

For the values $\sigma = -1$ or $\sigma = 0$, we can distinguish the future conformal infinity $\mathcal{I}^+$ and the past conformal infinity $\mathcal{I}^-$. For $\sigma = +1$ the future and past infinities of null geodesics are the same. The character of the conformal infinity is usually correlated with the sign of the cosmological constant,

$$\sigma = -\operatorname{sign}\Lambda. \tag{3.5}$$

In [3] a vector $\mathbf{n}$ normal to $\Omega =$ const. was introduced in the physical spacetime $(\mathcal{M}, \mathbf{g})$ with

$$\mathbf{g}_{ab}\, \mathbf{n}^a\, \mathbf{n}^b = \sigma, \tag{3.6}$$

which implies the relation

$$\mathbf{n}^a = \Omega\, \tilde{\mathbf{n}}^a. \tag{3.7}$$

The vector $\mathbf{n}$ normalized in the physical spacetime is not possible to introduce at the conformal infinity $\mathcal{I}$ directly because $\mathcal{I}$ does not belong to the physical spacetime. The physical metric $\mathbf{g}$ is not well defined on $\mathcal{I}$. However, it is important that the conformal transformation is *isotropic*. It means that all directions are rescaled in the same way. Practically, we can associate with any physical quantity (like tensors, vectors etc.) a conformal quantity transformed by proper power of Ω which is correctly defined at $\mathcal{I}$ and which is independent of direction along which $\mathcal{I}$ is approached.

In the following text we will use vectors (null tetrads) normalized in the physical geometry, in the sense mentioned above. However, we must be careful if the transformation is not isotropic, as in the case of interpretation null tetrad parallelly transported to $\mathcal{I}$.

3.2 Null geodesics

We can relate geodesics $z(\eta)$ in the physical spacetime $(\mathcal{M}, \mathbf{g})$ to geodesics $\tilde{z}(\tilde{\eta})$ in the conformal spacetime $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$. It is well-known that *null geodesics are conformally invariant.* The affine parameter $\tilde{\eta}$ for geodesics in conformal spacetime is related to the affine parameter η for geodesics in physical spacetime by

$$\frac{d\tilde{\eta}}{d\eta} = \Omega^2, \quad \text{i.e.,} \quad \frac{Dz}{d\,\eta} = \Omega^2 \frac{D\tilde{z}}{d\tilde{\eta}}. \tag{3.8}$$

We can set the affine parameter to $\tilde{\eta} = 0$ at the conformal infinity $\mathcal{I}$. When the null geodesic $\tilde{z}(\tilde{\eta})$ approaches a specific point $P \in \mathcal{I}$ in the conformal geometry, $\tilde{z}(0) = P$ and

$$\frac{Dz^a}{d\tilde{\eta}} \mathbf{d}_a \Omega \bigg|_{\mathcal{I}} \equiv \frac{d\,\Omega}{d\,\tilde{\eta}}\bigg|_{\mathcal{I}} = -\epsilon. \tag{3.9}$$

This geodesics is *outgoing* when $\epsilon = +1$, $\tilde{\eta} < 0$ in $\tilde{\mathcal{M}}$, or *ingoing* when $\epsilon = -1$, $\tilde{\eta} > 0$ in $\tilde{\mathcal{M}}$. The normalization of the affine parameter $\tilde{\eta}$ is thus fixed uniquely. Because of the assumption of smoothness of the conformal factor along $\tilde{z}(\tilde{\eta})$, we can expand Ω in powers of $\tilde{\eta}$ near $\mathcal{I}$. The equation (3.9) becomes (using $\tilde{\eta}|_{\mathcal{I}} = 0$)

$$\Omega = -\epsilon\, \tilde{\eta} + \Omega_2\, \tilde{\eta}^2 + \ldots, , \tag{3.10}$$

with Ω_2 constant. After substituting into equation (3.8) and integration we get the relation between the physical and conformal affine parameters as

$$\eta = -\frac{1}{\tilde{\eta}}(1 - 2\epsilon\, \Omega_2\, \tilde{\eta}\, \ln|\tilde{\eta}| - \eta_0\, \tilde{\eta} + \ldots), \tag{3.11}$$

where η_0 is a constant of integration. We thus obtain the leading terms as $\tilde{\eta} \approx -\eta^{-1}$ and $\Omega \approx \epsilon\eta^{-1}$ near $\mathcal{I}$. The null geodesic $z(\eta)$ reaches the point $P \in \mathcal{I}$ for an infinite value of the affine parameter, namely $z(\epsilon\infty) = P$. The leading term in these expressions (3.10) and (3.11) is sufficient for all calculations in the following text (as in [3]). The inversion of (3.10) and (3.11) to higher orders can be found in [3] in Appendix A.

3.3 Null tetrads

Our aim is to investigate the behavior of fields near conformal infinity. For this we need to define various tetrads. We denote the vectors of an *orthonormal tetrad* as $\mathbf{t}, \mathbf{q}, \mathbf{r}, \mathbf{s}$, where $\mathbf{t}$ is a future-orientated unit timelike vector and the remaining three are spacelike unit vectors. We associate with the vectors a *null tetrad* of null vectors $\mathbf{k}, \mathbf{l}, \mathbf{m}, \overline{\mathbf{m}}$ by expressions

$$\begin{aligned} \mathbf{k} &= \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{q}), \qquad \mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{t} - \mathbf{q}), \\ \mathbf{m} &= \frac{1}{\sqrt{2}}(\mathbf{r} - \mathrm{i}\mathbf{s}), \quad \overline{\mathbf{m}} = \frac{1}{\sqrt{2}}(\mathbf{r} + \mathrm{i}\mathbf{s}), \end{aligned} \tag{3.12}$$

the normalization conditions are

$$\mathbf{g}_{ab}\, \mathbf{t}^a\, \mathbf{t}^b = -1, \quad \mathbf{g}_{ab}\, \mathbf{q}^a\, \mathbf{q}^b = \mathbf{g}_{ab}\, \mathbf{r}^a\, \mathbf{r}^b = \mathbf{g}_{ab}\, \mathbf{s}^a\, \mathbf{s}^b = 1, \tag{3.13}$$

i.e.

$$\mathbf{g}_{ab}\, \mathbf{k}^a\, \mathbf{l}^b = -1, \quad \mathbf{g}_{ab}\, \mathbf{m}^a\, \overline{\mathbf{m}}^b = 1, \tag{3.14}$$

the other scalar products are zero. Transformations between orthonormal tetrads (null tetrads) form the Lorentz group. It is convenient to consider four simple transformations which can generate any Lorentz transformation:
null rotation with $\mathbf{k}$ *fixed*, parametrized by $L \in \mathbb{C}$,

$$\mathbf{k} = \mathbf{k}_\mathrm{o}, \quad \mathbf{l} = \mathbf{l}_\mathrm{o} + \bar{L}\, \mathbf{m}_\mathrm{o} + L\bar{\mathbf{m}}_\mathrm{o} + L\, \bar{L}\, \mathbf{k}_\mathrm{o}, \quad \mathbf{m} = \mathbf{m}_\mathrm{o} + L\mathbf{k}_\mathrm{o}, \tag{3.15}$$

null rotation with $\mathbf{l}$ *fixed*, parametrized by $K \in \mathbb{C}$

$$\mathbf{k} = \mathbf{k}_\mathrm{o} + \bar{K}\, \mathbf{m}_\mathrm{o} + K\, \bar{\mathbf{m}}_\mathrm{o} + K\, \bar{K}\, \mathbf{l}_\mathrm{o}, \quad \mathbf{l} = \mathbf{l}_\mathrm{o}, \quad \mathbf{m} = \mathbf{m}_\mathrm{o} + K\mathbf{l}_\mathrm{o}, \tag{3.16}$$

boost in the $\mathbf{k} - \mathbf{l}$ *plane*, $B \in \mathbb{R}$, and a *spatial rotation in the* $\mathbf{m} - \bar{\mathbf{m}}$ *plane*, $\phi \in \mathbb{R}$,

$$\mathbf{k} = B\mathbf{k}_\mathrm{o}, \quad \mathbf{l} = B^{-1}\, \mathbf{l}_\mathrm{o}, \quad \mathbf{m} = \exp{(\mathrm{i}\phi)}\, \mathbf{m}_\mathrm{o}. \tag{3.17}$$

We will use the *interpretation null tetrad* $\mathbf{k}_\mathrm{i}, \mathbf{l}_\mathrm{i}, \mathbf{m}_\mathrm{i}, \overline{\mathbf{m}}_\mathrm{i}$. This tetrad is parallelly transported along a null geodesic $z(\eta)$ in the physical spacetime, with $\mathbf{k}_\mathrm{i}$ tangent to the geodesics $z(\eta)$. We require that the vector $\mathbf{k}_i$ is proportional to $Dz/d\eta$ by the same factor. It means that the component of the vector $\mathbf{k}_i$ normal to $\mathcal{I}$ is the same for all interpretation tetrads approaching a given point on $\mathcal{I}$. We will define the radiative component of the field with respect to this tetrad. More precise definition and asymptotic behavior of this tetrad is described in [3].

Another important tetrad is the *reference null tetrad* $\mathbf{k}_o, \mathbf{l}_o, \mathbf{m}_o, \overline{\mathbf{m}}_o$. This tetrad helps us to identify the direction $\mathbf{k}_i$ of the null geodesics and orientation of the associated interpretation tetrad near $\mathcal{I}$ using suitable directional parametrization.

The reference tetrad is any tetrad adjusted to conformal infinity (at any point $P \in \mathcal{I}$) that satisfies the relation

$$\mathbf{n} = \epsilon_o \frac{1}{\sqrt{2}}(-\sigma \mathbf{k}_o + \mathbf{l}_o), \tag{3.18}$$

where the sign $\epsilon_o = \pm 1$ indicates the outgoing/ingoing orientation of the vector $\mathbf{k}_o$ and $\mathbf{l}_o$ with respect to $\mathcal{I}$. For $\sigma = -1$ or 0, the parameter ϵ_o will be $\epsilon_o = +1$ on $\mathcal{I}^+$ and $\epsilon_o = -1$ on $\mathcal{I}^-$. For $\sigma = +1$, the parameter ϵ_o can be chosen either $+1$ or -1 because it corresponds to $\mathbf{k}_o$ oriented outside or inside $\mathcal{M}$. The condition (3.18) guarantees that the vectors $\mathbf{k}_o$ and $\mathbf{l}_o$ are collinear with the normal $\mathbf{n}$ to $\mathcal{I}$, and then are normalized particularly as

$$\mathbf{n} = \begin{cases} \epsilon_o \mathbf{t}_o & \text{for a } \textit{spacelike} \text{ infinity} \quad (\sigma = -1), \\ -\epsilon_o \mathbf{q}_o & \text{for a } \textit{timelike} \text{ infinity} \quad (\sigma = +1), \\ \epsilon_o \mathbf{l}_o/\sqrt{2} & \text{for a } \textit{null} \text{ infinity} \quad (\sigma = 0). \end{cases} \tag{3.19}$$

The condition (3.18) also implies

$$\mathbf{g}_{ab}\, \mathbf{m}^a\, \mathbf{n}^b = 0 = \mathbf{g}_{ab}\, \bar{\mathbf{m}}^a\, \mathbf{n}^b, \tag{3.20}$$

so that the vectors $\mathbf{m}, \bar{\mathbf{m}}$ on the conformal infinity are always *tangent to* $\mathcal{I}$. The reference tetrad can be chosen arbitrarily so we can choose it conveniently. It can respect the spacetime geometry (the reference tetrad is adapted to the Killing vectors or to directions toward sources etc.). Alternatively, it can be adapted to the studied fields — it can be oriented along algebraically special directions of the fields. In [6] the authors showed possible privileged definitions of the reference tetrad for algebraically simple fields of type D. Normalization and adjustment conditions do not fix the reference tetrad uniquely, so further condition will be specified in section 3.6.

3.4 Parametrization of null directions

It is also necessary to parametrize suitably a general null direction $\mathbf{k}$ near $\mathcal{I}$. Naturally, we characterize this direction by a complex *directional parameter* R such that

$$\mathbf{k} \propto \mathbf{k}_o + \bar{R}\, \mathbf{m}_o + R\, \bar{\mathbf{m}}_o + R\bar{R}\, \mathbf{l}_o. \tag{3.21}$$

In other words, the direction $\mathbf{k}$ is obtained from $\mathbf{k}_o$ by the null rotation (3.16) with parameter $K = R$. The value $R = \infty$ is also permitted — it corresponds to $\mathbf{k}$ orientated along $\mathbf{l}_o$.

We can project a null vector $\mathbf{k}$ onto the corresponding conformal infinity. When the *infinity is spacelike* ($\Lambda > 0$), we perform (see [3]) a normalized spatial projection (since $\mathbf{q} \cdot \mathbf{n} = 0$) to a three-dimensional space orthogonal to $\mathbf{t}_o$,

$$\mathbf{q} = \frac{\mathbf{k} + (\mathbf{k} \cdot \mathbf{t}_o)\mathbf{t}_o}{|\mathbf{k} \cdot \mathbf{t}_o|}, \tag{3.22}$$

where $\mathbf{k} \cdot \mathbf{t}_o = \mathbf{g}_{ab}\mathbf{k}^a\mathbf{t}_o^b$. Such a null direction $\mathbf{k}$ is possible to parametrize by spherical angles. The normalized spatial projection $\mathbf{q}$ of the null direction $\mathbf{k}$ into $\mathcal{I}$ is given by

$$\mathbf{q} = \cos\theta\,\mathbf{q}_o + \sin\theta\,(\cos\phi\,\mathbf{r}_o + \sin\phi\,\mathbf{s}_o). \tag{3.23}$$

The complex parameter R is exactly the stereographic representation of the spatial direction $\mathbf{q}$:

$$R = \tan\left(\tfrac{\theta}{2}\right)\exp(-\mathrm{i}\phi). \tag{3.24}$$

When the *infinity is timelike* ($\Lambda < 0$), the normalized projection of $\mathbf{k}$ onto $\mathcal{I}$ (now $\mathbf{t} \cdot \mathbf{n} = 0$) is

$$\mathbf{t} = \frac{\mathbf{k} - (\mathbf{k} \cdot \mathbf{q}_o)\mathbf{q}_o}{|\mathbf{k} \cdot \mathbf{q}_o|}. \tag{3.25}$$

Analogously, we parametrize the null direction $\mathbf{k}$ by pseudo-spherical parameters. The normalized projection $\mathbf{t}$ of the null direction $\mathbf{k}$ into $\mathcal{I}$ is given by

$$\mathbf{t} = \cosh\psi\,\mathbf{t}_o + \sinh\psi\,(\cos\phi\,\mathbf{r}_o + \sin\phi\,\mathbf{s}_o). \tag{3.26}$$

However, these parameters do not specify the null direction $\mathbf{k}$ uniquely so there always exists one ingoing and one outgoing null direction with the same parameters ψ and ϕ. These directions need to be distinguished by the parameter ϵ. The outgoing direction is $\epsilon = +1$ and the ingoing direction is $\epsilon = -1$. We can write the complex parameter R in a closed formula for both cases $\epsilon = \pm 1$ as

$$R = \tanh^{\epsilon\epsilon_o}\left(\tfrac{\psi}{2}\right)\exp(-\mathrm{i}\phi). \tag{3.27}$$

The value $R = \infty$ is also allowed, it corresponds to $\psi = 0, \epsilon = -\epsilon_o$, i.e. $\mathbf{k} \propto (\mathbf{t}_o - \mathbf{q}_o)/\sqrt{2}$. These parametrizations can be introduced simultaneously, independently of the causal character of the infinity, depending on the reference tetrad. The relation of the parameters is

$$\tanh\psi = \sin\theta,\ \sinh\psi = \tan\theta,\ \cosh\psi = \cos^{-1}\theta,\ \tanh(\psi/2) = \tan(\theta/2). \tag{3.28}$$

3.5 Asymptotic directional structure of radiation

In [3] the authors investigated a general field of spin s, which transforms according to spin-s representation of the Lorentz group. These fields can be characterized by specific $2s+1$ complex components Υ_j, $j = 0, \ldots, 2s$ with respect to a null tetrad. The particular cases of gravitational ($s = 2$) and electromagnetic ($s = 1$) fields were studied in more detail. The components of gravitational and electromagnetic fields are standard Newman–Penrose coefficients Ψ_j, $j = 0, 1, 2, 3, 4$, and Φ_j, $j = 0, 1, 2$, respectively. The gravitational field is characterized by the Weyl tensor $\mathbf{C}_{abcd}$ and can be parametrized by five complex coefficients

$$\begin{aligned}
\Psi_0 &= \mathbf{C}_{abcd}\mathbf{k}^a\mathbf{m}^b\mathbf{k}^c\mathbf{m}^d, \\
\Psi_1 &= \mathbf{C}_{abcd}\mathbf{k}^a\mathbf{l}^b\mathbf{k}^c\mathbf{m}^d, \\
\Psi_2 &= \mathbf{C}_{abcd}\mathbf{k}^a\mathbf{m}^b\bar{\mathbf{m}}^c\mathbf{l}^d, \\
\Psi_3 &= \mathbf{C}_{abcd}\mathbf{l}^a\mathbf{k}^b\mathbf{l}^c\bar{\mathbf{m}}^d, \\
\Psi_4 &= \mathbf{C}_{abcd}\mathbf{l}^a\bar{\mathbf{m}}^b\mathbf{l}^c\bar{\mathbf{m}}^d.
\end{aligned} \tag{3.29}$$

The electromagnetic field is described by the tensor $\mathbf{F}_{ab}$ which is parametrized by three complex coefficients

$$\begin{aligned}
\Phi_0 &= \mathbf{F}_{ab}\mathbf{k}^a\mathbf{m}^b, \\
\Phi_1 &= \mathbf{F}_{ab}(\mathbf{k}^a\mathbf{l}^b - \mathbf{m}^c\bar{\mathbf{m}}^d), \\
\Phi_2 &= \mathbf{F}_{ab}\bar{\mathbf{m}}^c\mathbf{l}^d.
\end{aligned} \tag{3.30}$$

There exist *principal null directions* (PNDs) for gravitational, electromagnetic or, generally, for any field of spin s. They are privileged null directions $\mathbf{k}$ such that $\Upsilon_0 = 0$ in the null tetrad $\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}}$ (where choice of $\mathbf{l}, \mathbf{m}, \bar{\mathbf{m}}$ is irrelevant). The PND $\mathbf{k}$ can be obtained from a reference tetrad $\mathbf{k}_{\mathrm{o}}, \mathbf{l}_{\mathrm{o}}, \mathbf{m}_{\mathrm{o}}, \bar{\mathbf{m}}_{\mathrm{o}}$ by null rotation (3.21) given by a directional parameter $R \in \mathbb{C}$. We choose the remaining vectors $\mathbf{l}, \mathbf{m}, \bar{\mathbf{m}}$ by the same null rotation (3.16) with $K = R$. The field of spin s has $2s$ PNDs. There are thus 4 principal null directions for gravitational field and 2 for electromagnetic field which can be degenerate, i.e. some PNDs may coincide, and this degeneracy is called the special algebraic structure of the field. The classification of PNDs of the gravitational field is the well-known Petrov classification.

It was shown that these field components satisfy the standard *peeling-off* property. It means that they exhibit a different fall-off in η when approaching $\mathcal{I}$ where η is the affine parameter of the geodesic.

We will briefly describe the derivation of the final result, the directional structure of these fields, for precise discussion see [3]. First, the field components are expressed in the reference tetrad as Υ^{o}_j, for $j = 0, \ldots, 2s$, then it is possible to find the relation $\Upsilon^{\mathrm{o}}_0 = (-1)^{2s}\Upsilon^{\mathrm{o}}_{2s}\prod_{j=1}^{2s} R_j$ where R_j characterize the algebraically privileged principal null directions. These field components have to be evaluated with respect to the interpretation tetrad Υ^i_{2s}. Then, after assuming the fall-off typical for zero-rest-mass fields as

$$\Upsilon^{\mathrm{o}}_j \approx \frac{\Upsilon^{\mathrm{o}}_{j*}}{\eta^{s+1}}, \quad \Upsilon^{\mathrm{o}}_{j*} = const., \tag{3.31}$$

we obtain the final result (3.32) below. The leading component of the field represents the radiative component Υ^i_{2s}: it has the fall-off of the order η^{-1} and also depends on the direction R of the null geodesics along which a fixed point at the infinity $\mathcal{I}$ is approached.

Using the above notation, Krtouš and Podolský derived in [3] the explicit dependence of the radiative component of the field on the direction along which the infinity is approached. This dependence is called the *asymptotic directional structure of radiation*,

$$\Upsilon^{\mathrm{i}}_{2s} \approx \frac{1}{\eta}\epsilon_{\mathrm{o}}^{s}\Upsilon^{\mathrm{o}}_{2s*}\frac{(1-\sigma R_1\bar{R})(1-\sigma R_2\bar{R})\dots(1-\sigma R_{2s}\bar{R})}{(1-\sigma R\bar{R})^{s}}. \tag{3.32}$$

This expression fully characterizes the asymptotic behaviour on $\mathcal{I}$ of the dominant component of any massless fields of spin s in the normalized interpretation tetrad $\mathbf{k}_{\mathrm{i}}$, $\mathbf{l}_{\mathrm{i}}, \mathbf{m}_{\mathrm{i}}$, $\bar{\mathbf{m}}_{\mathrm{i}}$ which is parallelly propagated along a null geodesics $z(\eta)$. Only the modulus $|\Upsilon^{\mathrm{i}}_{2s}|$ has an invariant meaning because there is remaining freedom in spatial rotation in the transverse $\mathbf{m}_{\mathrm{i}} - \bar{\mathbf{m}}_{\mathrm{i}}$ plane. The phase of $\Upsilon^{\mathrm{i}}_{2s}$ describes polarization.

The complex constants $R_1, \dots, R_{2s}$ represent the principal null directions (PNDs) $\mathbf{k}_1, \dots, \mathbf{k}_{2s}$ of the spin-s field, the directional structure is thus completely determined by the algebraic (Petrov) type of the field. The sign $\sigma = \pm 1, 0$ specifies the causal character of the conformal infinity, ϵ_{o} denotes orientation of the reference tetrad and $\Upsilon^{\mathrm{o}}_{2s*}$ is a constant normalization factor of the field evaluated with respect to reference tetrad. The dependence of $\Upsilon^{\mathrm{i}}_{2s}$ on the direction R along which $P \in \mathcal{I}$ is approached occurs only for cases $\sigma = \pm 1$, i.e. at a spacelike $\mathcal{I}$ (de Sitter) or timelike $\mathcal{I}$ (anti-de Sitter). The directional dependence completely vanishes for null $\mathcal{I}$ ($\sigma = 0$).

3.6 Fields of type D

In this work we will concentrate on algebraically special spacetimes which are of type D. These fields have two distinct and equivalent algebraically special directions. This case may appear only for fields of an integer spin, $s \in \mathbb{N}$. The degeneracy of PNDs is thus

$$\mathbf{k}_1 = \dots = \mathbf{k}_s \quad \text{and} \quad \mathbf{k}_{s+1} = \dots = \mathbf{k}_{2s}. \tag{3.33}$$

The directional structure (3.32) then takes the simpler form

$$\Upsilon^{i}_{2s} \approx \frac{1}{\eta}\epsilon_{\mathrm{o}}^{s}\Upsilon^{\mathrm{o}}_{2s*}\frac{(1-\sigma R_1\bar{R})^{s}(1-\sigma R_{2s}\bar{R})^{s}}{(1-\sigma R\bar{R})^{s}}, \tag{3.34}$$

where the constants R_1 and R_{2s} parametrize the two distinct PNDs.

The magnitudes for gravitational and electromagnetic field are obviously

$$|\Psi^{\mathrm{i}}_4| \approx \frac{1}{|\eta|}|\Psi^{\mathrm{o}}_{4*}|\frac{|1-\sigma R_1\bar{R}|^2|1-\sigma R_4\bar{R}|^2}{|1-\sigma R\bar{R}|^2}, \tag{3.35}$$

$$|\Phi^{\mathrm{i}}_2| \approx \frac{1}{|\eta|}|\Phi^{\mathrm{o}}_{2*}|\frac{|1-\sigma R_1\bar{R}||1-\sigma R_2\bar{R}|}{|1-\sigma R\bar{R}|}. \tag{3.36}$$

The form of the directional structure (3.34) has been derived assuming that the normalization factor $\Upsilon^{\mathrm{o}}_{2s*}$ is non-vanishing. Particularly, the vanishing of this factor indicates that the reference tetrad is asymptotically aligned along some PND and we have to use a different normalization factor to express the field components, see [3]. For type D spacetimes there exists a more natural "symmetric" normalization of the field which is not degenerate and the only non-vanishing component with respect of the null tetrad associated with the PNDs (3.33) can be used (a canonical field component).

3.6.1 Directions of vanishing radiation and algebraically special null tetrad

In this section we will study the conditions under which the radiation vanishes. We are mostly interested in the asymptotic directional structure of gravitational ($s = 2$) a electromagnetic ($s = 1$) fields.

First, the conformal infinity for $\sigma = 0$ is *Minkovskian* and the field has no directional structure. It follows from (3.35) and (3.36) that the equation reduces to the forms $|\Psi^{\mathrm{i}}_4| \approx \frac{|\Psi_{4*}|}{|\eta|}$, $|\Phi^{\mathrm{o}}_2| \approx \frac{|\Phi^{\mathrm{o}}_{2*}|}{|\eta|}$, the form is the same for all geodesics approaching a given point $P \in \mathcal{I}$. It is thus possible to distinguish between the radiative and non-radiative fields for (locally) asymptotically flat spacetimes. Radiation is absent where the constants Ψ^{o}_{4*} or Φ^{o}_{2*} vanish. It occurs when the principal null direction is oriented along the vector $\mathbf{l}_{\mathrm{o}} \propto \mathbf{n}$.

The asymptotic directional structure near a *de Sitter-like* spacelike conformal infinity can be obtained by substituting $\sigma = -1$ into (3.34): for $s = 2$ and $s = 1$ we obtained

$$\Psi^{\mathrm{i}}_4 \approx \frac{\Psi^{\mathrm{o}}_{4*}}{\eta} \frac{1}{(1+|R|^2)^2} \left(1 - \frac{R_1}{R_a}\right)^2 \left(1 - \frac{R_4}{R_a}\right)^2, \tag{3.37}$$

$$\Phi^{\mathrm{i}}_2 \approx \epsilon_{\mathrm{o}} \frac{\Phi^{\mathrm{o}}_{2*}}{\eta} \frac{1}{(1+|R|^2)} \left(1 - \frac{R_1}{R_a}\right) \left(1 - \frac{R_2}{R_a}\right), \tag{3.38}$$

where by using (3.24) we get $(1 + |R|^2)^{-1} = \cos^2(\frac{\theta}{2})$ and the complex number R_a is

$$R_a = -\frac{1}{\bar{R}} = -\cot(\tfrac{\theta}{2}) \exp(-\mathrm{i}\phi). \tag{3.39}$$

The number R_a characterizes a spatial direction opposite to the direction which is given by R with $\theta_a = \pi - \theta$ and $\phi_a = \phi + \pi$. Generally, the radiative component of the gravitational field asymptotically vanishes for the directions which satisfy $R_a = R_n$, $n = 1, 2$, similarly, there are two directions for electromagnetic field. These directions of vanishing radiation are given by (3.21) with $R = R_n = R_a$, exactly *opposite to the projections of the principal null directions onto* $\mathcal{I}$.

Finally, the asymptotic directional structure near a *anti–de Sitter-like* timelike conformal infinity can be rewritten by substituting $\sigma = +1$ into (3.34), as

$$\Psi_4^{\mathrm{i}} \approx \frac{\Psi_{4*}^{\mathrm{o}}}{\eta} \frac{1}{(1-|R|^2)^2} \left(1 - \frac{R_1}{R_m}\right)^2 \left(1 - \frac{R_4}{R_m}\right)^2, \tag{3.40}$$

$$\Phi_2^{\mathrm{i}} \approx \epsilon_{\mathrm{o}} \frac{\Phi_{2*}^{\mathrm{o}}}{\eta} \frac{1}{(1-|R|^2)} \left(1 - \frac{R_1}{R_m}\right) \left(1 - \frac{R_2}{R_m}\right), \tag{3.41}$$

where by using (3.27) we get the complex number

$$R_m = -\frac{1}{\bar{R}} = \coth^{\epsilon\epsilon_{\mathrm{o}}}(\tfrac{\theta}{2}) \exp(-\mathrm{i}\phi). \tag{3.42}$$

The complex number R_m characterizes a direction obtained from the direction R by a reflection with respect to $\mathcal{I}$, the mirrored direction with $\psi_m = \psi$, $\phi_m = \phi$ but with opposite orientation $\epsilon_m = -\epsilon$. As in the previous case for gravitational field there are two special directions where the radiation vanishes: these are directions corresponding to PNDs *reflected with respect to* $\mathcal{I}$ which are given by $R = R_m = R_n$.

There also occurs a divergence at $|R| = 1$. These are null directions tangent to $\mathcal{I}$ which do not represent a direction of any geodesics approaching $\mathcal{I}$ from the interior of spacetime. The reason for divergence is "kinematic" because we have fixed the component $\mathbf{k}_i$ normal to $\mathcal{I}$. If $\mathbf{k}_i$ is tangent to $\mathcal{I}$, the fixing condition implies an infinite rescaling and it causes the divergence of $|\Psi_4^{\mathrm{i}}|$. This divergence splits the radiation pattern into two components—the pattern of outgoing geodesics ($\epsilon = +1$) and the separate pattern for ingoing geodesics ($\epsilon = -1$).

In [6] the *algebraically special null tetrad* $\mathbf{k}_{\mathrm{s}}, \mathbf{l}_{\mathrm{s}}, \mathbf{m}_{\mathrm{s}}, \bar{\mathbf{m}}_{\mathrm{s}}$ (and associated orthonormal tetrad $\mathbf{t}_{\mathrm{s}}, \mathbf{q}_{\mathrm{s}}, \mathbf{r}_{\mathrm{s}}, \bar{\mathbf{s}}_{\mathrm{s}}$) was employed. We require that $\mathbf{k}_{\mathrm{s}}, \mathbf{l}_{\mathrm{s}}$ are proportional to the PNDs, are future-oriented and that the spatial vector $\mathbf{s}_{\mathrm{s}}$ is tangent to $\mathcal{I}$,

$$\mathbf{k}_{\mathrm{s}} \propto \mathbf{k}_1, \qquad \mathbf{l}_{\mathrm{s}} \propto \mathbf{k}_{2\mathrm{s}}, \qquad \mathbf{s}_{\mathrm{s}} \cdot \mathbf{n} = 0. \tag{3.43}$$

These PNDs, which are not tangent to the conformal infinity, can be fixed by the normalization condition for null vectors $\mathbf{k}_{\mathrm{s}}, \mathbf{l}_{\mathrm{s}}$ as

$$\epsilon_1 \mathbf{k}_{\mathrm{s}} \cdot \mathbf{n} = \epsilon_{2s} \mathbf{l}_{\mathrm{s}} \cdot \mathbf{n}, \tag{3.44}$$

where $\epsilon_1, \epsilon_{2s} = \pm 1$ parametrize orientations of the PNDs with respect to $\mathcal{I}$.

The field components for type D fields have very special form with respect the algebraically special tetrad — only the component Υ_s^{s} is non-vanishing i. e. (Ψ_2^{s} for gravitational and Φ_1^{s} for electromagnetic fields). Moreover, this component is independent of the choice of the spatial vectors $\mathbf{r}_{\mathrm{s}}, \mathbf{s}_{\mathrm{s}}$ and then it does not depend on the normal vector $\mathbf{n}$, which is used in definition (3.43). It does not depend on the normalization (3.44), provided that the normalization (3.13) is satisfied. It is convenient to use this privileged component Υ_s^{s} of the field for the normalization of the directional structure of radiation for fields of type D. However, this algebraically special tetrad is not adjusted to the infinity because the condition (3.18) is not in general satisfied ($\mathbf{k}_\mathrm{s}$, $\mathbf{l}_\mathrm{s}$ are not collinear with $\mathbf{n}$) so it cannot be directly used as a reference tetrad.

However, in [6] privileged reference tetrads were defined which are basically aligned with the algebraically special tetrads. They share together some symmetries of the geometric situation. We shall always assume that the reference tetrad satisfies the normalization and adjustment conditions (3.13), (3.18) and we set $\mathbf{s}_\mathrm{o} = \mathbf{s}_\mathrm{s}$. But this still does not fix the reference tetrad completely. The remaining necessary condition cannot be prescribed generally. It must be discussed separately in various possible cases which depend on the character of the infinity $\mathcal{I}$ and on the orientation of the PNDs with respect to $\mathcal{I}$, particularly in the case of timelike $\mathcal{I}$.

In [6] the privileged reference tetrads were defined for all the possible cases. The relation between the reference frame component Υ^o_{2s} and the canonical frame component Υ^s_{s} which can be substituted into the directional structure (3.34) were also given. Let us summarize them here.

3.6.2 Spacelike $\mathcal{I}$

We will first describe the behavior of radiation near a spacelike conformal infinity ($\sigma = -1$). The two distinct future-orientated algebraically special directions are both ingoing or both outgoing, accordingly we set the orientation ϵ_o of the reference terad. In this case, the reference tetrad is defined by the conditions

$$\mathbf{q}_\mathrm{o} = \mathbf{q}_\mathrm{s}, \quad \mathbf{s}_\mathrm{o} = \mathbf{s}_\mathrm{s}, \quad \epsilon_\mathrm{o} = \epsilon_1 = \epsilon_{2s}, \tag{3.45}$$

and by adjustment condition (3.19). Afterwards, the algebraically special directions $\mathbf{k}_\mathrm{s}$ and $\mathbf{l}_\mathrm{s}$ are parametrized with respect to the reference tetrad by a single parameter θ_s as

$$\begin{aligned}\mathbf{k}_\mathrm{s} &= \frac{1}{\sqrt{2}}\cos^{-1}\theta_\mathrm{s}\,(\mathbf{r}_\mathrm{o} + \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}),\\ \mathbf{l}_\mathrm{s} &= \frac{1}{\sqrt{2}}\cos^{-1}\theta_\mathrm{s}\,(\mathbf{r}_\mathrm{o} - \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}).\end{aligned} \tag{3.46}$$

The algebraically special and reference tetrads are thus related by

$$\begin{aligned}\mathbf{t}_\mathrm{s} &= \cos^{-1}\theta_\mathrm{s}\,\mathbf{t}_\mathrm{o} + \tan\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o},\\ \mathbf{q}_\mathrm{s} &= \mathbf{q}_\mathrm{o},\\ \mathbf{r}_\mathrm{s} &= \cos^{-1}\theta_\mathrm{s}\mathbf{r}_\mathrm{o} + \tan\theta_\mathrm{s}\,\mathbf{t}_\mathrm{o},\\ \mathbf{s}_\mathrm{s} &= \mathbf{s}_\mathrm{o},\end{aligned} \tag{3.47}$$

which is a boost in $\mathbf{t}_\mathrm{o} - \mathbf{r}_\mathrm{o}$ plane with rapidity parameter ψ_s related to θ_s by (3.28), see Fig. 2 on p. 126 in [6]. Comparing the normalized projections of $\mathbf{k}_1 = \mathbf{k}_\mathrm{s}$ and $\mathbf{k}_{2s} = \mathbf{l}_\mathrm{s}$ with (3.23), we find that the spherical parameters θ, ϕ of the algebraically special directions are $\theta_1 = \theta_{\rm s},\ \phi_1$, and $\theta_{2s} = \pi - \theta_{\rm s},\ \phi_{2s} = 0$. Then the complex parameters R_1 and R_{2s} are

$$R_1 = \tan\left(\tfrac{1}{2}\theta_{\rm s}\right), \quad R_{2s} = \cot(\tfrac{1}{2}\theta_{\rm s}). \tag{3.48}$$

The transformation (3.47) from algebraically special to the reference tetrad can be decomposed into boost, subsequent null rotation with $\mathbf{k}$ fixed and null rotation with $\mathbf{l}$ fixed (see [6]). After this transformation the relation between the component $\Upsilon^{\rm o}_{2s}$ and the canonical component $\Upsilon^{\rm s}_{s}$ was found as

$$\Upsilon^{\rm o}_{2s} = (-1)^s \frac{(2s)!}{2^s(s!)^2} \tan^s \theta_{\rm s} \Upsilon^{\rm s}_{s}. \tag{3.49}$$

Explicit expressions for gravitational ($s = 2$) and electromagnetic ($s = 1$) fields components with respect to the reference tetrad are

$$\Psi^{\rm o}_0 = \Psi^{\rm o}_4 = \frac{3}{2} \tan^2 \theta_{\rm s}\, \Psi^{\rm s}_2, \quad \Phi^{\rm o}_0 = \Phi^{\rm o}_2 = -\tan^2 \theta_{\rm s}\, \Phi^{\rm s}_1, \tag{3.50}$$

We assume the asymptotic behaviour $\Upsilon^{\rm o}_s \approx \Upsilon^{\rm o}_{s*}\eta^{-s-1}$ with constant coefficients $\Upsilon^{\rm o}_{s*}$. The equivalent equations holds for $\Upsilon^{\rm s}_s$,

$$\Upsilon^{\rm s}_s \approx \Upsilon^{\rm s}_{s*}\eta^{-s-1}. \tag{3.51}$$

The relations (3.49)-(3.50) have also their “stared” version. Substituting (3.48), (3.24) with “stared” version of (3.49) and $\sigma = -1$ into (3.34), we obtain the asymptotic directional structure of radiation for type D fields

$$\Upsilon^{\rm i}_{2s} \approx \frac{(-\epsilon_{\rm o})^s}{\eta} \frac{(2s)!}{2^s(s!)^2} \Upsilon^{\rm s}_{s*} \left[\frac{\exp({\rm i}\phi)}{\cos\theta_{\rm s}} (\sin\theta + \sin\theta_{\rm s}\cos\phi - {\rm i}\sin\theta_{\rm s}\cos\theta\sin\phi)\right]^s. \tag{3.52}$$

The null direction along which the field is measured is parametrized by angles θ, ϕ, the field is characterized by the normalization component $\Upsilon^{\rm s}_{s*}$ and by the parameter $\theta_{\rm s}$ which specifies the directions of the algebraically special directions with respect to a spacelike infinity $\mathcal{I}$. We can finally write the magnitude of the component of the gravitational and electromagnetic fields as

$$|\Psi^{\rm i}_4| \approx \frac{1}{|\eta|} \frac{3}{2} \frac{|\Psi^{\rm s}_{2*}|}{\cos^2\theta_{\rm s}} \left|\sin\theta + \sin\theta_{\rm s}\cos\phi - {\rm i}\sin\theta_{\rm s}\cos\theta\sin\phi\right|^2, \tag{3.53}$$

$$|\Phi^{\rm i}_2|^2 \approx \frac{1}{\eta^2} \frac{|\Phi^{\rm s}_{1*}|^2}{\cos^2\theta_{\rm s}} \left|\sin\theta + \sin\theta_{\rm s}\cos\phi - {\rm i}\sin\theta_{\rm s}\cos\theta\sin\phi\right|^2. \tag{3.54}$$

The directional structure of radiation is illustrated in Fig. 1.

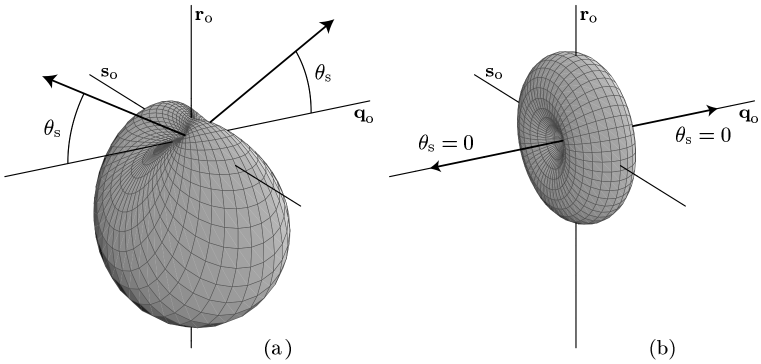

Figure 1: Directional structure of radiation near a spacelike infinity. Directions in the diagram parametrized by θ and ϕ correspond to spatial directions (projections onto $\mathcal{I}$) of null geodesics along which the infinity is approached. The diagrams show the directional dependence of the magnitude of the radiative component (3.53) or (3.54) of the field. The arrows depicts the directions which are spatially opposite to algebraically special directions PNDs, the radiation evaluated along these directions is asymptotically vanishing. The diagram (a) shows a general orientation of PNDs, the diagram (b) shows the situation when both PNDs are spatially opposite. These diagrams are taken over from [6].

3.6.3 Timelike $\mathcal{I}$ with non-tangent PNDs, $\epsilon_1 \neq \epsilon_{2s}$

Now we will study the behavior near a timelike conformal infinity ($\sigma = +1$), when both algebraic directions are not tangent to $\mathcal{I}$, such that *one of them is outgoing and the other one is ingoing*, $\epsilon_1 \neq \epsilon_{2s}$. For this case we require that the orientation parameter ϵ_o of the reference tetrad is adjusted to ϵ_1, and $\mathbf{t}_\mathrm{s}$ is aligned along $\mathbf{t}_\mathrm{o}$. The reference tetrad is then defined by conditions

$$\mathbf{t}_\mathrm{o} = \mathbf{t}_\mathrm{s}, \quad \mathbf{s}_\mathrm{o} = \mathbf{s}_\mathrm{s}, \quad \epsilon_\mathrm{o} = \epsilon_1 = -\epsilon_{2s}, \tag{3.55}$$

and by adjustment condition (3.19). Again, the algebraically special directions $\mathbf{k}_\mathrm{s}$ and $\mathbf{l}_\mathrm{s}$ are parametrized with respect to the reference tetrad by single parameter θ_s as

$$\begin{aligned} \mathbf{k}_\mathrm{s} &= \frac{1}{\sqrt{2}}\,(\mathbf{t}_\mathrm{o} + \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}), \\ \mathbf{l}_\mathrm{s} &= \frac{1}{\sqrt{2}}\,(\mathbf{t}_\mathrm{o} - \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} - \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}). \end{aligned} \tag{3.56}$$

The relation between the algebraically special and reference tetrad is given by

$$\begin{aligned} \mathbf{t}_\mathrm{s} &= \mathbf{t}_\mathrm{o}, \\ \mathbf{q}_\mathrm{s} &= \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}, \\ \mathbf{r}_\mathrm{s} &= -\sin\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \cos\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}, \\ \mathbf{s}_\mathrm{s} &= \mathbf{s}_\mathrm{o}, \end{aligned} \tag{3.57}$$

which is a spatial rotation in $\mathbf{q}_\mathrm{o} - \mathbf{r}_\mathrm{o}$ plane by angle θ_s, see Fig. 4a on p. 129 in [6]. The relation (3.57) between reference tetrad and algebraically special tetrad can be parametrized by the angle θ_s, which lies in the $\mathbf{t}_\mathrm{o} - \mathbf{r}_\mathrm{o}$ plane (the angle between $\mathbf{t}_\mathrm{o}$ and the projection of $\mathbf{k}_\mathrm{s}$ to $\mathcal{I}$) and by pseudospherical parameter ψ_s in cases 3.6.3 and 3.6.4. These parameters are related by (3.28). It is convenient to rewrite (3.56) in another form by pseudospherical parameter ψ_s to read out the normalized projection into $\mathcal{I}$ of the null vectors $\mathbf{k}_\mathrm{s}, \mathbf{l}_\mathrm{s}$ as

$$\begin{aligned} \mathbf{k}_\mathrm{s} &= \frac{1}{\sqrt{2}} \cosh^{-1}\psi_\mathrm{s}\,(\mathbf{q}_\mathrm{o} + \cosh\psi_\mathrm{s}\,\mathbf{t}_\mathrm{o} + \sinh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}), \\ \mathbf{l}_\mathrm{s} &= \frac{1}{\sqrt{2}} \cosh^{-1}\psi_\mathrm{s}\,(-\mathbf{q}_\mathrm{o} + \cosh\psi_\mathrm{s}\,\mathbf{t}_\mathrm{o} - \sinh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}). \end{aligned} \tag{3.58}$$

Comparing the normalized projections of $\mathbf{k}_1 = \mathbf{k}_\mathrm{s}$ and $\mathbf{k}_{2s} = \mathbf{l}_\mathrm{s}$ with (3.26), we find that the pseudospherical parameters ψ, ϕ, ϵ of the algebraically special directions are $\psi_1 = \psi_\mathrm{s}, \phi_1 = 0, \epsilon_1 = \epsilon_\mathrm{o}$, and $\psi_{2s} = \psi_\mathrm{s}, \phi_{2s} = \pi, \epsilon_{2s} = -\epsilon_\mathrm{o}$ respectively. The complex parameters R_1 and $R_{2\mathrm{s}}$ then are

$$R_1 = \tanh\left(\tfrac{1}{2}\psi_\mathrm{s}\right), \quad R_{2s} = -\coth\left(\tfrac{1}{2}\psi_\mathrm{s}\right). \tag{3.59}$$

The relation (3.57) can thus be expressed in pseudospherical parameter ψ_s as

$$\begin{aligned} \mathbf{t}_\mathrm{s} &= \mathbf{t}_\mathrm{o}, \\ \mathbf{q}_\mathrm{s} &= \cosh^{-1}\psi_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \tanh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}, \\ \mathbf{r}_\mathrm{s} &= -\tanh\psi_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \cosh^{-1}\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}, \\ \mathbf{s}_\mathrm{s} &= \mathbf{s}_\mathrm{o}, \end{aligned} \tag{3.60}$$

The transformation (3.57) from algebraically special to the reference tetrad can be decomposed into boost, null rotation with $\mathbf{k}$ fixed and null rotation with $\mathbf{l}$ fixed (see [6]). After this transformation it was found that the relation between the component Υ^o_{2s} and the canonical component Υ^s_s is

$$\Upsilon^\mathrm{o}_{2s} = \frac{(2s)!}{2^s(s!)^2}\tanh^s\psi_\mathrm{s}\Upsilon^\mathrm{s}_s, \tag{3.61}$$

and concretely for gravitational and electromagnetic fields

$$\Psi^\mathrm{o}_0 = \Psi^\mathrm{o}_4 = \frac{3}{2}\tanh^2\psi_\mathrm{s}\,\Psi^\mathrm{s}_2, \quad -\Phi^\mathrm{o}_0 = \Phi^\mathrm{o}_2 = \tanh\psi_\mathrm{s}\,\Phi^\mathrm{s}_1. \tag{3.62}$$

Assuming again the relation for $\Upsilon^\mathrm{s}_s \approx \Upsilon^\mathrm{s}_{s*}\eta^{-s-1}$ and substituting (3.59), (3.27) and 'stared' version of (3.61) into (3.34), we obtain the asymptotic directional structure of radiation for type D fields

$$\begin{aligned} \Upsilon^\mathrm{i}_{2s} \approx{}& \frac{(\epsilon)^s}{\eta}\frac{(2s)!}{2^s(s!)^2}\Upsilon^\mathrm{s}_{s*} \\ &\times\left[\frac{\exp(\mathrm{i}\phi)}{\cos\theta_\mathrm{s}}(\sinh\psi + \epsilon\epsilon_\mathrm{o}\sinh\psi_\mathrm{s}\cos\phi - \mathrm{i}\sinh\psi_\mathrm{s}\cosh\psi\sin\phi)\right]^s. \end{aligned} \tag{3.63}$$

The direction is given by pseudospherical parameters ψ, ϕ, ϵ, the field is characterized by the component $\Upsilon^{\rm s}_{s*}$ and by the parameter $\psi_{\rm s}$, which fixes the orientation of the algebraically special directions with respect to conformal infinity $\mathcal{I}$. Again, only the magnitude of the component Υ^{i}_{2s} has the physical meaning so

$$|\Psi^{\rm i}_4| \approx \frac{1}{|\eta|}\frac{3}{2}\frac{|\Psi^{\rm s}_{2*}|}{\cosh^2\psi_{\rm s}}\left|\sinh\psi + \epsilon\epsilon_{\rm o}\sinh\psi_{\rm s}\cos\phi - {\rm i}\sinh\psi_{\rm s}\cosh\psi\sin\phi\right|^2 , \quad (3.64)$$

$$|\Phi^{\rm i}_2|^2 \approx \frac{1}{\eta^2}\frac{|\Phi^{\rm s}_{1*}|^2}{\cosh^2\psi_{\rm s}}\left|\sinh\psi + \epsilon\epsilon_{\rm o}\sinh\psi_{\rm s}\cos\phi - {\rm i}\sinh\psi_{\rm s}\cosh\psi\sin\phi\right|^2 . \quad (3.65)$$

The directional structure is illustrated in Fig. 2a. The radiation asymptotically vanishes along mirrored reflections of the PNDs, this fact is valid also for all other timelike cases.

3.6.4 Timelike $\mathcal{I}$ with non-tangent PNDs, $\epsilon_1 = \epsilon_{2s}$

In the previous section the two PNDs were oriented in opposite directions with respect to the conformal infinity when we studied the directional structure of radiation near timelike $\mathcal{I}$. Now we will discuss the case when *both PNDs are outgoing or both outgoing*, $\epsilon_1 = \epsilon_{2s}$ in this section. The reference tetrad is fixed by conditions

$$\mathbf{r}_{\rm o} = \mathbf{q}_{\rm s}, \quad \mathbf{s}_{\rm o} = \mathbf{s}_{\rm s}, \quad \epsilon_{\rm o} = \epsilon_1 = \epsilon_{2s}, \quad (3.66)$$

and by adjustment condition (3.19). The algebraically special directions $\mathbf{k}_{\rm s}$ and $\mathbf{l}_{\rm s}$ are parametrized with respect to the reference tetrad by single parameter $\theta_{\rm s}$ or $\psi_{\rm s}$ related by (3.28) as

$$\begin{aligned}
\mathbf{k}_{\rm s} &= \frac{1}{\sqrt{2}}\sin^{-1}\theta_{\rm s}\,(\mathbf{t}_{\rm o} + \cos\theta_{\rm s}\,\mathbf{q}_{\rm o} + \sin\theta_{\rm s}\,\mathbf{r}_{\rm o})\\
&= \frac{1}{\sqrt{2}}\sinh^{-1}\psi_{\rm s}\,(\mathbf{q}_{\rm o} + \cosh\psi_{\rm s}\,\mathbf{t}_{\rm o} + \sinh\psi_{\rm s}\,\mathbf{r}_{\rm o}),\\
\mathbf{l}_{\rm s} &= \frac{1}{\sqrt{2}}\sin^{-1}\theta_{\rm s}\,(\mathbf{t}_{\rm o} + \cos\theta_{\rm s}\,\mathbf{q}_{\rm o} - \sin\theta_{\rm s}\,\mathbf{r}_{\rm o})\\
&= \frac{1}{\sqrt{2}}\sinh^{-1}\psi_{\rm s}\,(\mathbf{q}_{\rm o} + \cosh\psi_{\rm s}\,\mathbf{t}_{\rm o} - \sinh\psi_{\rm s}\,\mathbf{r}_{\rm o}).
\end{aligned} \quad (3.67)$$

The algebraically special and reference tetrad are related by

$$\begin{aligned}
\mathbf{t}_{\rm s} &= \sin^{-1}\theta_{\rm s}\,\mathbf{t}_{\rm o} + \cot\theta_{\rm s}\,\mathbf{q}_{\rm o} = \tanh^{-1}\psi_{\rm s}\,\mathbf{t}_{\rm o} + \sinh^{-1}\psi_{\rm s}\,\mathbf{q}_{\rm o},\\
\mathbf{q}_{\rm s} &= \mathbf{r}_{\rm o},\\
-\mathbf{r}_{\rm s} &= \cot\theta_{\rm s}\,\mathbf{t}_{\rm o} + \sin^{-1}\theta_{\rm s}\,\mathbf{q}_{\rm o} = \sinh^{-1}\psi_{\rm s}\,\mathbf{t}_{\rm o} + \tanh^{-1}\psi_{\rm s}\,\mathbf{q}_{\rm o},\\
\mathbf{s}_{\rm s} &= \mathbf{s}_{\rm o},
\end{aligned} \quad (3.68)$$

see Fig. 4b for details, p. 131 in [6]. The pseudospherical parameters ψ, ϕ, ϵ of projections of the PNDs into $\mathcal{I}$ are $\psi_1 = \psi_{\rm s}$, $\phi_1 = 0$, $\epsilon_1 = \epsilon_{\rm o}$, and $\psi_{2s} = \psi_{\rm s}$, $\phi_{2s} =$

π, $\epsilon_{2s} = \epsilon_{\rm o}$, respectively. The complex parameters R_1 and $R_{2\rm s}$ then are

$$R_1 = \tanh\left(\tfrac{1}{2}\psi_{\rm s}\right), \quad R_{2\rm s} = -\tanh\left(\tfrac{1}{2}\psi_{\rm s}\right). \tag{3.69}$$

The transformation (3.68) from algebraically special to the reference tetrad can be decomposed into boost, null rotation with $\mathbf{k}$ fixed and null rotation with $\mathbf{l}$ fixed, (see [6]). The relation between the component $\Upsilon^{\rm o}_{2s}$ and the canonical component $\Upsilon^{\rm s}_{s}$ is

$$\Upsilon^{\rm o}_{2s} = \frac{(2s)!}{2^s(s!)^2}\coth^s\frac{\psi_{\rm s}}{2}\Upsilon^{\rm s}_{s}, \tag{3.70}$$

in particular the expression for gravitational and electromagnetic fields are

$$\Psi^{\rm o}_4 = \frac{3}{2}\coth^2(\tfrac{1}{2}\psi_{\rm s})\,\Psi^{\rm s}_2, \quad \Phi^{\rm o}_2 = \coth(\tfrac{1}{2}\psi_{\rm s})\,\Phi^{\rm s}_1. \tag{3.71}$$

Substituting into expression (3.34), we obtain the asymptotic directional structure of radiation as

$$\begin{aligned}\Upsilon^{\rm i}_{2\rm s} \approx\ & \frac{\epsilon^s}{\eta}\frac{(2s)!}{2^s(s!)^2}\Upsilon^{\rm s}_{s*}\\ & \times\left[\frac{\exp({\rm i}\phi)}{\sinh\psi_{\rm s}}\left((\cosh\psi + \epsilon\epsilon_{\rm o}\cosh\psi_{\rm s})\cos\phi - {\rm i}(\epsilon\epsilon_{\rm o} + \cosh\psi_{\rm s}\cosh\psi)\sin\phi\right)\right]^s.\end{aligned} \tag{3.72}$$

Finally, we obtain magnitudes of components for gravitational and electromagnetic fields:

$$|\Psi^{\rm i}_4| \approx \frac{1}{|\eta|}\frac{3}{2}|\Psi^{\rm s}_{2*}|\,\left(\sinh^{-2}\psi_{\rm s}(\cosh\psi_{\rm s} + \epsilon\epsilon_{\rm o}\cosh\psi)^2 + \sinh^2\psi\sin^2\phi\right), \tag{3.73}$$

$$|\Phi^{\rm i}_2|^2 \approx \frac{1}{\eta^2}|\Phi^{\rm s}_{1*}|^2\,\left(\sinh^{-2}\psi_{\rm s}(\cosh\psi_{\rm s} + \epsilon\epsilon_{\rm o}\cosh\psi)^2 + \sinh^2\psi\sin^2\phi\right). \tag{3.74}$$

This is illustrated in Fig. 2b.

3.6.5 Timelike $\mathcal{I}$, one PND tangent to $\mathcal{I}$

In the following two sections we will study the special cases when the PNDs are tangent to $\mathcal{I}$. This can happen only for timelike and null conformal infinity. First, we will discuss the case when only *one of two distinct PNDs, say* $\mathbf{k}_{2s}$*, is tangent to* $\mathcal{I}$. We require the normalization condition (3.44) only for $\mathbf{k}_{\rm s} \propto \mathbf{k}_1$. The second PND $\mathbf{l}_{\rm s}$ is fixed by the normalization condition $\mathbf{k}_{\rm s}\cdot\mathbf{l}_{\rm s} = -1$. The reference tetrad is fixed by these conditions

$$\mathbf{k}_{\rm o} = \mathbf{k}_{\rm s}, \quad \mathbf{s}_{\rm o} = \mathbf{s}_{\rm s}, \quad \epsilon_1 = \epsilon_1, \tag{3.75}$$

and by adjustment condition (3.19). Then the algebraically special directions $\mathbf{k}_{\rm s}$ and $\mathbf{l}_{\rm s}$ are given by (see Fig. 4c on p. 133 in [6])

$$\mathbf{k}_{\rm s} = \frac{1}{\sqrt{2}}(\mathbf{r}_{\rm o} + \mathbf{q}_{\rm o}), \quad \mathbf{l}_{\rm s} = \frac{1}{\sqrt{2}}(\mathbf{t}_{\rm o} + \mathbf{r}_{\rm o}). \tag{3.76}$$

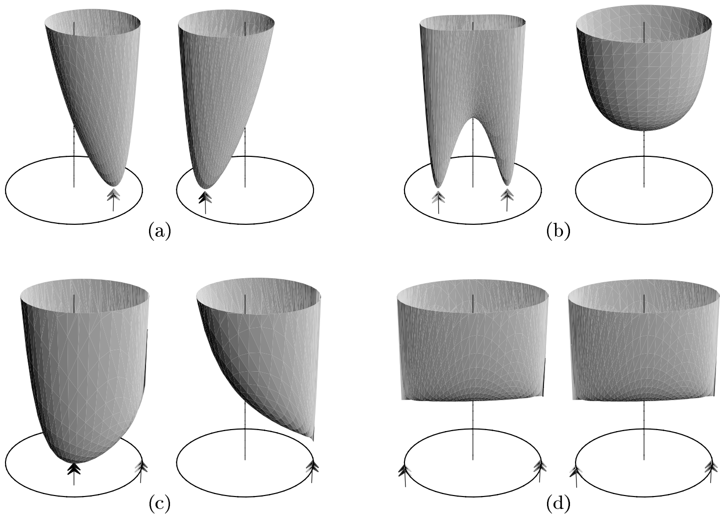

Figure 2: Directional structure of gravitational radiation near a timelike infinity. The four diagrams correspond to different orientation of algebraically special directions with respect to the infinity $\mathcal{I}$. Each consists from a pair of radiation patterns: (a) one PND outgoing and one ingoing (subsec. 3.6.3), (b) both PNDs outgoing, the case with both PNDs ingoing is analogous (subsec. 3.6.4), (c) one PND tangent to $\mathcal{I}$ and one outgoing (subsec. 3.6.5), (d) both PNDs tangent to $\mathcal{I}$ (subsec. 3.6.6). In each diagram the circles in horizontal plane represent spatial projections of hemispheres of ingoing (left circle) and outgoing (right circle) directions. The circles are parametrized by coordinates $\rho = \tanh\psi = \sin\theta$ and ϕ. The magnitude of the radiative field component is plotted on vertical axis. The arrows indicate mirror reflections with respect to $\mathcal{I}$ of PNDs. The radiative component evaluated along these geodesics in these directions is asymptotically vanishing. Diagrams are taken over from [6].

The algebraically special and the reference tetrads are related by

$$\begin{aligned} \mathbf{t}_{\mathrm{s}} &= \tfrac{3}{2}\mathbf{t}_{\mathrm{o}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{o}} + \mathbf{r}_{\mathrm{o}}, \\ \mathbf{q}_{\mathrm{s}} &= -\tfrac{1}{2}\mathbf{t}_{\mathrm{o}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{o}} - \mathbf{r}_{\mathrm{o}}, \\ \mathbf{r}_{\mathrm{s}} &= \mathbf{t}_{\mathrm{o}} + \mathbf{q}_{\mathrm{o}} + \mathbf{r}_{\mathrm{o}}, \\ \mathbf{s}_{\mathrm{s}} &= \mathbf{s}_{\mathrm{o}}, \end{aligned} \tag{3.77}$$

the corresponding complex parameters are

$$R_1 = 0, \quad R_{2s} = +1. \tag{3.78}$$

The transformation (3.77) from algebraically special to the reference tetrad is just the null rotation with $\mathbf{k}$ fixed (see [6]). Afterwards, the authors found the relation between the component $\Upsilon^{\mathrm{o}}_{2s}$ and the canonical component $\Upsilon^{\mathrm{s}}_{s}$ as

$$\Upsilon^{\mathrm{o}}_{2s} = (-1)^{s}\frac{(2s)!}{(s!)^2}\Upsilon^{\mathrm{s}}_{s}, \tag{3.79}$$

specifically for $s = 2, 1$
$$\Psi_4^{\mathrm{o}} = 6\Psi_2^{\mathrm{s}},\ \Phi_2^{\mathrm{o}} = -2\Phi_1^{\mathrm{s}}. \tag{3.80}$$
After substituting into (3.34), we obtain the asymptotic directional structure of radiation
$$\Upsilon_{2s}^{\mathrm{i}} \approx \frac{\epsilon^s}{\eta}\frac{(2s)!}{2^s(s!)^2}\Upsilon_{s*}^{\mathrm{s}}\left(\epsilon\epsilon_{\mathrm{o}} + \cosh\psi - \sinh\psi\exp(\mathrm{i}\phi)\right)^s. \tag{3.81}$$
And thus we obtain magnitudes of components for gravitational and electromagnetic fields
$$|\Psi_4^{\mathrm{i}}| \approx 3\frac{|\Psi_{2*}^{\mathrm{s}}|}{|\eta|}(\epsilon\epsilon_{\mathrm{o}} + \cosh\psi)(\cosh\psi - \sinh\psi\cos\phi), \tag{3.82}$$
$$|\Phi_2^{\mathrm{i}}|^2 \approx 2\frac{|\Phi_{1*}^{\mathrm{s}}|^2}{\eta^2}(\epsilon\epsilon_{\mathrm{o}} + \cosh\psi)(\cosh\psi - \sinh\psi\cos\phi), \tag{3.83}$$
for illustration see Fig. 2c.

3.6.6 Timelike $\mathcal{I}$, two PNDs tangent to $\mathcal{I}$

Now we will describe the remaining case when *both the PNDs are tangent to a timelike conformal infinity $\mathcal{I}$*. There does not exist such natural normalization of both PNDs analogous to the condition (3.44) which we have used above. It is related to an ambiguity in the choice of the timelike unit vector $\mathbf{t}_{\mathrm{s}}$ because we can choose any future oriented unit vector in the plane $\mathbf{k}_1 - \mathbf{k}_{2s}$. But the nonvanishing component Υ_s^{s} is independent of this ambiguity because the choices of the special tetrad differ only by a boost in $\mathbf{k}_1 - \mathbf{k}_{2s}$ plane and Υ_s^{s} does not change under that boost. So we arbitrarily choose one particular algebraically special tetrad with respect to which we define the reference tetrad. Let us note that the reference tetrad thus shares the same ambiguity as the algebraical tetrad. The reference tetrad is fixed by conditions
$$\mathbf{t}_{\mathrm{o}} = \mathbf{t}_{\mathrm{s}}, \quad \mathbf{s}_{\mathrm{o}} = \mathbf{s}_{\mathrm{s}}, \quad \epsilon_{\mathrm{o}} = \epsilon_1, \tag{3.84}$$
and by adjustment condition (3.19). The algebraically special directions $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ are given by
$$\mathbf{k}_{\mathrm{s}} = \frac{1}{\sqrt{2}}(\mathbf{t}_{\mathrm{o}} + \mathbf{r}_{\mathrm{o}}), \quad \mathbf{l}_{\mathrm{s}} = \frac{1}{\sqrt{2}}(\mathbf{t}_{\mathrm{o}} - \mathbf{r}_{\mathrm{o}}). \tag{3.85}$$
The algebraically special and the reference tetrads are related by
$$\begin{aligned}\mathbf{t}_{\mathrm{s}} &= \mathbf{t}_{\mathrm{o}},\\ \mathbf{q}_{\mathrm{s}} &= \mathbf{r}_{\mathrm{o}},\\ \mathbf{r}_{\mathrm{s}} &= -\mathbf{q}_{\mathrm{o}},\\ \mathbf{s}_{\mathrm{s}} &= \mathbf{s}_{\mathrm{o}},\end{aligned} \tag{3.86}$$
the corresponding complex parameters are
$$R_1 = +1, \quad R_{2s} = -1. \tag{3.87}$$

The transformation (3.86) from algebraically special to the reference tetrad is just the null rotation with **k** fixed and null rotation with **l** fixed. The relation between $\Upsilon^{\rm o}_{2s}$ and $\Upsilon^{\rm s}_{s}$ is now

$$\Upsilon^{\rm o}_{2s} = (-1)^s \frac{(2s)!}{2^s(s!)^2} \Upsilon^{\rm s}_{s}, \tag{3.88}$$

specifically for $s = 2, 1$ we obtain

$$\Psi^{\rm o}_0 = \Psi^{\rm o}_4 = \tfrac{3}{2}\Psi^{\rm s}_2,\ \Phi^{\rm o}_0 = -\Phi^{\rm o}_2 = 2\Phi^{\rm s}_1. \tag{3.89}$$

By substituting into (3.34), we obtain

$$\Upsilon^{\rm i}_{2s} \approx \frac{(-\epsilon_{\rm o})^s}{\eta} \frac{(2s)!}{2^s(s!)^2} \Upsilon^{\rm s}_{s*} \left(\exp({\rm i}\phi)(\cos\phi - {\rm i}\epsilon\epsilon_{\rm o}\cosh\psi\sin\phi)\right)^s. \tag{3.90}$$

So that for gravitational and electromagnetic fields,

$$|\Psi^{\rm i}_4| \approx \frac{3}{2}\frac{1}{|\eta|}\,|\Psi^{\rm s}_{2*}|(1 + \sinh^2\psi\sin^2\phi), \tag{3.91}$$

$$|\Phi^{\rm i}_2|^2 \approx \frac{|\Phi^{\rm s}_{1*}|^2}{\eta^2}(1 + \sinh^2\psi\sin^2\phi), \tag{3.92}$$

for illustration see Fig. 2d.

3.6.7 Null $\mathcal{I}$

Finally, we describe the case of *null conformal infinity.* We can see directly from (3.34) by setting $\sigma = 0$ that the directional structure of radiation near the null infinity is independent of the direction along which the infinity is approached. We will only be interested in the question whether the field is radiative or non-radiative (when the dominant field component is vanishing or not). From the definition of PNDs it follows that the normalization factor $\Upsilon^{\rm o}_{2s*}$ is vanishing if one of the PNDs is tangent to $\mathcal{I}$. We assume that one of the PNDs (it can be $\mathbf{l}_{\rm s}$) is tangent to $\mathcal{I}$. Following the previous section, we can not fix the normalization of the algebraically tetrad by condition (3.44), we can not select the algebraically tetrad uniquely. But the field component $\Upsilon^{\rm s}_{s}$ is still unique. If we choose one algebraically special tetrad, we can use it as the reference tetrad, because it satisfies the adjustment condition (3.19), see Fig. 6 at p. 135 in [6]. In other words, we take

$$\mathbf{t}_{\rm o} = \mathbf{t}_{\rm s}, \quad \mathbf{q}_{\rm o} = \mathbf{q}_{\rm s}, \quad \mathbf{r}_{\rm o} = \mathbf{r}_{\rm s}, \quad \mathbf{s}_{\rm o} = \mathbf{s}_{\rm s}. \tag{3.93}$$

It is clear that then $\mathbf{l}_{\rm o} = \mathbf{l}_{\rm s} \approx \mathbf{n}$, so the component $\Upsilon^{\rm o}_{2s*}$ is vanishing because one PND is tangent to $\mathcal{I}$:

$$\Upsilon^{\rm o}_{2s} = 0. \tag{3.94}$$

Now we will concentrate on another case when both PNDs are not tangent to $\mathcal{I}$. The PNDs can be normalized by condition (3.44) and we can fix the reference tetrad by condition

$$\mathbf{t}_{\mathrm{o}} = \mathbf{t}_{\mathrm{s}}, \quad \mathbf{s}_{\mathrm{o}} = \mathbf{s}_{\mathrm{s}}, \quad \epsilon_1 = \epsilon_1, \tag{3.95}$$

and by adjustment condition (3.19), see Fig. 6 in [6]. The algebraically special directions $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ in terms of the reference tetrad are given by

$$\mathbf{k}_{\mathrm{s}} = \frac{1}{\sqrt{2}}(\mathbf{t}_{\mathrm{o}} + \mathbf{r}_{\mathrm{o}}), \quad \mathbf{l}_{\mathrm{s}} = \frac{1}{\sqrt{2}}(\mathbf{t}_{\mathrm{o}} - \mathbf{r}_{\mathrm{o}}). \tag{3.96}$$

As in section 3.6.6, the algebraically special and the reference tetrads are related by

$$\begin{aligned} \mathbf{t}_{\mathrm{s}} &= \mathbf{t}_{\mathrm{o}}, \\ \mathbf{q}_{\mathrm{s}} &= \mathbf{r}_{\mathrm{o}}, \\ \mathbf{r}_{\mathrm{s}} &= -\mathbf{q}_{\mathrm{o}}, \\ \mathbf{s}_{\mathrm{s}} &= \mathbf{s}_{\mathrm{o}}, \end{aligned} \tag{3.97}$$

and their complex parameters are

$$R_1 = +1, \quad R_{2s} = -1. \tag{3.98}$$

Using (3.34), $\sigma = 0$, and relation (3.88), we find that the radiative component has no directional structure

$$\Upsilon^{\mathrm{i}}_{2s} \approx \epsilon_{\mathrm{o}}^{s} \frac{(2s)!}{(s!)^2} \Upsilon^{\mathrm{s}}_{s*} \frac{1}{\eta}. \tag{3.99}$$

The magnitudes of components for gravitational and electromagnetic fields are then

$$|\Psi^{\mathrm{i}}_4| \approx 3|\Psi^{\mathrm{s}}_{2*}| \frac{1}{|\eta|}, \tag{3.100}$$

$$|\Phi^{\mathrm{i}}_2|^2 \approx 2|\Phi^{\mathrm{s}}_{1*}|^2 \frac{1}{\eta^2}. \tag{3.101}$$

We have thus reviewed all possible directional structures which may occur on spacelike $\mathcal{I}$, timelike $\mathcal{I}$ and null $\mathcal{I}$.

4 Asymptotic structure of radiation for the Plebański–Demiański black holes

Now we have a complete framework needed to analyze the asymptotic directional properties of the gravitational field of exact models from the Plebański–Demiański family. In the first part 4.1 of this section we will investigate the Kerr–Newman black holes in de Sitter or anti-de Sitter universe, respectively, and then in 9.4 we will study the general Plebański–Demiański metric which describes the complete family of black hole spacetimes. We are mainly interested in the explicit dependence on the parameters of the metric, namely the mass, the charge, the rotation and the cosmological constant, in the final formula for the directional structure of radiation.

4.1 Radiation in the Kerr–Newman–de Sitter spacetime

The Kerr–Newman–de Sitter metric has the form (9.40); let us recall it again

$$\begin{aligned} \mathbf{g}_{ab} = \frac{-\Delta_r}{\Xi^2\rho^2}\left[\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\phi\right]^2 + \frac{\Delta_\vartheta\sin^2\vartheta}{\Xi^2\rho^2}\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2 \\ + \quad \frac{\rho^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\vartheta}\,\mathrm{d}\vartheta^2, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\vartheta, \\ \Delta_r &= (r^2+a^2)(1-\tfrac{\Lambda}{3}r^2) - 2mr + (e^2+g^2), \\ \Delta_\vartheta &= 1 + \tfrac{\Lambda}{3}a^2\cos^2\vartheta, \\ \Xi &= 1 + \tfrac{\Lambda}{3}a^2. \end{aligned} \tag{4.2}$$

First, to study the radiation structure following chapter 3, we have to identify the conformal infinity $\mathcal{I}$ of the spacetime (4.1). We will demostrate that the appropriate conformal factor is $\Omega = r^{-1}$ (as in the simplest Schwarzschild solution, which is the special case of (4.1) for $e = g = a = \Lambda = 0$). The conformal infinity $\mathcal{I}$ is then situated on $\Omega = 0$ which corresponds to $r = \infty$.

Indeed, by substituting the relations (4.2) into the metric (4.1), the metric can be written explicitly as

$$\begin{aligned}\mathbf{g}_{ab} =& r^2 \left\{ -\frac{\left(1+\frac{a^2}{r^2}\right)\left(\frac{1}{r^2}-\frac{\Lambda}{3}\right)-\frac{2m}{r^3}+\frac{(e^2+g^2)}{r^4}}{\left(1+\frac{a^2\cos^2\vartheta}{r^2}\right)\Xi^2}[\mathrm{d}t-a\sin^2\vartheta\mathrm{d}\phi]^2\right.\\ &+\frac{\Delta_\vartheta\sin^2\vartheta}{\left(1+\frac{a^2\cos^2\vartheta}{r^2}\right)\Xi^2}\frac{1}{r^4}[a\mathrm{d}t-(r^2+a^2)\mathrm{d}\phi]^2\\ &+\frac{\left(1+\frac{a^2\cos^2\vartheta}{r^2}\right)}{\left(1+\frac{a^2}{r^2}\right)\left(\frac{1}{r^2}-\frac{1}{3}\Lambda\right)-\frac{2m}{r^3}+\frac{(e^2+g^2)}{r^4}}\frac{\mathrm{d}r^2}{r^4}\\ &\left.+\frac{\left(1+\frac{a^2\cos^2\vartheta}{r^2}\right)}{\Delta_\vartheta}\mathrm{d}\vartheta^2\right\},\end{aligned}$$

where we factorized the term r^2 corresponding to the conformal factor Ω^{-2}. The expression in the curly brackets is the conformal metric $\tilde{\mathbf{g}}_{ab}$. We will investigate its behavior as $r \to \infty$:

$$\lim_{r\to\infty}\tilde{\mathbf{g}}_{ab} = \frac{\Lambda}{3\,\Xi^2}(\mathrm{d}t-a\sin^2\vartheta\mathrm{d}\phi)^2+\frac{\Delta_\vartheta\sin^2\vartheta}{\Xi^2}\mathrm{d}\phi^2-\frac{3}{\Lambda}\left[\mathrm{d}(r^{-1})\right]^2+\frac{1}{\Delta_\vartheta}\mathrm{d}\vartheta^2. \tag{4.3}$$

We observe that the relation (4.3) is regular for $\Lambda \neq 0$ (considering the new coordinate $\xi = r^{-1}$). The case $\Lambda = 0$. (The analytical extension of Kerr–Newman was studied in [14, 15] but still the explicit conformal factor was not presented. Thus, we will not investigate this case in this work.)

The conformal factor is thus exactly

$$\Omega = \frac{1}{r} \tag{4.4}$$

in the sense of the relation (3.1) between the conformal metric $\tilde{\mathbf{g}}_{ab}$ and the physical one $\mathbf{g}_{ab}$.

By differentiating (4.4) we get a gradient of Ω as

$$\mathrm{d}\Omega = -\frac{1}{r^2}\mathrm{d}r. \tag{4.5}$$

The inverse of the conformal Kerr–Newman–de Sitter metric is

$$\begin{aligned}\tilde{\mathbf{g}}^{ab}\partial_a\,\partial_b = r^2 &\left\{\frac{\rho^2\Xi^2}{\Delta_\vartheta\mathcal{M}\sin^2\vartheta}[a\sin^2\vartheta\,\partial_t+\partial_\phi]^2-\frac{\rho^2\Xi^2}{\Delta_r\mathcal{M}}[(r^2+a^2)\,\partial_t+a\,\partial_\phi]^2\right.\\ &\left.+\quad\frac{\Delta_\vartheta}{\rho^2}\partial_\vartheta\,\partial_\vartheta+\frac{\Delta_r}{\rho^2}\partial_r\,\partial_r\right\},\end{aligned}$$

where we denoted

$$\mathcal{M} = (r^2 + a^2) + a^2 \sin^2 \vartheta (a^2 \sin^2 \vartheta - 2r^2 - 2a^2). \tag{4.6}$$

Accordingly, we obtain $\tilde{N}$ for the case $\sigma = \pm 1$ by using the relations (4.6), (4.5) and the definition (3.3), as

$$\tilde{N} = \frac{\rho \, r}{\sqrt{|\Delta_r|}}. \tag{4.7}$$

Now we calculate $\tilde{\mathbf{n}}^a$ and σ by using (3.3)

$$\tilde{\mathbf{n}}^a = -\tilde{N} \frac{\Delta_r}{\rho^2} \partial_r = -\frac{\Delta_r}{\sqrt{|\Delta_r|}} \frac{r}{\rho} \partial_r, \tag{4.8}$$

$$\sigma = \frac{\Delta_r}{r^2 \rho^2} \tilde{N}^2 = \frac{\Delta_r}{|\Delta_r|}. \tag{4.9}$$

We may explicitly rewrite the last equation by using (4.2) which evaluated on the conformal infinity $\mathcal{I}$ gives

$$\sigma_{\mathcal{I}} = \lim_{r \to \infty} \frac{r^4 \left[\left(1 + \frac{a^2}{r^2}\right) \left(\frac{1}{r^2} - \frac{1}{3}\Lambda\right) - \frac{2m}{r^3} + \frac{(e^2 + g^2)}{r^4} \right]}{\left| r^4 \left[\left(1 + \frac{a^2}{r^2}\right) \left(\frac{1}{r^2} - \frac{1}{3}\Lambda\right) - \frac{2m}{r^3} + \frac{(e^2 + g^2)}{r^4} \right] \right|} = -\operatorname{sign} \Lambda. \tag{4.10}$$

The result (4.10) shows explicitly that the character of the conformal infinity is given by the sign of the cosmological constant, namely

$$\sigma|_{\mathcal{I}} = \begin{cases} -1: & \text{for} \quad \Lambda > 0, \\ 0: & \text{for} \quad \Lambda = 0, \\ +1: & \text{for} \quad \Lambda < 0. \end{cases} \tag{4.11}$$

(In fact, we have just confirmed the general relation (3.5) for the Kerr–Newman–de Sitter spacetime.)

We can now obtain the physical normal vector $\mathbf{n}$ using the conformal transformation (3.7) of the conformal normal vector $\tilde{\mathbf{n}}$ (4.8) for $\sigma = \pm 1$

$$\mathbf{n} = -\frac{\Delta_r}{\sqrt{|\Delta_r|}} \frac{1}{\rho} \, \partial_r. \tag{4.12}$$

This expression crucially depends on the value of Δ_r, which has the form (4.2). The normal vector (4.12) near $\mathcal{I}$ is thus

$$\mathbf{n} = \begin{cases} \frac{\sqrt{-\Delta_r}}{\rho} \, \partial_r & \text{for } \Lambda > 0 \quad (\Delta_r \text{ is negative}) \\ -\frac{\sqrt{\Delta_r}}{\rho} \, \partial_r & \text{for } \Lambda < 0 \quad (\Delta_r \text{ is positive}). \end{cases} \tag{4.13}$$

In the next sections (4.1.1) and (4.1.2) we will derive an explicit form for the radiative component of the Kerr–Newman–de Sitter gravitational field. We will separately discuss the cases $\Lambda > 0$ and $\Lambda < 0$.

4.1.1 Radiation in the Kerr–Newman–de Sitter spacetime with spacelike $\mathcal{I}$

We first consider the case $\Lambda > 0$ when the conformal infinity $\mathcal{I}$ is spacelike. The algebraically special null tetrad $(\mathbf{k}_\mathrm{s}, \mathbf{l}_\mathrm{s}, \mathbf{m}_\mathrm{s}, \bar{\mathbf{m}}_\mathrm{s})$ for this solution was already derived above in equations (9.45) and (9.46). Using this tetrad we easily identify the algebraically special orthonormal tetrad from the definition (3.12) as

$$
\begin{aligned}
\mathbf{t}_\mathrm{s} &= \frac{\sqrt{-\Delta_r}}{\rho}\,\partial_r, \\
\mathbf{q}_\mathrm{s} &= -\frac{\Xi}{\rho\sqrt{-\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right], \\
\mathbf{r}_\mathrm{s} &= \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta, \\
\mathbf{s}_\mathrm{s} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}(a\sin^2\vartheta\,\partial_t + \partial_\phi).
\end{aligned} \tag{4.14}
$$

Now we need to express the reference tetrad in terms of the algebraically special tetrad. By inverting the relations (3.47), we obtain

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\,\mathbf{t}_\mathrm{s} - \tan\theta_\mathrm{s}\,\mathbf{r}_\mathrm{s}, \\
\mathbf{q}_\mathrm{o} &= \mathbf{q}_\mathrm{s}, \\
\mathbf{r}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\,\mathbf{r}_\mathrm{s} - \tan\theta_\mathrm{s}\,\mathbf{t}_\mathrm{s}, \\
\mathbf{t}_\mathrm{o} &= \mathbf{t}_\mathrm{s}.
\end{aligned} \tag{4.15}
$$

Substituting (4.14) into (4.15), we find

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\,\frac{\sqrt{-\Delta_r}}{\rho}\,\partial_r - \tan\theta_\mathrm{s}\,\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta, \\
\mathbf{q}_\mathrm{o} &= -\frac{\Xi}{\rho\sqrt{-\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi], \\
\mathbf{r}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta - \tan\theta_s\frac{\sqrt{-\Delta_r}}{\rho}\,\partial_r, \\
\mathbf{s}_\mathrm{o} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}\left(a\sin^2\vartheta\,\partial_t + \partial_\phi\right).
\end{aligned} \tag{4.16}
$$

The reference tetrad (4.16) must satisfy the adjustment condition (3.19) for $\sigma = -1$ which is explicitly $\mathbf{n} = \epsilon_\mathrm{o}\mathbf{t}_\mathrm{o}$ and where $\mathbf{n}$ is given by (4.13). To satisfy this, it is necessary to choose the parameter

$$
\theta_\mathrm{s} = 0. \tag{4.17}
$$

Indeed, the component of the vector $\mathbf{t}_\mathrm{o}$ proportional to ∂_ϑ must not contribute to the normal $\mathbf{n}$ to $\mathcal{I}$ and thus we have to set $\tan\theta_\mathrm{s} = 0$. The first part of $\mathbf{t}_\mathrm{o}$ is then proportional to the normal (4.13) with $\cos\theta_{\rm s} = 1$. These two conditions on $\theta_{\rm s}$ give us the unique value of the parameter $\theta_{\rm s} = 0$. We also observe that for this case $\epsilon_{\rm o} = 1$. It means that the normal and the reference tetrad (both PNDs) are outgoing with respect to the future spacelike conformal infinity $\mathcal{I}^+$.

We can also use the following argument. We can calculate the projection (3.22) where $\mathbf{k}_{\rm s}\cdot\mathbf{t}_{\rm o} = \mathbf{g}_{ab}\,\mathbf{k}_{\rm s}^a\mathbf{t}_{\rm o}^b$. Explicitly, we get $\mathbf{k}_{\rm s}\cdot\mathbf{t}_{\rm o} = -\frac{1}{\sqrt{2}}\cos^{-1}\theta_{\rm s}$ by using (4.16), and then the projection is

$$\begin{aligned}\mathbf{q}_{\rm s} =& \cos\theta_{\rm s}\left(-\frac{\Xi}{\rho\sqrt{-\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right]\right) + \frac{\tan\theta_{\rm s}}{\rho}\sqrt{\Delta_\vartheta}\,\partial_\vartheta \\ &+ \frac{\sqrt{-\Delta_r}}{\rho}\cos\theta_{\rm s}(1-\cos^{-2}\theta_{\rm s})\,\partial_r.\end{aligned} \tag{4.18}$$

Because $\mathbf{q}_{\rm s}$ is the projection of $\mathbf{k}_{\rm s}$ onto $\mathcal{I}$, and $\mathbf{n}$ is vector (4.13) normal to $\mathcal{I}$, the scalar product must be $\mathbf{q}\cdot\mathbf{n} = 0$. This condition is satisfied when $(\cos^{-1}\theta_s - \cos\theta_{\rm s}) = 0$. It means that $\cos^2\theta_s = 1, \cos\theta_s \neq 0$, and it again contains the solution $\cos\theta_{\rm s} = 1$, i.e. $\theta_{\rm s} = 0$. The second solution $\cos\theta_{\rm s} = -1$ gives the value of the parameter $\theta_{\rm s} = \pi$ which would mean that the normal $\mathbf{n}$ and the reference tetrad (PNDs) are oriented to the past. This is not our case because we have chosen future oriented PNDs.

This again confirms that the appropriate choice of the parameter is $\theta_{\rm s} = 0$. In other words: in the present case *the reference tetrad is identical with the algebraically special tetrad* (4.14), see (4.15).

Now we can calculate the radiative component of the gravitational field. The only non-zero component of the Weyl tensor (following the procedure described in [6]) with respect to the algebraically special tetrad is (9.49), i.e.

$$\Psi_2^{\rm s} = -\frac{m}{(r+ia\cos\vartheta)^3} + \frac{e^2+g^2}{(r+ia\cos\vartheta)^3(r-ia\cos\vartheta)}. \tag{4.19}$$

The coefficient $\Psi_{2*}^{\rm s}$ is obtained from (3.51) for $s = 2$,

$$\Psi_2^{\rm s} \approx \Psi_{2*}^{\rm s}\,\eta^{-3}, \tag{4.20}$$

where (see section 3.2) $\eta \approx \epsilon\,\Omega^{-1}$ and $\Omega = r^{-1}$ so that $\eta \approx \epsilon\,r$. The leading factor of the gravitational field is thus

$$\Psi_{2*}^{\rm s} \approx \eta^3\Psi_2^{\rm s} = \epsilon\left(-m + \frac{e^2+g^2}{r-ia\cos\vartheta}\right)\left(1+\frac{ia\cos\vartheta}{r}\right)^{-3}, \tag{4.21}$$

as $r\to\infty$. We use this explicit function and the fact that $\theta_{\rm s} = 0$ to evaluate the expression (3.53),

$$|\Psi_4^{\rm i}| \approx \frac{3}{2}\frac{1}{|\eta|}\left|\left(m - \frac{e^2+g^2}{r-ia\cos\vartheta}\right)\left(1+\frac{ia\cos\vartheta}{r}\right)^{-3}\right|\sin^2\theta, \tag{4.22}$$

which can be written explicitly as

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2}\frac{1}{|\eta|}\sqrt{\left(m^2 - \frac{2mr(e^2+g^2)-(e^2+g^2)^2}{r^2+a^2\cos^2\vartheta}\right)\left(1+\frac{a^2\cos^2\vartheta}{r^2}\right)^{-3}}\,\sin^2\theta. \tag{4.23}$$

Considering the fact that the expansion of the field in powers of r^{-1} is only consistent here up to the first order, we finally obtain

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2}\frac{m}{|\eta|}\,\sin^2\theta. \tag{4.24}$$

This gives the explicit formula for the radiative component of gravitational field in the Kerr–Newman–de Sitter spacetime. From the equation (4.23) we observe that the contribution from the electric and magnetic charges e and g is negligible with respect to the mass m of the black hole in the order r^{-1}, and the influence of the rotation represented by a is even smaller of the order r^{-2}. The formula (4.24) is independent of ϕ so that the directional pattern is axially symmetric, see Fig. 1a. Radiation is vanishing in the directions where $\theta = 0, \pi$. The fact that generic observer detects radiation even for in non–accelerating case is intuitively caused by observer's asymptotic motion relative to the "static" black hole. Only for the special observers moving along the null geodesics radial with respect to the black hole ($\theta = 0, \pi$) the radiation vanishes as is expected for "static" sources.

The result (4.24) thus agrees with the radiative component of the gravitational field for the non–accelerating C–metric-de Sitter metric found in [4]. These results differ only in the constant factor due to the normalization condition of the component $\mathbf{k}_i$ normal to $\mathcal{I}$ which is different in [3], otherwise they fully agree. The result (4.23) is the same for the Schwarzschild–de Sitter solution because the parameter a does not contribute to the directional structure of the Kerr–Newman–de Sitter spacetime, only up to the order r^{-2}.

4.1.2 Radiation in the Kerr–Newman–anti–de Sitter spacetime with timelike $\mathcal{I}$

Analogously, we investigate the gravitational field in the anti-de Sitter background with $\Lambda < 0$. The algebraically special null tetrad $(\mathbf{k}_{\mathrm{s}}, \mathbf{l}_{\mathrm{s}}, \mathbf{m}_{\mathrm{s}}, \bar{\mathbf{m}}_{\mathrm{s}})$ for this solution was already derived above, see (9.47) and (9.48). In this section we have to analyze the directional structure of radiation near timelike conformal infinity $\mathcal{I}$. There are four possibilities of PNDs orientation with respect to $\mathcal{I}$. We will start our analysis with the case described in 3.6.3 where both PNDs are not tangent to $\mathcal{I}$ and *one PND is outgoing and one is ingoing* with respect to $\mathcal{I}$. Using the tetrad (9.47) we can easily identify the algebraically special orthonormal tetrad from its definition (3.12),

$$\begin{aligned}
\mathbf{t}_{\mathrm{s}} &= -\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right],\\
\mathbf{q}_{\mathrm{s}} &= \frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{r}_{\mathrm{s}} &= \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta,\\
\mathbf{s}_{\mathrm{s}} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}(a\sin^2\vartheta\,\partial_t + \partial_\phi).
\end{aligned} \tag{4.25}$$

This algebraically special orthonormal tetrad is clearly the same for all the cases possible on timelike $\mathcal{I}$. We need to express the reference tetrad in terms of this algebraicaly special tetrad. By inverting the relations (3.60) we obtain

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \mathbf{t}_{\mathrm{s}},\\
\mathbf{q}_{\mathrm{o}} &= -\tanh\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{s}} + \cosh^{-1}\psi_{\mathrm{s}}\,\mathbf{q}_{\mathrm{s}},\\
\mathbf{r}_{\mathrm{o}} &= \cosh^{-1}\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{s}} + \tanh\psi_{\mathrm{s}}\,\mathbf{q}_{\mathrm{s}},\\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.
\end{aligned} \tag{4.26}$$

After substituting (4.25) into (4.26), we obtain the reference tetrad expressed in terms of the coordinate frame

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= -\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right],\\
\mathbf{q}_{\mathrm{o}} &= \cosh^{-1}\psi_{\mathrm{s}}\,\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r - \tanh\psi_{\mathrm{s}}\,\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta,\\
\mathbf{r}_{\mathrm{o}} &= \cosh^{-1}\psi_{\mathrm{s}}\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta + \tanh\psi_{s}\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{s}_{\mathrm{o}} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}\left(a\sin^2\vartheta\,\partial_t + \,\partial_\phi\right).
\end{aligned} \tag{4.27}$$

The reference tetrad (4.27) must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_{\mathrm{o}}\mathbf{q}_{\mathrm{o}}$, and when we substitute the vector $\mathbf{q}_{\mathrm{o}}$, we directly get

$$\mathbf{n} = -\epsilon_{\mathrm{o}}\,(\cosh^{-1}\psi_{\mathrm{s}}\,\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r - \tanh\psi_{\mathrm{s}}\,\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta). \tag{4.28}$$

Generally, for $\Lambda < 0$ the normal vector $\mathbf{n}$ has the form (4.13). We see that the component of the vector $\mathbf{q}_{\mathrm{o}}$ proportional to ∂_ϑ must not contribute to the normal $\mathbf{n}$ near $\mathcal{I}$ so we have to set $\tanh\psi_{\mathrm{s}} = 0$. The other component of the vector $\mathbf{q}_{\mathrm{o}}$ is proportional to the normal $\mathbf{n}$ up to the factor $\cosh^{-1}\psi_{\mathrm{s}} = 1$. These two conditions imply

$$\psi_{\mathrm{s}} = 0.$$

We observe that the parameter ϵ_o which denotes orientation of the reference tetrad is $\epsilon_o = +1$. It tells us that the PND vector $\mathbf{l}_o$ is oriented into the manifold $\mathcal{M}$ while the other PND $\mathbf{k}_o$ is oriented outside the manifold on $\mathcal{I}$, and the normal $\mathbf{n}$ is oriented inside the manifold on $\mathcal{I}$.

As in the previous section we can also use another argument We can calculate the projection (3.25) where $\mathbf{k}_s \cdot \mathbf{q}_o = \mathbf{g}_{ab}\,\mathbf{k}_s^a \mathbf{q}_o^b$. Explicitly, we get $\mathbf{k}_s \cdot \mathbf{q}_o = \frac{1}{\sqrt{2}} \cosh^{-1}\psi_s$ by using (4.27), and the projection gives

$$\begin{aligned}\mathbf{t}_s = &- \cosh\psi_s \left(\frac{\Xi}{\rho\sqrt{\Delta_r}} \left[(r^2 + a^2)\,\partial_t + a\,\partial_\phi \right] \right) + \frac{\tanh\psi_s}{\rho} \sqrt{\Delta_\vartheta}\,\partial_\vartheta \\ &+ \frac{\sqrt{\Delta_r}}{\rho} \cosh\psi_s \left(1 - \cosh^{-2}\psi_s\right)\,\partial_r .\end{aligned} \tag{4.29}$$

Again, the vector $\mathbf{t}_s$ is the projection of $\mathbf{k}_s$ onto $\mathcal{I}$. Thus $\mathbf{n}$ is normal vector to $\mathcal{I}$ then the scalar product of $\mathbf{t}_s \cdot \mathbf{n}$ must be equal to zero. This condition is satisfied only when $\cosh\psi_s \left(1 - \cosh^{-2}\psi_s\right) = 0$. It means that $\cosh^2\psi = 1$, $\cosh\psi \neq 0$ and it again contains two solutions $\cosh\psi_s = \pm 1$. Only $\cosh\psi_s = 1$ makes sense. This again confirms that the appropriate choice of the parameter is $\psi_s = 0$.

To sum up: also in the present $\Lambda < 0$ case *the reference tetrad is identical to the algebraically special tetrad*, see (4.26).

Now we can calculate the radiative component of the gravitational field. The general derivation is described in the section 3.6.3. We use the explicit expression (4.21) and $\psi_s = 0$ to evaluate (3.64) which explicitly gives

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2} \frac{1}{|\eta|} \sqrt{\left(m^2 - \frac{2mr(e^2+g^2) - (e^2+g^2)^2}{r^2 + a^2\cos^2\vartheta} \right) \left(1 + \frac{a^2\cos^2\vartheta}{r^2} \right)^{-3}} \sinh^2\psi . \tag{4.30}$$

After proper expansion to the first order in r^{-1}, we finally obtain the result

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2} \frac{m}{|\eta|} \sinh^2\psi . \tag{4.31}$$

This gives the dominant component of radiation in the Kerr–Newman–anti-de Sitter spacetime. As in the previous case $\Lambda > 0$ it is independent of the charges e, g and rotation a of the black hole. The directional pattern is again axially symmetric. There are two directions of vanishing radiation. One direction is ingoing which is the mirrored reflection of outgoing PND $\mathbf{l}_s$ and the other is outgoing which is mirrored reflection of ingoing PND $\mathbf{k}_s$. The generic observer thus detects radiation except for these two directions. The result (4.31) agrees with non-accelerating C–metric–anti-de Sitter studied in [5]; again, the only difference is in the normalization condition of the component of the vector $\mathbf{k}_i$ normal to $\mathcal{I}$.

In the following we will discuss the remaining possible cases and we will prove that they *do not*, in fact, occur in the Kerr–Newman–anti-de Sitter spacetime.

First, we begin with the case 3.6.4, where *both PNDs are oriented to be outgoing or both ingoing*. The algebraically special orthonormal tetrad is (4.25).

We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.68); we obtain

$$
\begin{aligned}
\mathbf{t}_{\rm o} &= +\sinh^{-1}\psi_{\rm s}\,\mathbf{r}_{\rm s} + \coth\psi_{\rm s}\mathbf{t}_{\rm s},\\
\mathbf{q}_{\rm o} &= -\sinh^{-1}\psi_{\rm s}\,\mathbf{t}_{\rm s} - \coth\psi_{\rm s}\,\mathbf{r}_{\rm s},\\
\mathbf{r}_{\rm o} &= \mathbf{q}_{\rm s},\\
\mathbf{s}_{\rm o} &= \mathbf{s}_{\rm s}.
\end{aligned}
\tag{4.32}
$$

When we substitute (4.25) into (4.32), we obtain the reference tetrad in terms of the coordinate tetrad as

$$
\begin{aligned}
\mathbf{t}_{\rm o} &= \sinh^{-1}\psi_{\rm s}\frac{\sqrt{\Delta_\vartheta}}{\rho}\partial_\vartheta - \coth\psi_{\rm s}\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right],\\
\mathbf{q}_{\rm o} &= \frac{\Xi}{\rho\sqrt{\Delta_r}}\sinh^{-1}\psi_{\rm s}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right] - \tanh^{-1}\psi_{\rm s}\,\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta,\\
\mathbf{r}_{\rm o} &= \frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{s}_{\rm o} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}\left(a\sin^2\vartheta\,\partial_t +\ \partial_\phi\right).
\end{aligned}
\tag{4.33}
$$

This reference tetrad must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_{\rm o}\mathbf{q}_{\rm o}$.

Generally, $\mathbf{n}$ has the form (4.13) for $\sigma = +1$. We have to choose the parameter $\psi_{\rm s}$ in (4.33) to obtain the vector $\mathbf{q}_{\rm o}$ proportional to the normal $\mathbf{n}$. However, we see that *none* of the components of the vector $\mathbf{q}_{\rm o}$ is proportional to ∂_r which occurs in the normal $\mathbf{n}$ near $\mathcal{I}$. This reference tetrad thus does *not* satisfy the adjustment condition.

Alternatively, we can also calculate the projection of $\mathbf{k}_{\rm s}$ into $\mathcal{I}$ to check this result. Beginning from (3.25), we obtain $\mathbf{k}_{\rm s}\cdot\mathbf{q}_{\rm o} = \frac{1}{\sqrt{2}}\sinh^{-1}\psi_{\rm s}$ by using (4.33), and the projection gives

$$
\begin{aligned}
\mathbf{t}_{\rm s} =& -\sinh\psi_{\rm s}(1+\sinh^{-2}\psi_{\rm s})\left(\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right]\right)\\
&+\coth\psi_{\rm s}\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta + \sinh\psi_{\rm s}\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r.
\end{aligned}
\tag{4.34}
$$

Again, the scalar product must be $\mathbf{t}_{\rm s}\cdot\mathbf{n} = 0$. The condition is satisfied only when $\sinh\psi_{\rm s} = 0$ but this does not make sense when substituting this result into $\mathbf{q}_{\rm o}$ in (4.32) because $\mathbf{q}_{\rm o}$ becomes infinite. This case thus does *not* appear as a possible asymptotic behavior of the gravitational field of the Kerr–Newman–anti-de Sitter spacetime.

We will investigate the next possibility described in section 3.6.5, i.e. the case when *one PND is tangent to* $\mathcal{I}$. The algebraically special orthonormal tetrad is again (4.25). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.77), we obtain

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \tfrac{3}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}} - \mathbf{r}_{\mathrm{s}},\\
\mathbf{q}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \tfrac{1}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}},\\
\mathbf{r}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \mathbf{t}_{\mathrm{s}} - \mathbf{q}_{\mathrm{s}},\\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.
\end{aligned} \tag{4.35}$$

Substituting (4.25) into (4.35), we obtain the reference tetrad

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= -\frac{3}{2}\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right] + \frac{1}{2}\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r - \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta,\\
\mathbf{q}_{\mathrm{o}} &= \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta + \frac{1}{2}\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right] + \frac{1}{2}\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{r}_{\mathrm{o}} &= \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta + \frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right] - \frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{s}_{\mathrm{o}} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}(a\sin^2\vartheta\,\partial_t + \partial_\phi).
\end{aligned} \tag{4.36}$$

Of course, this reference tetrad must satisfy the adjustment condition (3.19). When we compare the vector $\mathbf{q}_{\mathrm{o}}$ with the normal vector $\mathbf{n}$, we observe that the vector $\mathbf{q}_{\mathrm{o}}$ has a component ∂_r pointing in the same direction as the normal $\mathbf{n}$, but $\mathbf{q}_{\mathrm{o}}$ has two other components which cannot disappear. This reference tetrad thus gain does *not* satisfy the adjustment condition. We can also calculate the projection of $\mathbf{k}_{\mathrm{s}}$ into $\mathcal{I}$. Using (3.25), since $\mathbf{k}_{\mathrm{s}}\cdot\mathbf{q}_{\mathrm{o}} = \frac{1}{\sqrt{2}}$, the projection gives

$$\mathbf{t}_{\mathrm{s}} = \frac{1}{2}\frac{\sqrt{\Delta_r}}{\rho}\,\partial_r - \frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta - \frac{3}{2}\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right]. \tag{4.37}$$

and the scalar product must be $\mathbf{t}_{\mathrm{s}}\cdot\mathbf{n} = 0$. This condition is *not* satisfied since we obtain $\mathbf{t}_{\mathrm{s}}\cdot\mathbf{n} = -\frac{1}{2}$. This case again does not represent the possible asymptotic behavior of the gravitational field of the Kerr–Newman –anti-de Sitter spacetime.

The last possible case is described in the section 3.6.6, when *two PNDs are tangent to $\mathcal{I}$*. It is the simplest case: we only write the inverse relation between the reference tetrad and the algebraically special tetrad (3.86) as

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \mathbf{t}_{\mathrm{s}},\\
\mathbf{q}_{\mathrm{o}} &= -\mathbf{r}_{\mathrm{s}},\\
\mathbf{r}_{\mathrm{o}} &= \mathbf{q}_{\mathrm{s}},\\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}},
\end{aligned} \tag{4.38}$$

and after substituting from (4.25) into (4.38) we get

$$\begin{aligned}
\mathbf{t}_{\rm o} &= -\frac{\Xi}{\rho\sqrt{\Delta_r}}\left[(r^2+a^2)\,\partial_t + a\,\partial_\phi\right],\\
\mathbf{q}_{\rm o} &= -\frac{\sqrt{\Delta_\vartheta}}{\rho}\,\partial_\vartheta,\\
\mathbf{r}_{\rm o} &= \frac{\sqrt{\Delta_r}}{\rho}\,\partial_r,\\
\mathbf{s}_{\rm o} &= \frac{\Xi}{\rho\sin\vartheta\sqrt{\Delta_\vartheta}}(a\sin^2\vartheta\,\partial_t + \partial_\phi).
\end{aligned} \tag{4.39}$$

This reference tetrad (4.39) must satisfy the adjustment condition (3.19) for $\sigma = +1$, with $\mathbf{n}$ given by (4.13). The vector $\mathbf{q}_{\rm o}$ does *not* point into the ∂_r direction as the normal vector $\mathbf{n}$, and again the adjustment condition is not satisfied. This case thus also does *not* occur as a possible asymptotic directional structure of the Kerr–Newman–anti-de Sitter spacetime.

We may thus conclude this section by summarizing that of 4 possible directional structures of radiation near the timelike conformal infinity $\mathcal{I}$ for the Kerr–Newman–anti-de Sitter spacetime, the only actual possibility is characterized by the directional structure (4.31) corresponding to the situation when one PND $\mathbf{k}_{\rm s}$ is outgoing while the other PND $\mathbf{l}_{\rm s}$ is ingoing with respect to $\mathcal{I}$. This is fully consistent with the results presented in [5] for non-accelerated limit of the C–metric with $\Lambda < 0$.

4.2 Radiation in the complete family of the Plebański–Demiański black hole spacetimes

Now we will generalize the above results to the $\Lambda \neq 0$ case of *accelerating* charged black holes with rotation a and NUT parameter l. We will use the alternative form (2.52) of the general metric for our investigation. Let us recall that

$$\begin{aligned}
\mathbf{g}_{ab} = \frac{1}{\mathbf{\Omega}^2}\Bigg[&\frac{-\mathcal{Q}}{\varrho^2}(\mathrm{d}\tilde{t} - (a\sin^2\vartheta + 4l\sin^2\frac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2 + \frac{\varrho^2}{\mathcal{P}}\mathrm{d}\vartheta^2 + \frac{\varrho^2}{\mathcal{Q}}\mathrm{d}q^2\\
&+ \frac{\mathcal{P}\sin^2\vartheta}{\varrho^2}(aq^2\mathrm{d}\tilde{t} - [1+(a+l)^2q^2]\mathrm{d}\tilde{\phi})^2\Bigg],
\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}
\mathbf{\Omega} &= -(q + \frac{\alpha}{\omega}(l + a\cos\vartheta)),\\
\varrho^2 &= 1 + q^2(l + a\cos\vartheta)^2,\\
\mathcal{P}(\vartheta) &= 1 - a_3\cos\vartheta - a_4\cos\vartheta^2,\\
\mathcal{Q}(q) &= -(\alpha^2 k + \Lambda/3) + 2\alpha\omega^{-1}nq + \epsilon q^2 + 2mq^3 + (\omega^2 k + e^2 + g^2)q^4,
\end{aligned} \tag{4.41}$$

α is the acceleration parameter, the coefficients a_3 and a_4 are given by (9.54) and (9.55), and the parameters ϵ, n are defined by the constraints (9.18), (9.19)

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}m - (a^2 + 3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right], \tag{4.42}$$

$$n = \frac{\omega^2 kl}{a^2 - l^2} - \alpha\frac{a^2 - l^2}{\omega}m + (a^2 - l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right]. \tag{4.43}$$

The parameter k for $a_0 = 1$ is

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right) k = 1 + 2\alpha\frac{l}{\omega}m - 3\alpha^2\frac{l^2}{\omega^2}(e^2 + g^2) - l^2\Lambda. \tag{4.44}$$

These constraints define the values of k, n and ϵ, the remaining scaling freedom is in the parameter ω which we can set to any convenient value with assumption that a and l do not both vanish (otherwise $\omega = 0$).

We observe that $\mathbf{\Omega}^{-2}$ is factorized from the metric (9.62). The remaining part of the metric in the square brackets is the conformal metric $\tilde{\mathbf{g}}_{ab}$. The conformal factor is in the sense of the relation (3.1) between the conformal metric $\tilde{\mathbf{g}}_{ab}$ and the physical one $\mathbf{g}_{ab}$ given exactly as

$$\mathbf{\Omega} = -(q + \frac{\alpha}{\omega}(l + a\cos\vartheta)). \tag{4.45}$$

The conformal factor must be positive, so we assume $q + \frac{\alpha}{\omega}(l + a\cos\vartheta) < 0$. Obviously, when we perform a limit $q \to 0$ of the conformal factor, there remains the term $\frac{\alpha}{\omega}(l + a\cos\vartheta)$ with the acceleration parameter which is non-trivial. This metric is thus convenient from the point of view of the conformal structure and it is more useful for our analyses.

The conformal infinity is localized on $\mathbf{\Omega} = 0$, and from (9.67) it gives an important equation

$$q = -\frac{\alpha}{\omega}(l + a\cos\vartheta). \tag{4.46}$$

This relation enables us to evaluate our results directly on $\mathcal{I}$.

We easily obtain a gradient of $\mathbf{\Omega}$ by differentiating (9.67) as

$$\mathrm{d}\mathbf{\Omega} = -(\mathrm{d}q - \frac{\alpha}{\omega}a\sin\vartheta\mathrm{d}\vartheta), \tag{4.47}$$

which, evaluated on $\Omega = \text{const.}$, gives the relation

$$\mathrm{d}q = \frac{\alpha}{\omega}a\sin\vartheta\mathrm{d}\vartheta. \tag{4.48}$$

For further purposes, it is convenient to investigate the general expression

$$\mathcal{Q} + \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta. \tag{4.49}$$

We find, surprisingly, that this expression (4.49) already has an interesting form in the initial metric (9.1) where we have substituted $r = -1/q$. Then the quartic $\mathcal{Q}$ is the same as in (9.63) and the quartic $\mathcal{P}$ is connected to P in the initial metric (9.1) by several scalings which lead to the general form of the metric (9.62) as (see (9.23) and (9.53))

$$P = \frac{a^2}{\omega^2}\sin^2\vartheta\,\mathcal{P}. \tag{4.50}$$

After substituting (4.50) into (4.49), we simply get $\mathcal{Q}+\alpha^2 P$. When we substitute P and $\mathcal{Q}$ (with $q = -1/r$) from the initial metric (9.1), we obtain

$$\begin{aligned}\mathcal{Q}+\alpha^2 P = -\tfrac{\Lambda}{3}(1+\alpha^2\omega^2p^4) + \mathbf{\Omega}\,\big[&(\omega^2k+e^2+g^2)(q-\alpha p)(q^2+\alpha^2p^2)\\ &+2m(q^2-\alpha qp+\alpha^2p^2)+\epsilon(q-\alpha p)+2\alpha n\omega^{-1}\big]\,.\end{aligned} \tag{4.51}$$

The expression (9.71) has two parts, one with the parameter Λ and the second part with a factorized conformal factor. The second part evaluated on $\mathcal{I}$ gives zero. Now, it is clear that this nice structure remains even after we perform all the transformations (9.16) in the coordinate p leading to the general metric (9.62); we obtain

$$\begin{aligned}&\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta = -\tfrac{\Lambda}{3}(1+\tfrac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)\\ &+\mathbf{\Omega}\Big[-\tfrac{\Lambda}{3}((a^2+3l^2)(q-\tfrac{\alpha}{\omega}(l+a\cos\vartheta))-\tfrac{2\alpha}{\omega}(a^2-l^2)l)\\ &+(\omega^2k+e^2+g^2)[q^3-q^2\tfrac{\alpha}{\omega}(l-a\cos\vartheta)-q\tfrac{\alpha^2}{\omega^2}((l+a\cos\vartheta)^2)-(a^2+3l^2)]\\ &+\tfrac{\alpha^3}{\omega^3}[(l+a\cos\vartheta)(a^2+3l^2-(l+a\cos\vartheta)^2)+2(a^2-l^2)l^2]\\ &+2m[q^2+q\tfrac{\alpha}{\omega}(l-a\cos\vartheta)+\tfrac{\alpha^2}{\omega^2}[(l+a\cos\vartheta)^2-2l(l+a\cos\vartheta)-(a^2-l^2)]\\ &+\tfrac{k\omega^2}{a^2-l^2}(1+2\alpha l\omega^{-1})]\Big],\end{aligned} \tag{4.52}$$

where k has the form (9.66). Finally, we evaluate the expression (4.52) directly on $\mathcal{I}$, by setting $\mathbf{\Omega} = 0$,

$$\mathcal{Q}_{\mathcal{I}}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta = -\frac{\Lambda}{3}\left(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4\right). \tag{4.53}$$

This expression will be very useful in evaluating our results directly on $\mathcal{I}$.

Now we will examine the conformal metric $\tilde{\mathbf{g}}_{ab}$ on $\mathcal{I}$. When we substitute (9.68), (9.70) into $\tilde{\mathbf{g}}_{ab}$ from (9.62) and (9.63), we obtain

$$\begin{aligned}\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}} = \Bigg[&\frac{-\mathcal{Q}_{\mathcal{I}}}{\varrho^2_{\mathcal{I}}}(\mathrm{d}\tilde{t}-(a\sin^2\vartheta+4l\sin^2\frac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2+\varrho^2_{\mathcal{I}}\frac{\mathcal{Q}_{\mathcal{I}}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}{\mathcal{P}\mathcal{Q}_{\mathcal{I}}}\mathrm{d}\vartheta^2\\ &+\frac{\mathcal{P}\sin^2\vartheta}{\varrho^2_{\mathcal{I}}}(a\tfrac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^2\mathrm{d}\tilde{t}-[1+(a+l)^2\tfrac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^2]\mathrm{d}\tilde{\phi})^2\Bigg],\end{aligned} \tag{4.54}$$

where the expressions for ϱ and $\mathcal{Q}$ changed to

$$\varrho_{\mathcal{I}} = 1 + \frac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^4 \tag{4.55}$$

and

$$\begin{aligned} \mathcal{Q}_{\mathcal{I}} = &-(\alpha^2 k + \tfrac{\Lambda}{3}) - 2\tfrac{\alpha^2}{\omega^2} n(l + \cos\vartheta) + \epsilon\tfrac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^2 \\ &- 2m\tfrac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^3 + \tfrac{\alpha^3}{\omega^3}(\omega^2 k + e^2 + g^2)(l + a\cos\vartheta)^4, \end{aligned} \tag{4.56}$$

where n, ϵ and k are the constraints (9.64), (9.65) and (9.66). The coordinate q is not present in (9.73) because we made the substitution (9.68) and therefore only the term proportional to $\mathrm{d}\vartheta^2$ appears in the conformal metric $\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}$. Furthermore, when we use the expression (9.72), it is possible to evaluate the conformal metric (9.73) on $\mathcal{I}$ more explicitly as

$$\begin{aligned} \tilde{\mathbf{g}}_{ab}|_{\mathcal{I}} = &\Bigg[\frac{-\mathcal{Q}_{\mathcal{I}}}{\varrho|_{\mathcal{I}}^2}(\mathrm{d}\tilde{t} - (a\sin^2\vartheta + 4l\sin^2\frac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2 + \varrho|_{\mathcal{I}}^2 \frac{-\frac{\Lambda}{3}(1 + \frac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^4)}{\mathcal{P}\mathcal{Q}_{\mathcal{I}}}\mathrm{d}\vartheta^2 \\ &+ \frac{\mathcal{P}\sin^2\vartheta}{\varrho|_{\mathcal{I}}^2}(a\tfrac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^2\mathrm{d}\tilde{t} - [1 + (a+l)^2\tfrac{\alpha^2}{\omega^2}(l + a\cos\vartheta)^2]\mathrm{d}\tilde{\phi})^2 \Bigg]. \end{aligned} \tag{4.57}$$

We observe from the last expression that $\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}$ is regular on $\mathcal{I}$ for $\sigma = \pm 1$ ($\Lambda \neq 0$). The case $\Lambda = 0$ has to be studied in another way, like the Kerr–Newman–de Sitter solution in section 4.1.

We also calculated the inverse form of the physical metric (9.62) (using the Maple software)

$$\begin{aligned} \mathbf{g}^{ab}\partial_a\,\partial_b = \mathbf{\Omega}^2 &\Bigg[\frac{\mathcal{Q}}{\varrho^2}\partial_q\partial_q + \frac{\mathcal{P}}{\varrho^2}\partial_\vartheta\partial_\vartheta - \frac{\varrho^2}{\mathcal{Q}\mathcal{W}}[(1 + (a+l)^2 q^2)\partial_{\tilde{t}} + aq^2\partial_{\tilde{\phi}}]^2 \\ &+ \frac{\varrho^2}{\mathcal{P}\sin^2\vartheta\,\mathcal{W}}[(a\sin^2\vartheta + 4l\sin^2(\vartheta/2))\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]^2 \Bigg], \end{aligned} \tag{4.58}$$

where we denoted

$$\mathcal{W} = \left[(1 + (a+l)^2 q^2) - aq^2(a\sin^2\vartheta + 4l\sin^2(\vartheta/2))\right]^2. \tag{4.59}$$

The trace of metric (9.62) and (9.77) is $\mathbf{g}_{ab}\mathbf{g}^{ab} = 4$. This confirms that (9.77) is really the inverse metric of (9.62).

We are now able to calculate the function $\tilde{N}$ and the conformal normal $\tilde{\mathbf{n}}^a$ to $\mathcal{I}$, and σ from the definition (3.3). The function $\tilde{N}$ is, for the cases $\sigma = \pm 1$ and for every $\Omega = \text{const}$, given by

$$\tilde{N} = \frac{\varrho}{\sqrt{|\mathcal{Q} + \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta|}}. \tag{4.60}$$

Then the conformal normal $\tilde{\mathbf{n}}^a$ to $\mathcal{I}$ and σ are

$$\tilde{\mathbf{n}}^a = -\frac{1}{\sqrt{|\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta|}}\left(\frac{\mathcal{Q}}{\varrho}\partial_q - \alpha\frac{a}{\omega}\frac{\mathcal{P}}{\varrho}\sin\vartheta\partial_\vartheta\right), \tag{4.61}$$

$$\sigma = \frac{\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}{|\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta|}. \tag{4.62}$$

Clearly, we observe that if $\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta > 0$ then $\sigma = 1$, and also conversely $\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta < 0$ for $\sigma = -1$. Obviously, we can just explicitly evaluate the expression (9.81) on $\mathcal{I}$ using (9.72). It gives

$$\sigma|_{\mathcal{I}} = \frac{-\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{|-\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)|} = \frac{-\Lambda}{|-\Lambda|} = -\mathrm{sign}\Lambda. \tag{4.63}$$

We have thus confirmed the relation (3.5), $\sigma = -\mathrm{sign}\,\Lambda$ which tells us that the character of $\mathcal{I}$ is related to the sign of cosmological constant.

The physical normal vector is according to (3.7) defined as $\mathbf{n}^a = \mathbf{\Omega}\tilde{\mathbf{n}}^a$, thus for $\sigma = \pm 1$

$$\mathbf{n}^a = -\frac{\mathbf{\Omega}}{\sqrt{|\mathcal{Q}+\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta|}}(\frac{\mathcal{Q}}{\varrho}\partial_q - \alpha\frac{a}{\omega}\frac{\mathcal{P}}{\varrho}\sin\vartheta\partial_\vartheta). \tag{4.64}$$

In the next sections we will derive explicit form of the radiation in the complete family of Plebański–Demiański black holes spacetimes. We will discuss separately the cases $\Lambda > 0$ and $\Lambda < 0$.

4.2.1 Radiation in the general metric with spacelike $\mathcal{I}$

We will investigate the radiative properties of the general metric (9.62). The separate cases can be divided according to the sign of the cosmological constant as in the Kerr–Newman spacetimes. The case $\Lambda > 0$ occurs for $\mathcal{Q} < 0$, it means that the leading term in $\mathcal{Q}$ gives the condition $\alpha^2 k > -\frac{\Lambda}{3}$ near the conformal infinity, while $\Lambda < 0$ occurs for $\mathcal{Q} > 0$, $\alpha^2 k < -\frac{\Lambda}{3}$ and for $\mathcal{Q} < 0$, $\alpha^2 k > -\frac{\Lambda}{3}$. The sign of the term $\mathcal{Q}$ specifies which null tetrad from the section (9.3) is convenient to use.

First, we will study the case when $\Lambda > 0$ with the tetrad chosen for $\mathcal{Q} < 0$. The algebraically special null tetrad is the tetrad (2.56) and (2.57) for the case $\mathcal{Q} < 0$ of the general metric. We can easily identify the algebraically special orthonormal tetrad using the definition (3.12) from the vectors (2.56) as

$$
\begin{aligned}
\mathbf{t}_{\rm s} &= \frac{\boldsymbol{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\,\partial_q, \\
\mathbf{q}_{\rm s} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{-1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}), \\
\mathbf{r}_{\rm s} &= \frac{\boldsymbol{\Omega}}{\varrho}\sqrt{\mathcal{P}}\partial_\vartheta, \\
\mathbf{s}_{\rm s} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned}
\tag{4.65}
$$

We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.47), as

$$
\begin{aligned}
\mathbf{t}_{\rm o} &= \cos^{-1}\theta_{\rm s}\,\mathbf{t}_{\rm s} - \tan\theta_{\rm s}\,\mathbf{r}_{\rm s}, \\
\mathbf{q}_{\rm o} &= \mathbf{q}_{\rm s}, \\
\mathbf{r}_{\rm o} &= \cos^{-1}\theta_{\rm s}\,\mathbf{r}_{\rm s} - \tan\theta_{\rm s}\,\mathbf{t}_{\rm s}, \\
\mathbf{t}_{\rm o} &= \mathbf{t}_{\rm s}.
\end{aligned}
\tag{4.66}
$$

Substituting (9.84) into (9.85), we obtain the reference tetrad for spacelike $\mathcal{I}$

$$
\begin{aligned}
\mathbf{t}_{\rm o} &= \frac{\boldsymbol{\Omega}}{\varrho}\left(\cos^{-1}\theta_{\rm s}\,\sqrt{-\mathcal{Q}}\partial_q - \tan\theta_{\rm s}\,\sqrt{\mathcal{P}}\partial_\vartheta\right), \\
\mathbf{q}_{\rm o} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{-1}{\sqrt{-\mathcal{Q}}}\left(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}\right), \\
\mathbf{r}_{\rm o} &= \frac{\boldsymbol{\Omega}}{\varrho}\left(\cos^{-1}\theta_{\rm s}\,\sqrt{\mathcal{P}}\partial_\vartheta - \tan\theta_{\rm s}\,\sqrt{-\mathcal{Q}}\partial_q\right), \\
\mathbf{s}_{\rm o} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned}
\tag{4.67}
$$

The reference tetrad (9.86) must satisfy the adjustment condition (3.19) for $\sigma = -1$ which is explicitly $\mathbf{n} = \epsilon_{\rm o}\mathbf{t}_{\rm o}$, i.e.

$$
\mathbf{n} = \epsilon_{\rm o}\boldsymbol{\Omega}\left(\cos^{-1}\theta_{\rm s}\,\frac{\sqrt{-\mathcal{Q}}}{\varrho}\partial_q - \tan\theta_{\rm s}\,\frac{\sqrt{\mathcal{P}}}{\varrho}\partial_\vartheta\right). \tag{4.68}
$$

On the other hand, the normal vector $\mathbf{n}$ is given by (9.83) for $\sigma = -1$,

$$
\mathbf{n} = -\frac{\boldsymbol{\Omega}}{\sqrt{-\mathcal{Q} - \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}}\left(\frac{\mathcal{Q}}{\varrho}\partial_q - \alpha\frac{a}{\omega}\frac{\mathcal{P}}{\varrho}\sin\vartheta\,\partial_\vartheta\right). \tag{4.69}
$$

The normal vectors (9.87) and (9.88) have to be equal. By comparing them we observe that it is necessary to set and then evaluate on $\mathcal{I}$ the following expressions (which we have already done by the substitution (9.68)),

$$\cos\theta_{\rm s} = \sqrt{1+\alpha^2\tfrac{a^2}{\omega^2}\tfrac{\mathcal{P}}{\mathcal{Q}_{\mathcal{I}}}\sin^2\vartheta}, \tag{4.70}$$

$$\tan\theta_{\rm s} = -\frac{\alpha\frac{a}{\omega}\sqrt{\mathcal{P}}\sin\vartheta}{\sqrt{-\mathcal{Q}_{\mathcal{I}}-\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}}, \tag{4.71}$$

$$\sin\theta_{\rm s} = -\frac{\alpha\frac{a}{\omega}\sqrt{\mathcal{P}}\sin\vartheta}{\sqrt{-\mathcal{Q}_{\mathcal{I}}}}, \tag{4.72}$$

where $\mathcal{Q}_{\mathcal{I}}(\vartheta)$ is the expression (9.75), $\mathcal{P}(\vartheta)$ is given by (9.63), and we also observe that $\epsilon_{\rm o} = 1$. In other words, the normal vector and the reference tetrad are outgoing with respect to the future spacelike conformal infinity $\mathcal{I}^+$. We evaluated the above expressions on $\mathcal{I}$ because we are interested in the form of $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ on $\mathcal{I}$. The parameter $\theta_{\rm s}$ *encodes the orientation of the algebraically special tetrad with respect to the spacelike* $\mathcal{I}$. The expressions $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ and $\tan\theta_{\rm s}$ are functions of only one coordinate ϑ on the conformal infinity $\mathcal{I}$.

Now we calculate the radiative component of the gravitational field. The only non-zero component of the Weyl tensor (following the procedure described in [6]) with respect to the algebraically special tetrad is (9.60), let us recall it again,

$$\Psi_2^{\rm s} = \left[(m+in)+(e^2+g^2)\left(\frac{q-\frac{\alpha}{\omega}(l+a\cos\vartheta)}{1+iq(l+a\cos\vartheta)}\right)\right]\left(\frac{q+\frac{\alpha}{\omega}(l+a\cos\vartheta)}{1-iq(l+a\cos\vartheta)}\right)^3. \tag{4.73}$$

The coefficient $\Psi_{2*}^{\rm s}$ is obtained from (3.51) for $s=2$ as

$$\Psi_2^{\rm s} \approx \Psi_{2*}^{\rm s}\,\eta^{-3}, \tag{4.74}$$

where $\eta \approx \epsilon\,\Omega^{-1}$ and $\Omega = -(q+\frac{\alpha}{\omega}(l+a\cos\vartheta))$ so that $\eta \approx \epsilon\,(-q-\frac{\alpha}{\omega}(l+a\cos\vartheta))^{-1}$. The leading factor of the gravitational field is thus

$$\Psi_{2*}^{\rm s} \approx \eta\Psi_2^{\rm s} = \epsilon(-q-\tfrac{\alpha}{\omega}(l+a\cos\vartheta))^{-3}\Psi_2^{\rm s}, \tag{4.75}$$

for $q \to -\frac{\alpha}{\omega}(l+a\cos\vartheta)$ when $\Omega \to 0$. After substitution of (9.92) into (9.94) we get

$$\Psi_{2*}^{\rm s} \approx \frac{\epsilon}{(1-iq(l+a\cos\vartheta))^3}\left[(m+in)+(e^2+g^2)\frac{q-\frac{\alpha}{\omega}(l+a\cos\vartheta)}{1+iq(l+a\cos\vartheta)}\right]. \tag{4.76}$$

The asymptotic directional structure has a standard form for spacelike $\mathcal{I}$ described by (3.53), namely

$$|\Psi_4^{\rm i}| \approx \frac{3}{2}\frac{1}{|\eta|}\frac{|\Psi_{2*}^{\rm s}|}{\cos^2\theta_{\rm s}}\mathcal{A}(\theta,\phi,\theta_{\rm s}). \tag{4.77}$$

where we have conveniently denoted the angular directional dependence as

$$\mathcal{A}(\theta,\phi,\theta_{\rm s}) = (\sin\theta+\sin\theta_{\rm s}\cos\phi)^2+\sin^2\theta_{\rm s}\cos^2\theta\sin^2\phi. \tag{4.78}$$

The asymptotic directional structure is determined by $\Psi^{\rm s}_{2*}$ evaluated on $\mathcal{I}$ and by $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ which are given by (9.89) and (9.91), both evaluated on $\mathcal{I}$. The parameter $\sin\theta_{\rm s}$ can be rewritten from (9.91) using (9.72) as

$$\sin\theta_{\rm s} = -\frac{\alpha\frac{a}{\omega}\sqrt{\mathcal{P}}\sin\vartheta}{\sqrt{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}}, \tag{4.79}$$

and we can also rewrite $\cos\theta_{\rm s}$ from (9.89) and (9.72) as

$$\cos^2\theta_{\rm s} = \frac{\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}, \tag{4.80}$$

where the function $\mathcal{P}$ is given by (9.63). We are mainly interested in the concrete form of $\Psi^{\rm s}_{2*}$ evaluated on $\mathcal{I}$. This normalization component (9.95) can be rewritten as

$$|\Psi^{\rm s}_{2*}| = \frac{\sqrt{\mathcal{D}}}{(1+q^2(l+a\cos\vartheta)^2)^{3/2}}, \tag{4.81}$$

where

$$\begin{aligned}\mathcal{D} =& m^2+n^2+2(e^2+g^2)\frac{(q-\frac{\alpha}{\omega}(l+a\cos\vartheta))(m-nq(l+a\cos\vartheta))}{1+q^2(l+a\cos\vartheta)^2}\\ &+(e^2+g^2)^2\frac{(q-\frac{\alpha}{\omega}(l+a\cos\vartheta))^2}{1+q^2(l+a\cos\vartheta)^2},\end{aligned} \tag{4.82}$$

and the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65). The particular form of $|\Psi^{\rm s}_{2*}|$ on $\mathcal{I}$ can be obtained by substituting (9.68) into the previous expression (9.101) as

$$|\Psi^{\rm s}_{2*}|_{\mathcal{I}} = \frac{\sqrt{\mathcal{D}_{\mathcal{I}}}}{(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)^{3/2}}, \tag{4.83}$$

where

$$\begin{aligned}\mathcal{D}_{\mathcal{I}} =& m^2+n^2-4\frac{\alpha}{\omega}(e^2+g^2)(l+a\cos\vartheta)\frac{m+n\frac{\alpha}{\omega}(l+a\cos\vartheta^2)}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4}\\ &+4\frac{\alpha^2}{\omega^2}(e^2+g^2)^2\frac{(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4}.\end{aligned} \tag{4.84}$$

We observe again that $|\Psi^{\rm s}_{2*}|_{\mathcal{I}}$ depends only on the coordinate ϑ. Then it is useful to denote the factor of the radiation as

$$B(\vartheta) \equiv \frac{|\Psi^{\rm s}_{2*}|_{\mathcal{I}}}{\cos^2\theta_{\rm s}}. \tag{4.85}$$

The final form of the asymptotic directional structure of radiation is thus

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2}\frac{1}{|\eta|}B(\vartheta)\mathcal{A}(\theta,\phi,\theta_{\mathrm{s}}), \tag{4.86}$$

where $B(\vartheta)$ can be rewritten by (9.99), (9.102) and (9.103) as

$$B(\vartheta) = \frac{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)^{5/2}}\sqrt{\mathcal{D}_{\mathcal{I}}}, \tag{4.87}$$

with

$$\begin{aligned}\mathcal{D}_{\mathcal{I}} =& m^2+n^2-4\tfrac{\alpha}{\omega}(e^2+g^2)(l+a\cos\vartheta)\frac{m+n\frac{\alpha}{\omega}(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4}\\ &+4\tfrac{\alpha^2}{\omega^2}(e^2+g^2)^2\frac{(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4},\end{aligned} \tag{4.88}$$

and the function $\mathcal{P}$ given by (9.63). The expressions (9.105), (9.106) and (9.107) give the explicit formula for the radiative component of gravitational field in the complete Plebański–Demiański black hole spacetimes in the de Sitter universe. The null direction along which the field is measured is parametrized by angles θ, ϕ in expression $\mathcal{A}$ in (9.97). The field itself is characterized by the normalization component $|\Psi^{\mathrm{s}}_{2*}|_{\mathcal{I}}$ or $B(\vartheta)$ and the parameter $\theta_{\mathrm{s}}\vartheta)$ fixes the orientation of the algebraically special directions with respect to a spacelike infinity $\mathcal{I}$. Two PNDs are both oriented outside the manifold on future spacelike $\mathcal{I}^+$. There are two directions in which the radiation vanish as which are spatially opposite to PNDs.

The result is consistent with [4] where the C-metric was investigated. The difference is in the sign of $\sin\theta_{\mathrm{s}}$ that it is probably caused by the choice of the reference tetrad. The other minor difference is again in the normalization of the normal component of the vector $\mathbf{k}_{\mathrm{i}}$ which is defined slightly differently in [4] and in the [3]. The result for $\alpha = 0$, $l = 0$ reduces to our previous result for Kerr–Newman–de Sitter presented in the section 4.1.1.

The amplitude of radiation $B(\vartheta)$ is represented by expression which depends on a single coordinate ϑ and on the parameters m, α, ω, a, l, Λ, e, g. The coordinate ϑ specifies one point on the conformal infinity $\mathcal{I}$ while the parameters characterize the sources (black holes) which generate the radiation. The dependence of the amplitude $B(\vartheta)$ on the parameters of the black holes is discussed in section 9.5. Surprisingly, we will observe that the amplitude $B(\vartheta)$ is quite similar for spacelike $\mathcal{I}$ and also for the cases with timelike $\mathcal{I}$, except of one special case. Thus all these cases can be investigated together.

4.2.2 Radiation in the general metric with timelike $\mathcal{I}$

In this section we will subsequently discuss four possible orientations of PNDs on timelike $\mathcal{I}$ for $\Lambda < 0$ in the tetrad for $\mathcal{Q} > 0$ which gives the condition for the acceleration $\alpha^2 k < -\frac{\Lambda}{3}$. In other words, the acceleration α is smaller than the cosmological constant and k is a positive constant.

We start with the case when *one PND is outgoing and one is ingoing with respect to* $\mathcal{I}$, see the section 3.6.3. The algebraically special null tetrad is represented by (2.58) and (2.59) for the case $\mathcal{Q} > 0$ of the general metric (9.62). We can easily identify the algebraically special orthonormal tetrad using the definition (3.12) from the vectors (2.58)

$$\begin{aligned}
\mathbf{t}_\mathrm{s} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1 + (l+a)^2q^2]\partial_{\tilde{t}}),\\
\mathbf{q}_\mathrm{s} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q,\\
\mathbf{r}_\mathrm{s} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{P}}\partial_\vartheta,\\
\mathbf{s}_\mathrm{s} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned} \tag{4.89}$$

This algebraically orthonormal tetrad will be used troughout this section for all cases which may appear on timelike $\mathcal{I}$ for $\Lambda < 0$ ($\mathcal{Q} > 0$).

We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.60),

$$\begin{aligned}
\mathbf{t}_\mathrm{o} &= \mathbf{t}_\mathrm{s},\\
\mathbf{q}_\mathrm{o} &= -\tanh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \cosh^{-1}\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s},\\
\mathbf{r}_\mathrm{o} &= \cosh^{-1}\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \tanh\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s},\\
\mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}.
\end{aligned} \tag{4.90}$$

We find the reference tetrad by substituting (9.108) into (9.109) as

$$\begin{aligned}
\mathbf{t}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{-1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1 + (l+a)^2q^2]\partial_{\tilde{t}}),\\
\mathbf{q}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\tanh\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta + \cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q\right),\\
\mathbf{r}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta + \tanh\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q\right),\\
\mathbf{s}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned} \tag{4.91}$$

The reference tetrad must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_\mathrm{o}\mathbf{q}_\mathrm{o}$, i.e.

$$\mathbf{n} = -\epsilon_\mathrm{o}\frac{\mathbf{\Omega}}{\varrho}\left(\cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q - \tanh\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta\right). \tag{4.92}$$

On the other hand, the normal vector $\mathbf{n}$ is given by (9.83) for $\sigma = +1$,

$$\mathbf{n} = -\frac{\boldsymbol{\Omega}}{\sqrt{\mathcal{Q} + \alpha^2 \frac{a^2}{\omega^2} \mathcal{P} \sin^2 \vartheta}} \left(\frac{\mathcal{Q}}{\varrho} \partial_q - \alpha \frac{a}{\omega} \frac{\mathcal{P}}{\varrho} \sin \vartheta \partial_\vartheta \right). \tag{4.93}$$

The normal vectors (9.111) and (9.112) have to be equal. By comparing them we observe that it is necessary to set and evaluate on $\mathcal{I}$ the expressions as

$$\cosh \psi_{\mathrm{s}} = \sqrt{1 + \alpha^2 \frac{a^2}{\omega^2} \frac{\mathcal{P}}{\mathcal{Q}_{\mathcal{I}}} \sin^2 \vartheta}, \tag{4.94}$$

$$\tanh \psi_{\mathrm{s}} = \frac{\alpha \frac{a}{\omega} \sqrt{\mathcal{P}} \sin \vartheta}{\sqrt{\mathcal{Q}_{\mathcal{I}} + \alpha^2 \frac{a^2}{\omega^2} \mathcal{P} \sin^2 \vartheta}}, \tag{4.95}$$

$$\sinh \psi_{\mathrm{s}} = \frac{\alpha \frac{a}{\omega} \sqrt{\mathcal{P}} \sin \vartheta}{\sqrt{\mathcal{Q}_{\mathcal{I}}}}, \tag{4.96}$$

where $\mathcal{Q}_{\mathcal{I}}$ is the expression (9.75), $\mathcal{P}$ is (9.63), and we also observe that $\epsilon_{\mathrm{o}} = +1$. It means, the normal vector is ingoing and one PND $\mathbf{k}_{\mathrm{s}}$ is outgoing while $\mathbf{l}_{\mathrm{s}}$ is ingoing with respect to the timelike conformal infinity $\mathcal{I}$.

Again, we evaluated the above expressions on $\mathcal{I}$ because then the parameter ψ_{s} encodes the orientation of the algebraically special tetrad with respect to $\mathcal{I}$. The expressions $\cos \psi_{\mathrm{s}}$, $\sin \psi_{\mathrm{s}}$ and $\tan \psi_{\mathrm{s}}$ are functions of coordinate ϑ only.

We are able to calculate the radiative component of the gravitational field. The procedure is almost the same as in the previous spacelike case, the normalization factor is (9.95) and its evaluation on $\mathcal{I}$ is (9.102). The asymptotic directional structure of radiation has a form (3.64), namely

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2} \frac{1}{|\eta|} \frac{|\Psi_{2*}^{\mathrm{s}}|}{\cosh^2 \psi_{\mathrm{s}}} \mathcal{A}_1(\psi, \phi, \psi_{\mathrm{s}}) \tag{4.97}$$

and it is determined by Ψ_{2*}^{s} evaluated on $\mathcal{I}$, and by $\cosh \psi_{\mathrm{s}}$, $\sinh \psi_{\mathrm{s}}$ which are given by (9.113) and (9.115), both evaluated on $\mathcal{I}$, and the directional angular dependence is

$$\mathcal{A}_1(\psi, \phi, \psi_{\mathrm{s}}) = (\sinh \psi + \epsilon \sinh \psi_{\mathrm{s}} \cos \phi)^2 + \sinh^2 \psi_{\mathrm{s}} \cosh^2 \psi \sin^2 \phi,$$

where we substituted $\epsilon_{\mathrm{o}} = +1$.
The term $\sinh \psi_{\mathrm{s}}$ can be rewritten from (9.115) by (9.72) as

$$\sinh \psi_{\mathrm{s}} = \frac{\alpha \frac{a}{\omega} \sqrt{\mathcal{P}} \sin \vartheta}{\sqrt{-\alpha^2 \frac{a^2}{\omega^2} \mathcal{P} \sin^2 \vartheta - \frac{\Lambda}{3} (1 + \frac{\alpha^2}{\omega^2} (l + a \cos \vartheta)^4)}}, \tag{4.98}$$

and we can also rewrite $\cosh \psi_{\mathrm{s}}$ from (9.113) by (9.72) as

$$\cosh^2 \psi_{\mathrm{s}} = \frac{\frac{\Lambda}{3} (1 + \frac{\alpha^2}{\omega^2} (l + a \cos \vartheta)^4)}{\alpha^2 \frac{a^2}{\omega^2} \mathcal{P} \sin^2 \vartheta + \frac{\Lambda}{3} (1 + \frac{\alpha^2}{\omega^2} (l + a \cos \vartheta)^4)}, \tag{4.99}$$

where the function $\mathcal{P}$ is given by(9.63). If we denote the amplitude of the radiation as

$$B(\vartheta) = \frac{|\Psi^{\rm s}_{2*}|_{\mathcal{I}}}{\cosh^2 \psi_{\rm s}}, \tag{4.100}$$

the asymptotic directional structure has the form

$$|\Psi^{\rm i}_4| \approx \frac{3}{2}\frac{1}{|\eta|}B(\vartheta)\mathcal{A}_1(\psi,\phi,\psi_{\rm s}) \tag{4.101}$$

where $B(\vartheta)$ can be rewritten by (9.102),(9.118) and (9.107) as

$$B(\vartheta) = \frac{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)^{5/2}}\sqrt{\mathcal{D}_{\mathcal{I}}} \tag{4.102}$$

and

$$\begin{aligned}\mathcal{D}_{\mathcal{I}} =& m^2+n^2-4\tfrac{\alpha}{\omega}(e^2+g^2)(l+a\cos\vartheta)\frac{m+n\frac{\alpha}{\omega}(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4}\\ &+4\tfrac{\alpha^2}{\omega^2}(e^2+g^2)^2\frac{(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4},\end{aligned} \tag{4.103}$$

where the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65).

We obtained the explicit formula for the radiative component of gravitational field of the black hole spacetimes in the anti-de Sitter background for $\alpha^2 k < -\Lambda/3$. The null direction is parametrized by ψ, ϕ in expression (9.120). The field itself is characterized by the normalization component $|\Psi^{\rm s}_{2*}|_{\rm s}$ or $B(\vartheta)$ and the parameter $\psi_{\rm s}$ fixes the orientation of the algebraically special directions with respect to a timelike infinity $\mathcal{I}$.

The PNDs are oriented on $\mathcal{I}$ such that one PND $\mathbf{k}$ is outgoing and the other PND $\mathbf{l}$ is ingoing with respect to $\mathcal{I}$. There exists just one ingoing direction of vanishing radiation which is a mirror reflection of PND $\mathbf{k}$ and one outgoing direction of mirrored reflection of PND $\mathbf{l}$. The result is fully consistent with [5] where the C-metric was investigated for $\alpha < \sqrt{-\Lambda/3}$. For this range of acceleration, the C-metric represents a single accelerating black hole.

We observe that the amplitude $B(\vartheta)$ is identical with the amplitude from previous section for spacelike $\mathcal{I}$, see equations (9.106), (9.107). The dependence of the amplitude $B(\vartheta)$ will be investigated in the section 9.5 together with other cases.

In the following we will discuss the remaining possible cases.

First, we first consider the case 3.6.4 where *both PNDs are oriented to be outgoing or both ingoing.* The identification of the algebraically special orthonormal tetrad from its definition (3.12) for the vectors of the tetrad (2.58) is the same as in the previous case (9.108). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.68) as

$$\begin{aligned}
\mathbf{t}_{\rm o} &= +\sinh^{-1}\psi_{\rm s}\,\mathbf{r}_{\rm s} + \coth\psi_{\rm s}\mathbf{t}_{\rm s},\\
\mathbf{q}_{\rm o} &= -\sinh^{-1}\psi_{\rm s}\,\mathbf{t}_{\rm s} - \coth\psi_{\rm s}\,\mathbf{r}_{\rm s},\\
\mathbf{r}_{\rm o} &= \mathbf{q}_{\rm s},\\
\mathbf{s}_{\rm o} &= \mathbf{s}_{\rm s}.
\end{aligned} \tag{4.104}$$

When we substitute (9.108) into (9.123), we obtain the reference tetrad

$$\begin{aligned}
\mathbf{t}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sinh^{-1}\psi_{\rm s}\sqrt{\mathcal{P}}\partial_\vartheta - \coth\psi_{\rm s}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde\phi} + [1+(l+a)^2q^2]\partial_{\tilde t})\right),\\
\mathbf{q}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sinh^{-1}\psi_{\rm s}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde\phi} + [1+(l+a)^2q^2]\partial_{\tilde t}) - \coth\psi_{\rm s}\sqrt{\mathcal{P}}\partial_\vartheta\right),\\
\mathbf{r}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q\\
\mathbf{s}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde t} + \partial_{\tilde\phi}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned} \tag{4.105}$$

This reference tetrad must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_{\rm o}\mathbf{q}_{\rm o}$ as

$$\mathbf{n} = -\epsilon_{\rm o}\frac{\mathbf{\Omega}}{\varrho}\left(\sinh^{-1}\psi_{\rm s}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde\phi} + [1+(l+a)^2q^2]\partial_{\tilde t}) - \coth\psi_{\rm s}\sqrt{\mathcal{P}}\partial_\vartheta\right). \tag{4.106}$$

The normal vector $\mathbf{n}$ calculated directly from the metric is given by (9.112). These two vectors have to be equal again. We have to choose the parameter $\psi_{\rm s}$ in (9.125) to obtain the vector $\mathbf{q}_{\rm o}$ proportional to the normal $\mathbf{n}$. However, we observe that the component ∂_q of the normal vector $\mathbf{n}$ (9.112) is not present in the vector (9.125). This reference tetrad thus does *not* satisfy the adjustment condition, so that this case does not appear as a possible asymptotic behavior of the gravitational field of the studied family of black hole spacetime in the anti de Sitter universe.

We can also calculate the projection of $\mathbf{k}_{\rm s}$ into $\mathcal{I}$, to check this result. Beginning from (3.25), using (9.124) and (3.67), we obtain $\mathbf{k}_{\rm s}\cdot\mathbf{q}_{\rm o} = \frac{1}{\sqrt{2}}\sinh^{-1}\psi_{\rm s}$, and the projection is thus

$$\begin{aligned}
\mathbf{t}_{\rm s} =&\frac{\mathbf{\Omega}}{\varrho}\left(-\sinh\psi_{\rm s}(1+\sinh^{-2}\psi_{\rm s})\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde\phi} + [1+(l+a)^2q^2]\partial_{\tilde t})\right.\\
&\left. + \sinh\psi_{\rm s}\sqrt{\mathcal{Q}}\partial_q + \coth\psi_{\rm s}\sqrt{\mathcal{P}}\partial_\vartheta\right).
\end{aligned} \tag{4.107}$$

Again, because the vector $\mathbf{t}_{\rm s}$ is a projection of $\mathbf{k}_{\rm s}$ onto $\mathcal{I}$ and $\mathbf{n}$ is normal vector to $\mathcal{I}$, the scalar product must be $\mathbf{t}_{\rm s}\cdot\mathbf{n} = 0$. The scalar product has a form

$$\mathbf{t}_{\rm s}\cdot\mathbf{n} = \frac{-\sinh\psi_{\rm s}}{\sqrt{\mathcal{Q} + \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}}\left(\sqrt{\mathcal{Q}} - \frac{\sqrt{\mathcal{P}}\frac{\alpha}{\omega}a\sin\vartheta}{\cosh\psi_{\rm s}}\right). \tag{4.108}$$

The scalar product is zero only when $\sinh\psi_{\rm s} = 0$ or when $\cosh\psi_{\rm s} = \frac{\sqrt{\mathcal{P}}\frac{\alpha}{\omega}a\sin\vartheta}{\sqrt{\mathcal{Q}}}$. When we substitute these conditions into $\mathbf{q}_{\rm o}$, the first condition implies that the first part of $\mathbf{q}_{\rm o}$ diverges, the second condition implies that the second part of the vector $\mathbf{q}_{\rm o}$ is proportional to the normal vector $\mathbf{n}$ (9.112), but still the first part of $\mathbf{q}_{\rm o}$ is not. Therefore, these two conditions which satisfy $\mathbf{t}_{\rm s}\cdot\mathbf{n} = 0$ do not give consistent results.

Now we will investigate the case described in section 3.6.5, *the case when one PND is tangent to $\mathcal{I}$.* The identification of the algebraically special orthonormal tetrad is (9.108). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.77). We obtain

$$\begin{aligned} \mathbf{t}_{\rm o} &= \tfrac{3}{2}\mathbf{t}_{\rm s} + \tfrac{1}{2}\mathbf{q}_{\rm s} - \mathbf{r}_{\rm s}, \\ \mathbf{q}_{\rm o} &= \mathbf{r}_{\rm s} - \tfrac{1}{2}\mathbf{t}_{\rm s} + \tfrac{1}{2}\mathbf{q}_{\rm s}, \\ \mathbf{r}_{\rm o} &= \mathbf{r}_{\rm s} - \mathbf{t}_{\rm s} - \mathbf{q}_{\rm s}, \\ \mathbf{s}_{\rm o} &= \mathbf{s}_{\rm s}. \end{aligned} \tag{4.109}$$

Substituting (9.108) into (9.128), we obtain the reference tetrad

$$\begin{aligned} \mathbf{t}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\frac{3}{2}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) + \frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q - \sqrt{\mathcal{P}}\partial_\vartheta\right), \\ \mathbf{q}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sqrt{\mathcal{P}}\partial_\vartheta + \frac{1}{2}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) + \frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q\right), \\ \mathbf{r}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sqrt{\mathcal{P}}\partial_\vartheta + \frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) - \sqrt{\mathcal{Q}}\,\partial_q\right), \\ \mathbf{s}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}. \end{aligned} \tag{4.110}$$

This reference tetrad must satisfy the adjustment condition (3.19) When we compare the vector $\mathbf{q}_{\rm o}$ with such normal $\mathbf{n}$ (9.112), we observe that the vector $\mathbf{q}_{\rm o}$ has components ∂_q and ∂_ϑ pointing in the same direction as the normal $\mathbf{n}$, but $\mathbf{q}_{\rm o}$ has one another component $\frac{1}{2}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}}+[1+(l+a)^2q^2]\partial_{\tilde{t}})$ which cannot disappear. This reference tetrad thus again does *not* satisfy the adjustment condition. We can also calculate the projection of $\mathbf{k}_{\rm s}$ into $\mathcal{I}$. Using (3.25), (9.129) and (3.76), we obtain $\mathbf{k}_{\rm s}\cdot\mathbf{q}_{\rm o} = \frac{1}{\sqrt{2}}$, and the the projection is

$$\mathbf{t}_{\rm s} = \frac{\mathbf{\Omega}}{\varrho}\left(\frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q - \sqrt{\mathcal{P}}\partial_\vartheta - \frac{3}{2}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}})\right). \tag{4.111}$$

Again, the scalar product must be $\mathbf{t}_{\rm s}\cdot\mathbf{n} = 0$ and thus this condition is *not* satisfied: we obtain $\mathbf{t}_{\rm s}\cdot\mathbf{n} = -(\sqrt{\mathcal{P}}\frac{\alpha}{\omega}a\sin\vartheta + \frac{1}{2}\sqrt{\mathcal{Q}})/(\sqrt{\mathcal{Q}+\alpha^2\mathcal{P}\frac{a^2}{\omega^2}\sin^2\vartheta})$. When this vanishes, the situation is unphysical.

This case thus again does not represent the asymptotic behavior of the gravitational field of the complete family of black hole solutions in anti-de Sitter universe.

The last possible case is described in the section 3.6.6, when *two PNDs are tangent to $\mathcal{I}$*. We only write the inverse relation between the reference tetrad and the algebraically special tetrad (3.86) as

$$\begin{aligned} \mathbf{t}_{\rm o} &= \mathbf{t}_{\rm s}, \\ \mathbf{q}_{\rm o} &= -\mathbf{r}_{\rm s}, \\ \mathbf{r}_{\rm o} &= \mathbf{q}_{\rm s}, \\ \mathbf{s}_{\rm o} &= \mathbf{s}_{\rm s}, \end{aligned} \tag{4.112}$$

and after substituting from (9.108) into (9.131) we get

$$\begin{aligned} \mathbf{t}_{\rm o} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}), \\ \mathbf{q}_{\rm o} &= -\frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{P}}\partial_{\vartheta}, \\ \mathbf{r}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q, \\ \mathbf{s}_{\rm o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}. \end{aligned} \tag{4.113}$$

The vector $\mathbf{q}_{\rm o}$ does not point into the both directions ∂_q and ∂_ϑ as the normal vector $\mathbf{n}$. But when we set $\mathcal{Q} = 0$, the normal vector (9.112) becomes (the function $\mathcal{P} > 0$ is positive)

$$\mathbf{n} = -\frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{P}}\partial_{\vartheta}. \tag{4.114}$$

and the vector $\mathbf{q}_{\rm o}$ is proportional to (9.133). The adjustment condition is thus *satisfied on horizons* where $\mathcal{Q} = 0$, and we observe that $\epsilon_{\rm o} = +1$. But we investigate the radiation at infinity, so that the adjustment condition is satisfied on roots of $\mathcal{Q}_{\mathcal{I}}$ given by (9.75). The directional structure is given by (3.91), where the amplitude of the radiation will be fully determined by $|\Psi^{\rm s}_{2*}|_{\mathcal{I}}$. When we substitute the roots of $\mathcal{Q}_{\mathcal{I}}$ into $|\Psi^{\rm s}_{2*}|_{\mathcal{I}}$, it becomes constant, of course, depending on the parameters of the sources.

This special case thus also occurs as a possible directional structure of radiation of general metric. It is a new feature, which does not occur in the C–metric for *small* acceleration.

To summarize the asymptotic directional structure of radiation near the timelike conformal infinity $\mathcal{I}$ for the general metric (9.62) in anti-de Sitter spacetime with small acceleration: it is characterized by the directional structure (9.121) corresponding to the situation when one PND $\mathbf{k}$ is ingoing and the other PND $\mathbf{l}$ is outgoing with respect to $\mathcal{I}$, which is fully consistent with the results presented in [5] for the C–metric, and by the structure where both PNDs are tangent to $\mathcal{I}$ on roots of $\mathcal{Q}_{\mathcal{I}}$ that is not present in the C–metric and is completely new. Again, there is a small difference in the normalization of the normal component of the vector $\mathbf{k}_\mathrm{i}$. We also observe that this section for timelike $\mathcal{I}$ of the complete family of black hole spacetimes for small acceleration is quite similar to the section 4.1.2 of Kerr–Newman–anti-de Sitter spacetime. This section is thus a generalization of the section on the Kerr–Newman–anti-de Sitter solution which includs acceleration.

4.2.3 Radiation in the general metric with timelike $\mathcal{I}$

Again, we will gradually discuss all four possible orientations of PNDs on timelike $\mathcal{I}$ for $\Lambda < 0$ in tetrad where $\mathcal{Q} < 0$, and it gives the condition for the acceleration $\alpha^2 k > -\frac{\Lambda}{3}$. It means that the acceleration α is larger than the cosmological constant. This situation does not appear in the directional structure of the Kerr–Newman–de Sitter spacetime.

First, we start with the case when *one PND is outgoing and one is ingoing with respect to* $\mathcal{I}$, see section 3.6.3. We have previously identified the algebraically special orthonormal tetrad (9.84) and the normal vector $\mathbf{n}$ for $\mathcal{Q} < 0$, (9.88) in the section 9.4.1 for spacelike $\mathcal{I}$. This tetrad and the normal vector are the same for all cases in this section. We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.60), we obtain

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \mathbf{t}_\mathrm{s}, \\ \mathbf{q}_\mathrm{o} &= -\tanh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \cosh^{-1}\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s}, \\ \mathbf{r}_\mathrm{o} &= \cosh^{-1}\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \tanh\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s}, \\ \mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}. \end{aligned} \tag{4.115}$$

Substituting (9.84) into (9.134), we find the reference tetrad

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\partial_q, \\ \mathbf{q}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\tanh\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta - \cosh^{-1}\psi_\mathrm{s}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}})\right), \\ \mathbf{r}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(\cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta - \tanh\psi_\mathrm{s}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}\right), \\ \mathbf{s}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}. \end{aligned} \tag{4.116}$$

The reference tetrad must satisfy the adjustment condition (3.19) which is explicitly $\mathbf{n} = -\epsilon_\mathrm{o}\mathbf{q}_\mathrm{o}$, i.e.

$$\mathbf{n} = -\epsilon_\mathrm{o}\frac{\mathbf{\Omega}}{\varrho}\left(-\tanh\psi_\mathrm{s}\sqrt{\mathcal{P}}\partial_\vartheta - \cosh^{-1}\psi_\mathrm{s}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}})\right). \tag{4.117}$$

The normal vector obtained from the metric (9.112) and (9.136) have to be equal. By comparing them we observe that the component ∂_q of the normal vector $\mathbf{n}$

does not appear in the normal from reference tetrad (9.136). This reference tetrad thus does *not* satisfy the adjustment condition, this case does not appear as a possible asymptotic behavior of the gravitational field of the complete family of black hole spacetime in anti-de Sitter universe.

We can also calculate the projection of $\mathbf{k}_\mathrm{s}$ into $\mathcal{I}$, to prove this result. Beginning from (3.25), using (9.135) and (3.58), we obtain $\mathbf{k}_\mathrm{s} \cdot \mathbf{q}_\mathrm{o} = \frac{1}{\sqrt{2}} \cosh^{-1} \psi_\mathrm{s}$, and the projection then is

$$\begin{aligned} \mathbf{t}_\mathrm{s} =& \frac{\boldsymbol{\Omega}}{\varrho} \Big(\cosh \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q + \tanh \psi_\mathrm{s} \sqrt{\mathcal{P}} \partial_\vartheta \\ &+ (\frac{1}{\cosh \psi_\mathrm{s}} - \cosh \psi_\mathrm{s}) \frac{1}{\sqrt{-\mathcal{Q}}} (a q^2 \partial_{\tilde{\phi}} + [1 + (l+a)^2 q^2] \partial_{\tilde{t}}) \Big) . \end{aligned} \tag{4.118}$$

Again, the vector $\mathbf{t}_\mathrm{s}$ is a projection of $\mathbf{k}_\mathrm{s}$ onto $\mathcal{I}$ and $\mathbf{n}$ is normal vector $\mathbf{n}$ (9.88) to $\mathcal{I}$, the scalar product must be $\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = 0$. When we calculate the scalar product,

$$\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = - \frac{1}{\sqrt{\mathcal{Q} + \alpha^2 \frac{a^2}{\omega^2} \mathcal{P} \sin^2 \vartheta}} \Big(\cosh \psi_\mathrm{s} \sqrt{-\mathcal{Q}} - \tanh \psi_\mathrm{s} \alpha \tfrac{a}{\omega} \sqrt{\mathcal{P}} \sin \vartheta \Big) , \tag{4.119}$$

we observe that when $\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = 0$ is satisfied, the implied expression does not give useful results. This case again does not represent the asymptotic behavior of the gravitational field of the complete family of black hole solutions in anti-de Sitter background.

Now we will investigate the case 3.6.4, in which *both PNDs are oriented to be outgoing or both ingoing.* The algebraically special orthonormal tetrad is (9.84). The reference tetrad expressed in terms of the algebraically special tetrad is the inverse of relations (3.68),

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= + \sinh^{-1} \psi_\mathrm{s} \, \mathbf{r}_\mathrm{s} + \coth \psi_\mathrm{s} \mathbf{t}_\mathrm{s} , \\ \mathbf{q}_\mathrm{o} &= - \sinh^{-1} \psi_\mathrm{s} \, \mathbf{t}_\mathrm{s} - \coth \psi_\mathrm{s} \, \mathbf{r}_\mathrm{s} , \\ \mathbf{r}_\mathrm{o} &= \mathbf{q}_\mathrm{s} , \\ \mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s} . \end{aligned} \tag{4.120}$$

When we substitute (9.84) into (9.139), we obtain the reference tetrad in terms of the coordinate tetrad as

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \Big(\sinh^{-1} \psi_\mathrm{s} \sqrt{\mathcal{P}} \partial_\vartheta + \coth \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q \Big) , \\ \mathbf{q}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \Big(- \sinh^{-1} \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q - \coth \psi_\mathrm{s} \sqrt{\mathcal{P}} \partial_\vartheta \Big) , \\ \mathbf{r}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \frac{-1}{\sqrt{-\mathcal{Q}}} (a q^2 \partial_{\tilde{\phi}} + [1 + (l+a)^2 q^2] \partial_{\tilde{t}} , \\ \mathbf{s}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \frac{[(4l \sin^2(\vartheta/2) + a \sin^2 \vartheta) \partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}} \sin \vartheta} . \end{aligned} \tag{4.121}$$

This reference tetrad must satisfy the condition

$$\mathbf{n} = -\epsilon_{\rm o}\frac{\mathbf{\Omega}}{\varrho}\left(-\sinh^{-1}\psi_{\rm s}\sqrt{-\mathcal{Q}}\partial_q - \coth\psi_{\rm s}\sqrt{\mathcal{P}}\partial_\vartheta\right). \tag{4.122}$$

By comparing (9.141) with (9.112), we observe that it is necessary to set and evaluate on $\mathcal{I}$ the following expressions

$$\cosh\psi_{\rm s} = \frac{\alpha\frac{a}{\omega}\sqrt{\mathcal{P}}\sin\vartheta}{\sqrt{-\mathcal{Q}_{\mathcal{I}}}}, \tag{4.123}$$

$$\coth\psi_{\rm s} = \frac{\alpha\frac{a}{\omega}\sqrt{\mathcal{P}}\sin\vartheta}{\sqrt{\mathcal{Q}_{\mathcal{I}} + \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}}, \tag{4.124}$$

$$\sinh\psi_{\rm s} = \sqrt{\frac{\mathcal{Q} + \alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta}{-\mathcal{Q}_{\mathcal{I}}}}, \tag{4.125}$$

where $\mathcal{Q}_{\mathcal{I}}$ is the expression (9.75), $\mathcal{P}$ is (9.63) and we also observe that $\epsilon_{\rm o} = +1$. It means that the normal vector is outgoing and both PNDs are outgoing with respect to the timelike conformal infinity $\mathcal{I}$. Again, we evaluated the above expressions on $\mathcal{I}$ and they are dependent only on the coordinate ϑ.

We may calculate the radiative component of the gravitational field. The procedure is almost the same as in the previous spacelike case, the normalization factor is (9.95) and its evaluation on $\mathcal{I}$ is (9.102). The asymptotic directional structure has a form (3.73),

$$|\Psi_4^{\rm i}| \approx \frac{1}{|\eta|}\frac{3}{2}\frac{|\Psi_{2*}^{\rm s}|}{\sinh^2\psi_{\rm s}}\mathcal{A}_2(\psi,\phi,\psi_{\rm s}). \tag{4.126}$$

Where we denoted the angular dependence as

$$\mathcal{A}_2(\psi,\phi,\psi_{\rm s}) = (\cosh\psi_{\rm s} + \epsilon\cosh\psi)^2 + \sinh^2\psi_{\rm s}\sinh^2\psi\sin^2\phi,$$

and we substituted $\epsilon_{\rm o} = +1$. Then $\sinh\psi_{\rm s}$ can be rewritten from (9.144) using (9.72) as

$$\sinh^2\psi_{\rm s} = -\frac{\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)} \tag{4.127}$$

and we can also rewrite $\cosh\psi_{\rm s}$ from (9.142) as

$$\cosh\psi_{\rm s} = \frac{\sqrt{\mathcal{P}}\alpha\frac{a}{\omega}\sin\vartheta}{\sqrt{\alpha^2\frac{a^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}}, \tag{4.128}$$

where the function $\mathcal{P}$ is given by (9.63).
We also denote the amplitude of the radiation as

$$B(\vartheta) = \frac{|\Psi_{2*}^{\rm s}|_{\mathcal{I}}}{\sinh^2\psi_{\rm s}}. \tag{4.129}$$

Consequently, the asymptotic directional structure has a form

$$|\Psi_4^{\mathrm{i}}| \approx \frac{3}{2}\frac{1}{|\eta|}B(\vartheta)\mathcal{A}_2(\psi,\phi,\psi_{\mathrm{s}}), \tag{4.130}$$

where $B(\vartheta)$ can be rewritten by (9.102), (9.147) and (9.107) as

$$B(\vartheta) = -\frac{\alpha^2\frac{\alpha^2}{\omega^2}\mathcal{P}\sin^2\vartheta + \frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4)}{\frac{\Lambda}{3}(1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4))^{5/2}}\sqrt{\mathcal{D}_{\mathcal{I}}} \tag{4.131}$$

with

$$\begin{aligned}\mathcal{D}_{\mathcal{I}} =& m^2+n^2-4\tfrac{\alpha}{\omega}(e^2+g^2)(l+a\cos\vartheta)\frac{m+n\frac{\alpha}{\omega}(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4}\\ &+4\tfrac{\alpha^2}{\omega^2}(e^2+g^2)^2\frac{(l+a\cos\vartheta)^2}{1+\frac{\alpha^2}{\omega^2}(l+a\cos\vartheta)^4},\end{aligned} \tag{4.132}$$

where the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65).

We obtained the explicit formula for the radiative component of gravitational field of the accelerated black holes in the anti-de Sitter universe for $\alpha^2 k > -\Lambda/3$. Both PNDs are oriented outwards with respect to $\mathcal{I}$. There is no *outgoing* direction along which the radiation vanishes because mirrored reflections of both PNDs are ingoing. But there are two *ingoing* vanishing-radiation directions given by the mirrored reflections of both PNDs. This result is consistent with [5] where the C-metric was investigated for $\alpha > \sqrt{-\Lambda/3}$.

We observe that the amplitude $B(\vartheta)$ given by (9.149) is *identical* up to a sign with the magnitudes from previous sections for spacelike $\mathcal{I}$ and for timelike $\mathcal{I}$ with small acceleration, see equations (9.106), (9.107) and (9.121), (9.122). The dependence of the amplitude $B(\vartheta)$ will be investigated in section 9.5.

We now investigate the next case described in section 3.6.5, *the case when one PND is tangent to* $\mathcal{I}$. The identification of the algebraically special orthonormal tetrad is (9.84). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.77). We obtain

$$\begin{aligned}\mathbf{t}_{\mathrm{o}} &= \tfrac{3}{2}\mathbf{t}_{\mathrm{s}}+\tfrac{1}{2}\mathbf{q}_{\mathrm{s}}-\mathbf{r}_{\mathrm{s}},\\ \mathbf{q}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}}-\tfrac{1}{2}\mathbf{t}_{\mathrm{s}}+\tfrac{1}{2}\mathbf{q}_{\mathrm{s}},\\ \mathbf{r}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}}-\mathbf{t}_{\mathrm{s}}-\mathbf{q}_{\mathrm{s}},\\ \mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.\end{aligned} \tag{4.133}$$

Substituting (9.84) into (9.152), we obtain the reference tetrad

$$
\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\frac{3}{2}\sqrt{-\mathcal{Q}}\,\partial_q - \frac{1}{2}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) - \sqrt{\mathcal{P}}\partial_\vartheta\right), \\
\mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sqrt{\mathcal{P}}\partial_\vartheta - \frac{1}{2}\sqrt{-\mathcal{Q}}\,\partial_q - \frac{1}{2}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}})\right), \\
\mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sqrt{\mathcal{P}}\partial_\vartheta + \frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}) - \sqrt{-\mathcal{Q}}\,\partial_q\right), \\
\mathbf{s}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned}
\tag{4.134}
$$

This reference tetrad must satisfy the adjustment condition (3.19). When we compare the vector $\mathbf{q}_{\mathrm{o}}$ with the normal $\mathbf{n}$ (9.112), we observe that the vector $\mathbf{q}_{\mathrm{o}}$ has components ∂_q and ∂_ϑ pointing in the same direction as the normal $\mathbf{n}$, but $\mathbf{q}_{\mathrm{o}}$ has another component which cannot disappear. This reference tetrad thus again does *not* satisfy the adjustment condition. We can also calculate the projection of $\mathbf{k}_{\mathrm{s}}$ into $\mathcal{I}$ as in the section for small acceleration and its discussion is quite similar. This case again does not represent the asymptotic behaviour of the gravitational field of black hole solutions in anti-de Sitter.

The last possible case is described in the section 3.6.6 when *two PNDs are tangent to* $\mathcal{I}$. We write the inverse relation between the reference tetrad and the algebraically special tetrad (3.86) as

$$
\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \mathbf{t}_{\mathrm{s}}, \\
\mathbf{q}_{\mathrm{o}} &= -\mathbf{r}_{\mathrm{s}}, \\
\mathbf{r}_{\mathrm{o}} &= \mathbf{q}_{\mathrm{s}}, \\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}},
\end{aligned}
\tag{4.135}
$$

and after substituting from (9.84) into (9.154), we get

$$
\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\,\partial_q, \\
\mathbf{q}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{P}}\partial_\vartheta, \\
\mathbf{r}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{-\mathcal{Q}}}(aq^2\partial_{\tilde{\phi}} + [1+(l+a)^2q^2]\partial_{\tilde{t}}), \\
\mathbf{s}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\frac{[(4l\sin^2(\vartheta/2) + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]}{\sqrt{\mathcal{P}}\sin\vartheta}.
\end{aligned}
\tag{4.136}
$$

As in the $\alpha^2 k < -\Lambda/3$ case, this reference tetrad satisfy the adjustment condition $\mathbf{n} = -\epsilon_{\mathrm{o}}\mathbf{q}_o$ for $\sigma = +1$ because the vector $\mathbf{q}_{\mathrm{o}}$ points into the ∂_ϑ direction as the normal for special case when $\mathcal{Q}_{\mathcal{I}} = 0$. We will not discuss this special case in this work. This degenerate case thus also occur as a possible asymptotic directional structure of the complete family of black hole spacetimes in anti-de Sitter space for $\alpha^2 k > -\Lambda/3$.

To sum up: the asymptotic directional structure of radiation near the timelike conformal infinity $\mathcal{I}$ for the general metric (9.62) with large acceleration $\alpha^2 > -\Lambda/3$ is characterized by the case (9.149) corresponding to the situation when *both PNDs* are oriented outward with respect to $\mathcal{I}$. The other possible exceptional structure is when both PNDs are tangent to $\mathcal{I}$.

This result is not yet completely consistent with the results presented in [5] for the C-metric. The C-metric has two possible directional structures, first when one PND $\mathbf{k}$ is outgoing and the other one PND $\mathbf{l}$ is ingoing, second when two PNDs are oriented outwards with respect to $\mathcal{I}$. The conformal infinity in C–metric is divided into domains with different structures of PNDs, on the boundaries the PNDs are tangent to $\mathcal{I}$. The C–metric represents two accelerating black holes for $\alpha > \sqrt{-\Lambda/3}$ which are causally separated.

The first case does not occur in the general metric studied here for large acceleration. Most probably, the coordinates which we are using here do *not cover completely* the manifold of the general metric near $\mathcal{I}$. The possible structure of both PNDs tangent to $\mathcal{I}$ on horizons of $\mathcal{Q}$ on $\mathcal{I}$ also does not appear in the C–metric. More work is necessary to investigate this particular feature.

4.3 Discussion of the amplitude $B(\vartheta)$ of radiation

In previous sections we found that the amplitude of radiation $B(\vartheta)$ has an identical form for spacelike $\mathcal{I}$ as (9.106), (9.107) with $\alpha^2 k > -\Lambda/3$, and for timelike $\mathcal{I}$ as (9.121), (9.122) with $\alpha^2 k < -\Lambda/3$. The amplitude $B(\vartheta)$ for timelike $\mathcal{I}$ given by (9.150), (9.151) with $\alpha^2 k > -\Lambda/3$ is similar up to a sign.

The amplitude $B(\vartheta)$ is very similar for all these cases because it is generally given by the same field component $|\Psi^{\rm s}_{2*}|_{\mathcal{I}}$ and by $\cos\theta_{\rm s}$ (9.99) for spacelike $\mathcal{I}$, or $\cosh\psi_{\rm s}$ (9.118) for timelike $\mathcal{I}$ ($\alpha^2 k < -\Lambda/3$) which have, quite surprisingly, the same form. The term $\sinh\psi_{\rm s}$ (9.146) for timelike $\mathcal{I}$ ($\alpha^2 k > -\Lambda/3$) is the same up to a sign. To have this just one expression for amplitude $B(\vartheta)$ for all cases mentioned above. For convenience, we introduce *parameter* b which distinguishes these cases. The parameter is $b = 1$ applies for spacelike $\mathcal{I}$ and timelike $\mathcal{I}$ for $\alpha^2 k < -\Lambda/3$, while $b = -1$ for timelike $\mathcal{I}$ with large acceleration $\alpha^2 k > -\Lambda/3$.

The amplitude $B(\vartheta)$ depends on a single coordinate $\vartheta \in [0, \pi]$ and on the physical parameters α, m, a, l, e, g, Λ (and ω). We will now investigate the shape of the function $B(\vartheta)$ and the influence of these parameters. The conformal infinities are distinguished by different ranges of parameters α and Λ and also by the parameter b.

First, we will study the physically most interesting case of accelerating Kerr–Newman–de Sitter black holes (see section 9.2.2): we set the NUT parameter $l = 0$. This implies $\omega = a$ ($k = 1$). The parameter n is then $n = -\alpha a m$, and when we put these parameters into (9.150) and (9.151), we get the amplitude

$B(\vartheta)$ in the form

$$B(\vartheta) = b\frac{(1+\alpha^2a^2\cos^4\vartheta)+\frac{3}{\Lambda}\alpha^2\mathcal{P}\sin^2\vartheta}{(1+\alpha^2a^2\cos^4\vartheta)^{5/2}} \times \sqrt{m^2(1+\alpha^2a^2)-4m\alpha(e^2+g^2)\frac{(1-\alpha^2a^2\cos^2\vartheta)\cos\vartheta}{1+\alpha^2a^2\cos^4\vartheta}+4(e^2+g^2)^2\frac{\alpha^2\cos^2\vartheta}{1+\alpha^2a^2\cos^4\vartheta}}, \tag{4.137}$$

where

$$\mathcal{P} = 1-2\alpha m\cos\vartheta+\left[\alpha^2(a^2+e^2+g^2)+\frac{\Lambda}{3}a^2\right]\cos^2\vartheta \tag{4.138}$$

which must be positive to retain the Lorentzian signature. Notice that for $\vartheta=\frac{\pi}{2}$ one gets simply

$$B(\vartheta=\tfrac{\pi}{2}) = b\,m(1+\tfrac{3}{\Lambda}\alpha^2)\sqrt{1+\alpha^2a^2}. \tag{4.139}$$

For bigger rotational parameter a, the radiation is thus stronger when ($\alpha\neq 0$).

In particular, to visualize the results we have chosen the values of parameters of the Kerr–Newman black holes as $m=1.1$, $e=g=0.5$. The parameter a then can not be chosen arbitrarily. It is necessary to respect the location of horizon of the black hole. The function Q has a form (see section (9.2.2))

$$Q = (\omega^2k+e^2+g^2)-2mr+r^2-\frac{1}{3}\Lambda r^2(r^2+a^2). \tag{4.140}$$

The horizons are determined by $Q=0$ and the above function is cubic equation for r and it is not easy to solve. If we consider horizons for $\Lambda=0$ the function (9.160) factorizes as

$$Q = \left((a^2+e^2+g^2)-2mr+r^2\right)(1-\alpha^2r^2). \tag{4.141}$$

The inner/outer horizons of the black hole occurs for $\Lambda=0$ on

$$r_\pm = m\pm\sqrt{m^2-a^2-e^2-g^2} \tag{4.142}$$

and the acceleration horizon is located on $r=1/\alpha$. To avoid a naked singularity case, we thus have to satisfy the condition

$$m^2-a^2-e^2-g^2>0. \tag{4.143}$$

which follows from (9.162). According to this relation we set $a\in[0,0.7]$. Let us remark that we have used the coordinate $r=-1/q$ instead of q, but these conclusions are obviously valid in general.

The *spacelike* infinity is further characterized by $\Lambda>0$ (we set $\Lambda=1$) and $\alpha\geq 0$. The amplitude $B(\vartheta)$ for spacelike $\mathcal{I}$ ($b=+1$) is visualized in Figs. 3–8. There are always figures without charges ($e=g=0$) and with charges ($e=g=0.5$). For comparison, the figures also include the constant line corresponding to $\alpha=0$.

This line represents the constant value of m and it is exactly the Kerr–Newman-de Sitter limit of (9.157) and non-accelerating limit of C-metric corresponding to expression (4.24).

Fig. 3 exhibits the same pattern as Fig. 4. The curves start at $\vartheta = 0$ under the $\alpha = 0$ line, then they reach considerably high maximum and finally fall again under the $\alpha = 0$ line at the end of the axes $\vartheta = \pi$. There is a difference in the amplitudes. The charged case on Fig. 4 has *higher* amplitude than uncharged one on Fig. 3. Also Fig. 5 and Fig. 6 have similar patterns as on the previous figures, but the amplitudes of these figures are higher than in Figs. 3 and 4. It shows that *the acceleration has stronger* influence than rotation even for small values. In Fig. 7 the curves start and end near the red line corresponding to $\alpha = 0$. The charged version of the previous figure, Fig. 8, is shifted, the curves start under this line and end above it.

The *timelike* infinity has $\Lambda < 0$ (we set $\Lambda = -1$). We have studied timelike $\mathcal{I}$ twice, i.e. for different ranges of acceleration α. For small acceleration $\alpha^2 k < -\Lambda/3$ and for $l = 0$ the parameter k is $k = 1$, so it implies $\alpha^2 < -\Lambda/3$. It has two roots and we have chosen it according to [5] as $\alpha < \sqrt{-\Lambda/3}$. For large acceleration the range of acceleration is thus $\alpha > \sqrt{-\Lambda/3}$.

The amplitude $B(\vartheta)$ for small acceleration with $(b = +1)$ is visualized in Figs. 9–14. The figures are basically the same as for spacelike $\mathcal{I}$, but they are turned upside down. Thus the cosmological constant strongly determines the shape the amplitude of radiation on $\mathcal{I}$. The amplitude $B(\vartheta)$ with large acceleration $(b = -1)$ is visualized in Figs. 15–18. The fact that the radiation appears negative in these figures is caused by the violation of the condition (9.163) which we are considering and which is satisfied here only for $\Lambda = 0$. The parameters are thus chosen not completely correctly to describe a physically realistic situation.

Secondly, we will discuss the important special case namely, the non–rotating charged C–metric with cosmological constant. It can be obtained from the previous amplitude $B(\vartheta)$ given by (9.157) and (9.158) when we simply set $a = 0$. Then we obtain the following expression for the amplitude of radiation,

$$B(\vartheta) = b\,(1 + \tfrac{3}{\Lambda}\alpha^2\mathcal{P}\sin^2\vartheta)\sqrt{m^2 - 4m\alpha(e^2+g^2)\cos\vartheta + 4\alpha^2(e^2+g^2)^2\cos^2\vartheta}, \tag{4.144}$$

where

$$\mathcal{P} = 1 - 2\alpha m\cos\vartheta + \alpha^2(e^2+g^2)\cos^2\vartheta. \tag{4.145}$$

The values of the parameters used in the corresponding figures are the same as before, except for the values of charges. We can now afford higher values (see (9.163)) when we have $a = 0$. We set $e = g = 0.7$.

The amplitude $B(\vartheta)$ for *spacelike* $\mathcal{I}$ is visualized in two figures Fig. 19 and charged version is in Fig. 20. We observe that when the rotational parameter is zero or is small, charges have greater influence.

The amplitude $B(\vartheta)$ for *timelike* $\mathcal{I}$ with small acceleration is visualized in Fig. 21 and the charged version is in Fig. 22. The amplitude for *timelike* $\mathcal{I}$ with large acceleration is visualized in Fig. 23 and the charged version is in Fig. 24.

To conclude our discussion, we investigated influence of the physical parameters of the sources, namely α, a, m, e, g and Λ on the amplitude of the radiation (9.157) which appears at spacelike and timelike conformal infinity in the subcase $l = 0$ of the general metric (9.62) and its special subcase of the C-metric. Generally, we observe that the acceleration parameter α and the cosmological constant Λ have *dominant* influence on the amplitude. The acceleration parameter α is stronger than the rotational parameter a. The charges e and g have a smaller influence than acceleration but their influence is quite stronger when the rotational parameter a is small or vanishes.

5 Conclusion

In our work we analyzed the asymptotic directional structure of gravitational field in the family of solutions of algebraic type D, namely in the Plebański-Demiański black-hole spacetimes.

We demonstrated that two different choices of algebraically special tetrads are needed to study the directional structure near the de Sitter-like ($\Lambda > 0$) and anti-de Sitter-like ($\Lambda < 0$) conformal infinity $\mathcal{I}$, cf. subsection 9.1.1.

First, we investigated the asymptotic directional structure of the non-accelerating Kerr–Newman–de Sitter solution. The directional pattern (4.24) is determined by two PNDs oriented outwards with respect to the future spacelike conformal infinity $\mathcal{I}^+$. The pattern of radiation (4.31) near the timelike $\mathcal{I}$ is given by only one structure of PNDs of four general possibilities. In these cases, the explicit forms are axially symmetric and the radiation vanishes in the directions where $\theta = 0$ and $\theta = \pi$. From these results we also observed that the contribution of the electric and magnetic charges e and g is negligible compared to the mass m of the black hole to the order r^{-1}, and that the influence of the rotation represented by a is even smaller of the order r^{-2}. The fact that generic observer detects radiation in this case of non–accelerating sources is intuitively caused by observer's asymptotic motion relative to the "static" black hole. We have thus basically verified results for non–rotating C-metric obtained in [4] and in [5] for $\Lambda > 0$ and $\Lambda < 0$, respectively.

Subsequently, the general form of Plebański–Demiański metric which represents the complete family of black hole spacetimes was investigated. The expressions (9.105), (9.106) and (9.107) give the explicit formula for the radiative component of gravitational field in the accelerated black holes with mass m, charges e and g, rotation a and NUT parameter l in the de Sitter universe. The pattern is determined by two PNDs which are both oriented outside the manifold on future spacelike $\mathcal{I}^+$. We also investigated the structure near timelike $\mathcal{I}$ for both cases where $\alpha^2 k > -\Lambda/3$ and $\alpha^2 k < -\Lambda/3$. Thus we obtained the explicit expressions for the radiative component of gravitational field for small and large acceleration for negative cosmological constant.

The directional structure for $\alpha^2 k < -\Lambda/3$ is described by (9.120) which is determined by one PND $\mathbf{k}$ outgoing and the other PND $\mathbf{l}$ ingoing with respect to $\mathcal{I}$. A special structure of both PNDs tangent to $\mathcal{I}$ occurs which is only possible on the horizon of the function $\mathcal{Q}$ evaluated on $\mathcal{I}$. This exceptional situation was not mentioned in [5] where the C-metric was investigated for $\alpha < \sqrt{-\Lambda/3}$.

The directional pattern in the case of large acceleration $\alpha > \sqrt{-\Lambda/3}$ is given by (9.149) corresponding to the situation when *both PNDs* are oriented outward with respect to $\mathcal{I}$. There again occurs the case when two PND are tangent to $\mathcal{I}$. This is not yet completely in agreement with the results presented in [5] for C–metric because one possible case (one PND $\mathbf{k}$ outgoing and the other PND $\mathbf{l}$ ingoing) does not occur in the general metric for large acceleration. We think that the reason is that the coordinates which we use here most probably do not cover completely the region near the conformal infinity of the general metric.

We have observed that the amplitude $B(\vartheta)$ given by (9.121) is up to a sign the same for *all* the possible directional structures which occur on spacelike $\mathcal{I}$ and for timelike $\mathcal{I}$ with small acceleration, and for timelike $\mathcal{I}$ for large acceleration (except for one special case). Thus, they were investigated together in section 9.5. The amplitude of radiation $B(\vartheta)$ is given by the expression which depends on a single coordinate ϑ and on the physical parameters m, α, ω, a, Λ, e and g. The coordinate ϑ specifies a point on the conformal infinity $\mathcal{I}$, while the parameters characterize the black-hole sources of the radiation. The dependence of the radiation on the parameters of sources is discussed and presented in several figures which are contained in appendix.

6 Appendix A: The dependence of the amplitude $B(\vartheta)$ of radiation on the choices of parameters

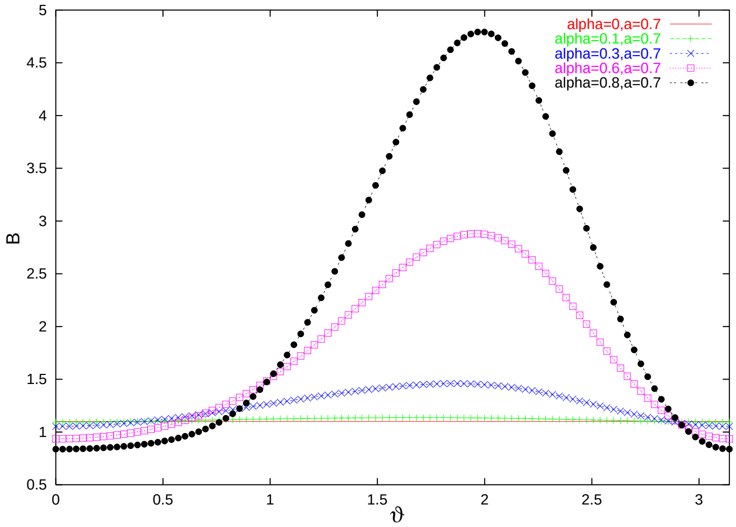

Figure 3: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ without charges, where $a = 0.7$ is fixed and α varies, the constant line for $\alpha = 0$ is shown for comparison.

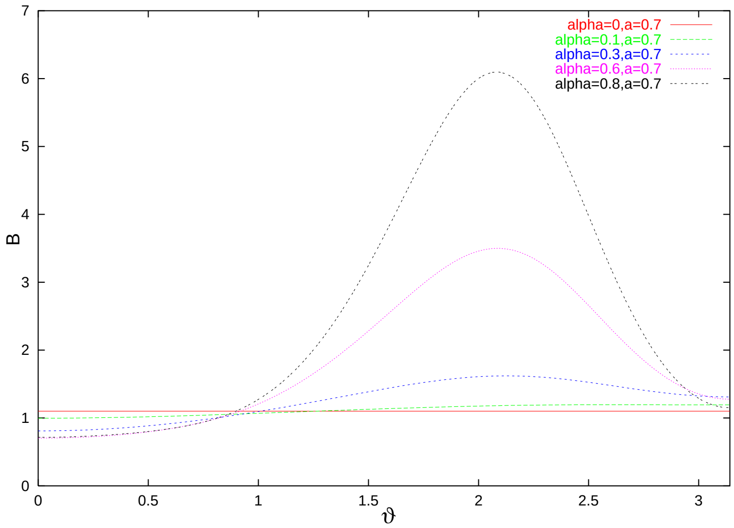

Figure 4: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ with charges, where $a = 0.7$ is fixed and α varies. The constant line for $\alpha = 0$ is shown for comparison.

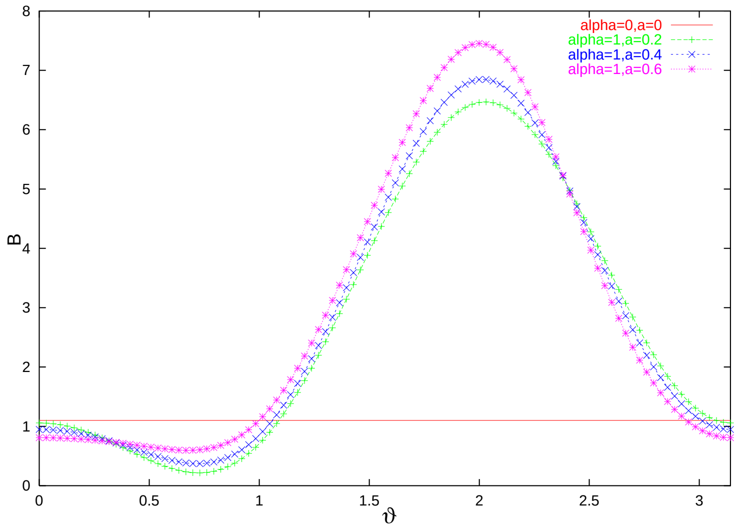

Figure 5: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ without charges, where $a \in [0, 0.7]$ varies and $\alpha = 1$ is fixed. The constant line for $\alpha = 0$ is shown for comparison.

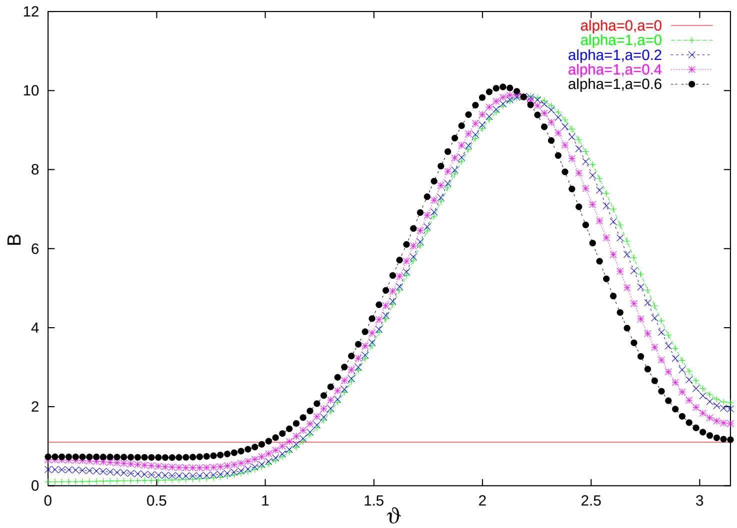

Figure 6: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ with charges, where $a \in [0, 0.7]$ varies and $\alpha = 1$ is fixed. The constant line for $\alpha = 0$ is shown for comparison.

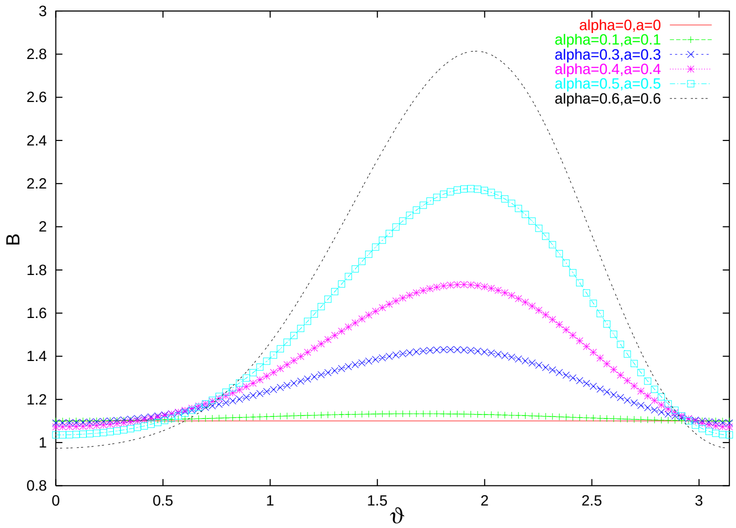

Figure 7: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ without charges, where the parameters $\alpha = a \in [0, 0.7]$ vary. The constant line for $\alpha = 0$ is shown for comparison.

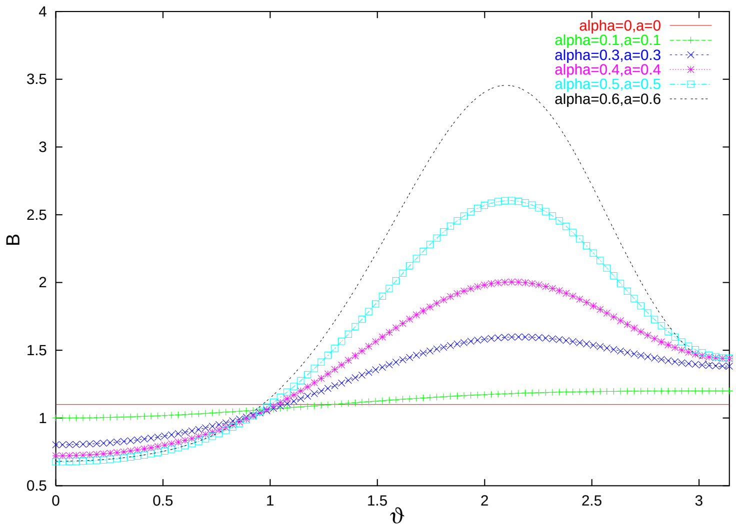

Figure 8: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ with charges, where the parameters $\alpha = a \in [0, 0.7]$ vary. The constant line for $\alpha = 0$ is shown for comparison.

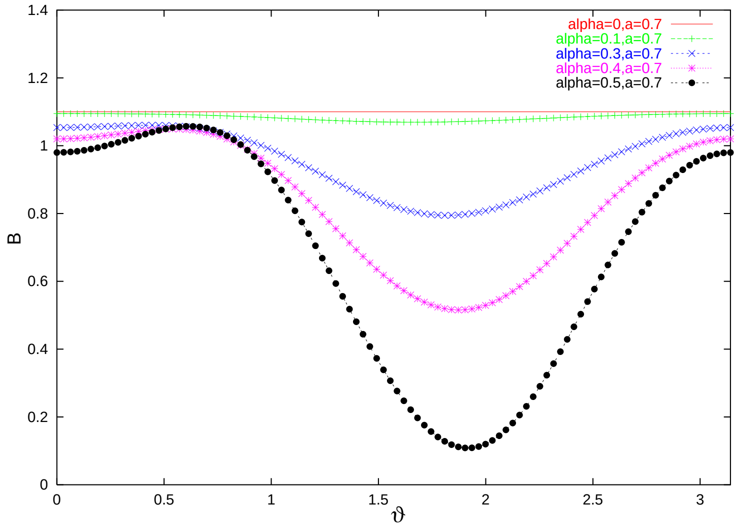

Figure 9: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ without charges, where $a = 0.7$ is fixed and $\alpha \in [0, 0.5]$ varies.

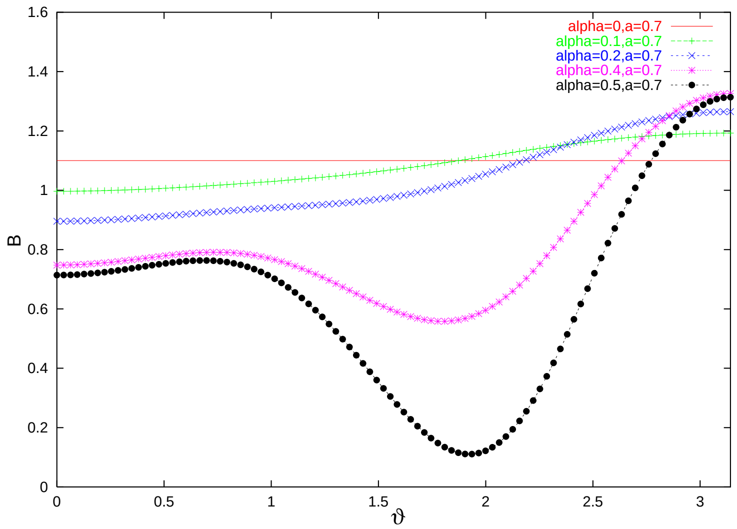

Figure 10: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ with charges, where $a = 0.7$ is fixed and $\alpha \in [0, 0.5]$ varies.

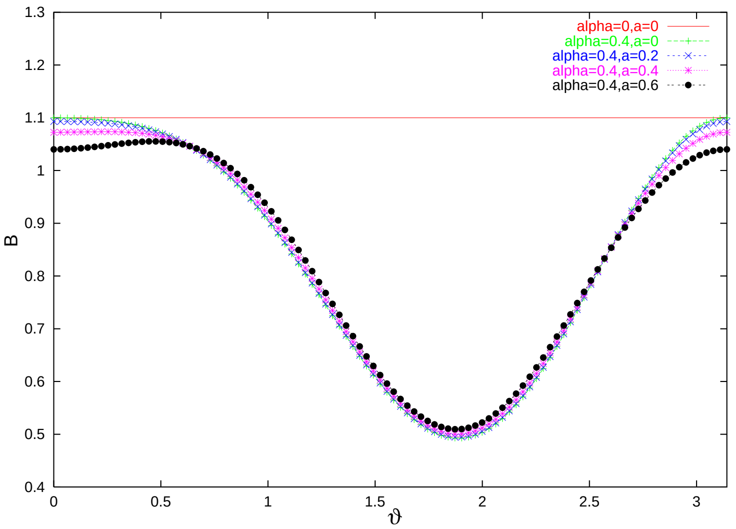

Figure 11: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ without charges, where $a \in [0, 0.7]$ varies and $\alpha = 0.4$ is fixed.

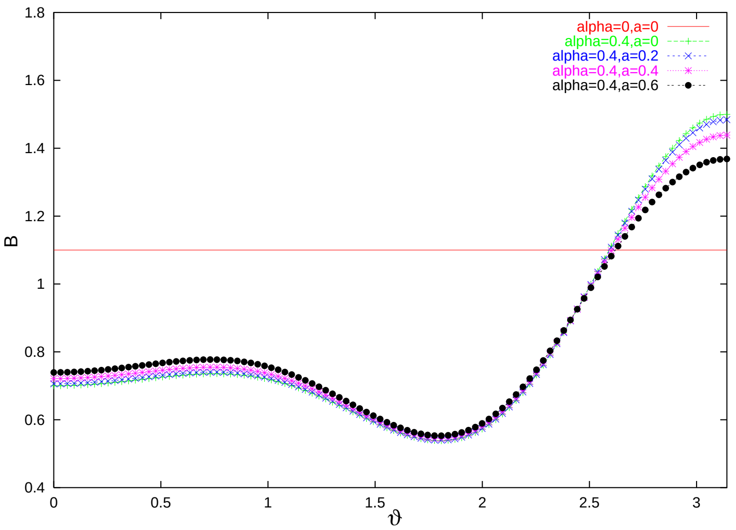

Figure 12: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ with charges, where $a \in [0, 0.7]$ varies and $\alpha = 0.4$ is fixed.

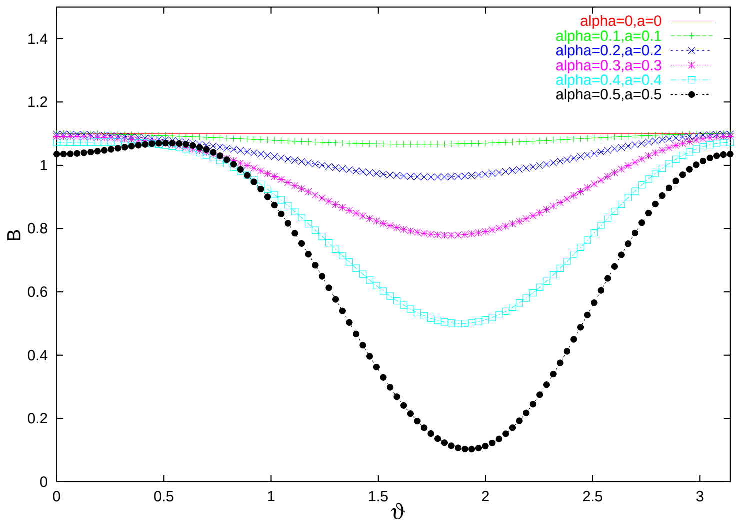

Figure 13: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ without charges, where $a \in [0, 0.5]$ and $\alpha \in [0, 0.5]$ varies.

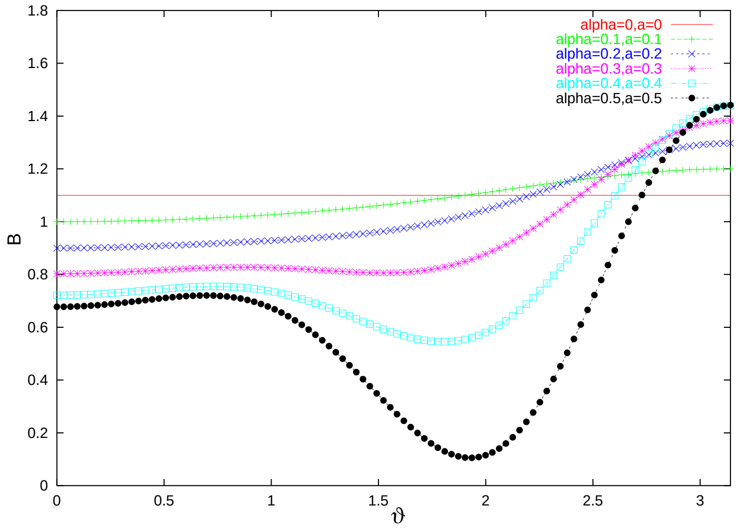

Figure 14: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$ with charges, where $a \in [0, 0.5]$ and $\alpha \in [0, 0.5]$ varies.

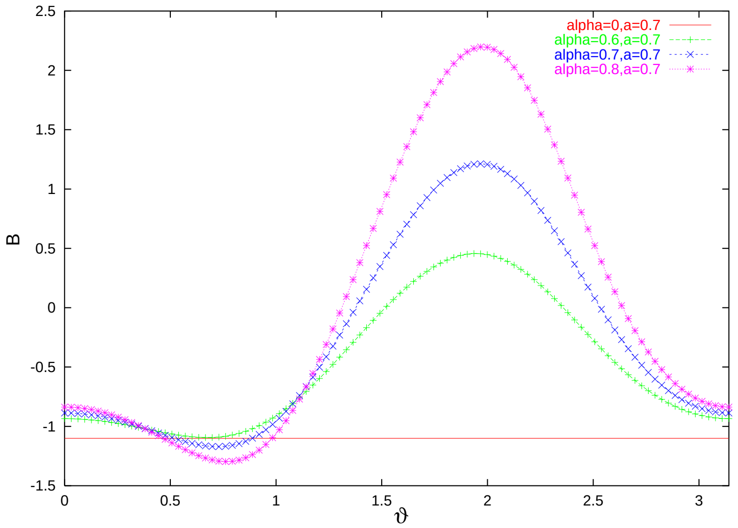

Figure 15: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$ without charges, where $a = 0.7$ is fixed and α varies.

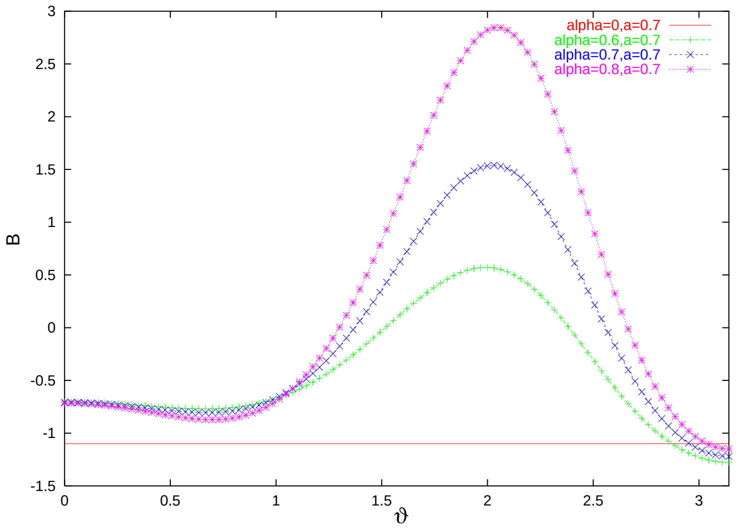

Figure 16: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$ with charges, where $a = 0.7$ is fixed and α varies.

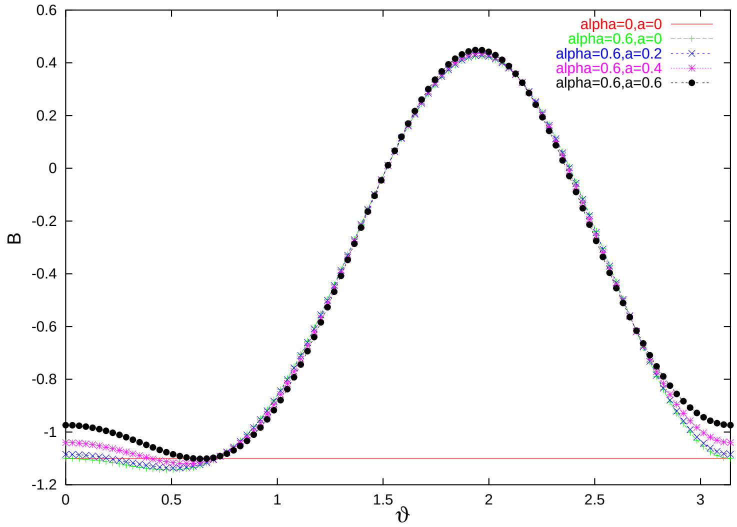

Figure 17: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$ without charges, where $a \in [0, 0.7]$ varies and $\alpha = 0.6$ is fixed.

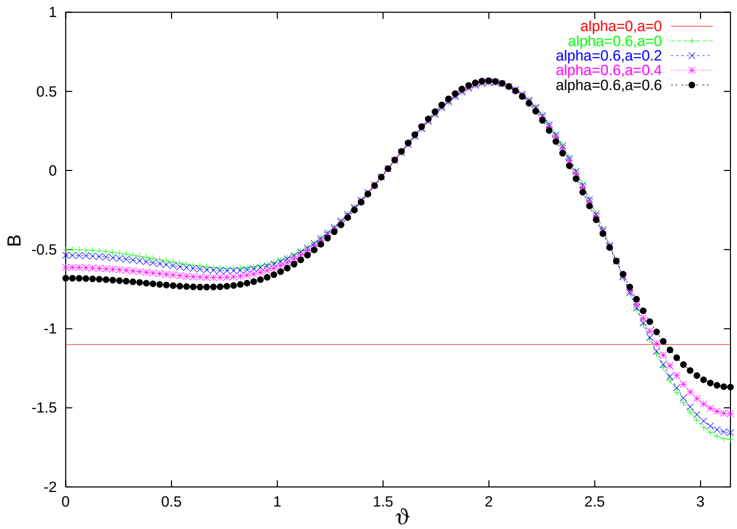

Figure 18: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$ with charges, where $a \in [0, 0.7]$ varies and $\alpha = 0.6$ is fixed.

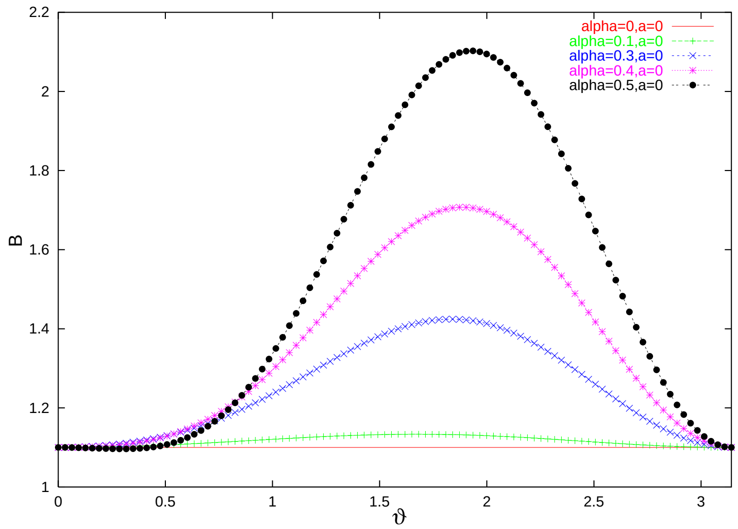

Figure 19: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ for C–metric without charges, where $a = 0$ and α varies.

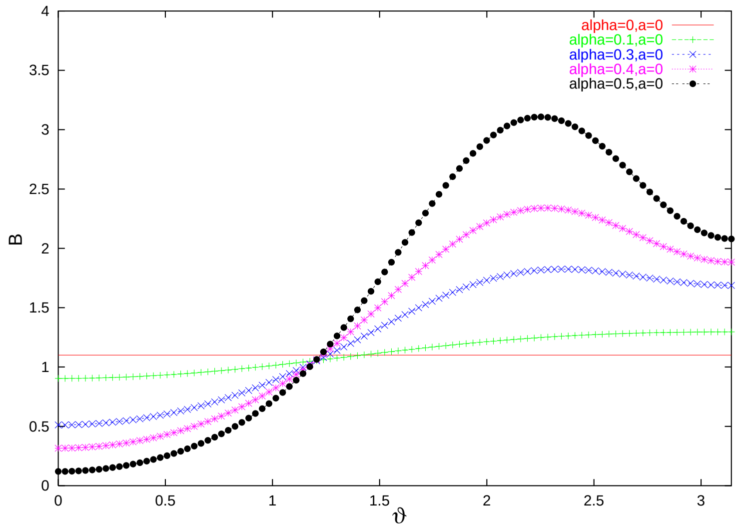

Figure 20: The magnitude of radiation $B(\vartheta)$ on spacelike $\mathcal{I}$ for charged C–metric, where $a = 0$ and α varies.

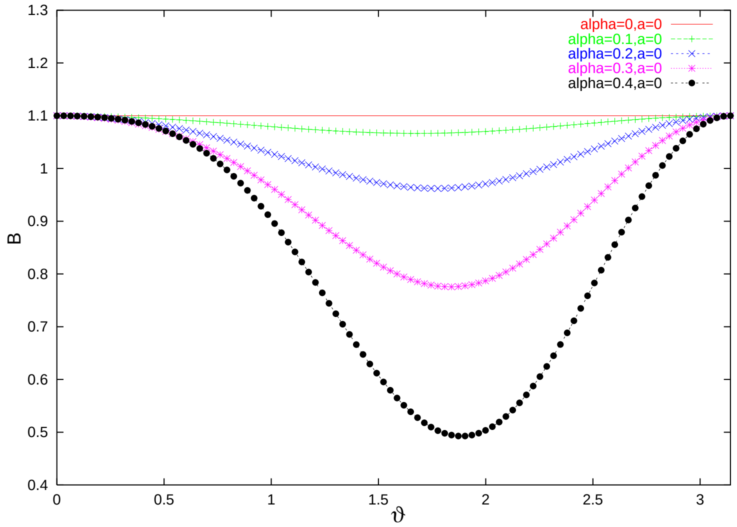

Figure 21: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$, C–metric without charges, where $a = 0$ and α varies.

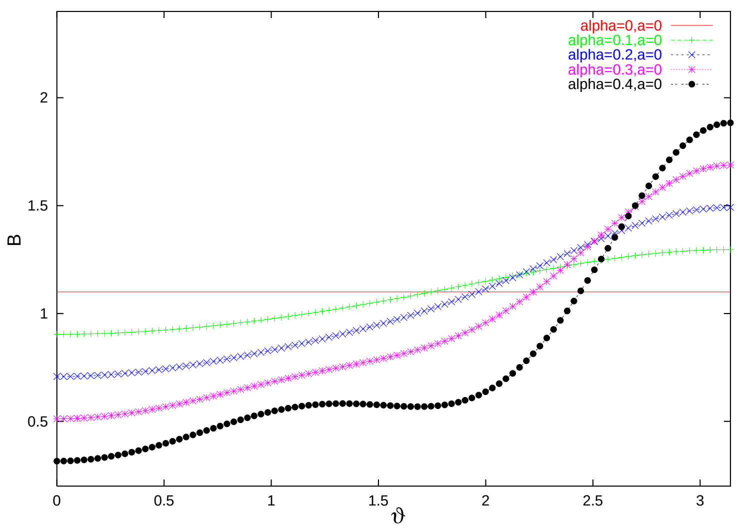

Figure 22: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha < \sqrt{-\Lambda/3}$, charged C–metric, where $a = 0$ and α varies.

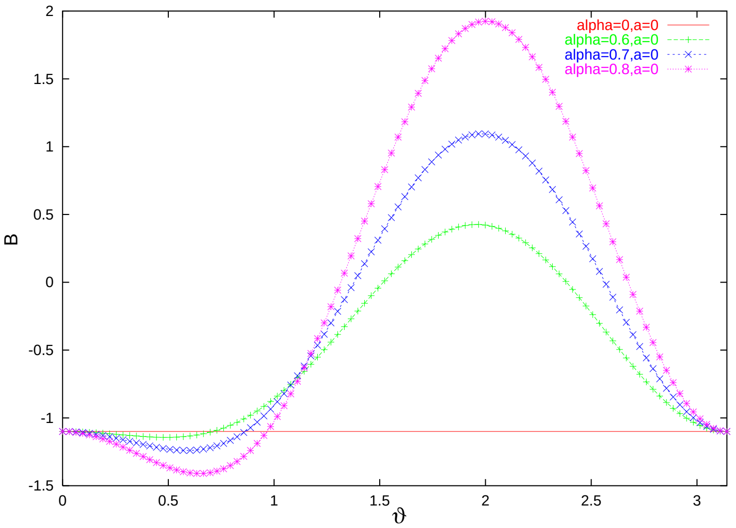

Figure 23: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$, C–metric without charges, where $a = 0$ and α varies.

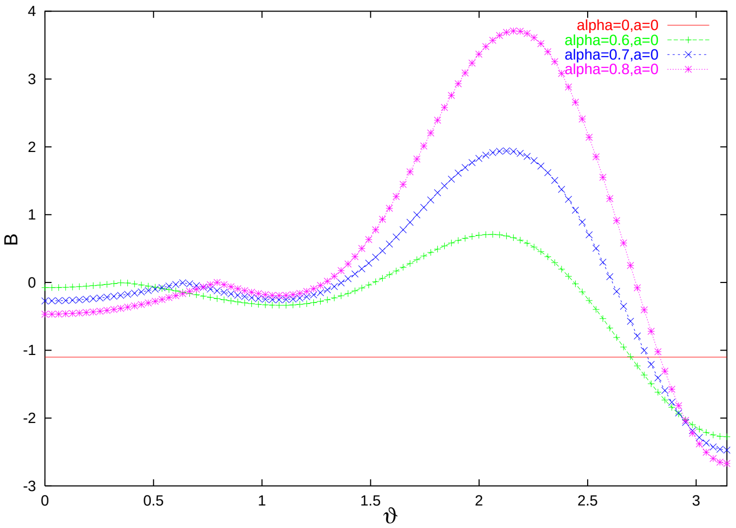

Figure 24: The magnitude of radiation $B(\vartheta)$ on timelike $\mathcal{I}$ for $\alpha > \sqrt{-\Lambda/3}$, charged C–metric, where $a = 0$ and α varies.

7 Appendix B: The version of diploma thesis in other coordinates

8 Introduction

Many rigorous theoretical studies have been devoted to investigation of gravitational waves within the full Einstein theory since 1950s. These are described in various review articles, such as [1, 2] and [3]. In many existing analyses the asymptotic flatness has been naturally assumed and the presence of a non-vanishing cosmological constant Λ was not usually considered. The importance of such studies rises due to the fact that the possible presence of a positive Λ has been indicated by recent observations. Moreover, the spacetimes with cosmological constant are now used in various branches of theoretical research, e.g. brane cosmologies, supergravity or string theories.

Krtouš and Podolský recently analyzed the asymptotic directional properties of electromagnetic and gravitational fields in spacetimes with a non-vanishing cosmological constant Λ. It had been known for a long time that the dominant (radiative) component of the fields in spacetimes with $\Lambda \neq 0$ depends on the direction along which a null geodesics approaches a given point at conformal infinity $\mathcal{I}$, contrary to the asymptotically flat spacetimes where the dominant term of radiation is unique.

First, they studied the radiation in the C-metric spacetime with Λ which represents accelerated black holes in de Sitter ($\Lambda > 0$) [4] and anti-de Sitter ($\Lambda < 0$) [5] universe. Their results were summarized and discussed in the topical review [3] where they presented the general asymptotic directional behaviour of any massless field of spin s. The fields of algebraic type D were investigated more explicitly in [6]. Furthermore, the generalization to higher dimensions was recently presented in [7].

The authors demonstrated in [3] that the directional structure of radiation has a universal character which is determined by the algebraic (Petrov) type of the field, namely by the number, degeneracy and specific orientation of the principal null directions with respect to $\mathcal{I}$. It covers all three possibilities $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$, corresponding to a spacelike, timelike or null conformal infinity.

The main intention of this diploma thesis is to apply the above general theory on particular exact model spacetimes of type D and to find the relation between the structure of the sources (namely the mass, the charge, the NUT parameter, the rotation and the acceleration of the corresponding black holes) and the properties of radiation which is generated by them, as observed at spacelike or timelike conformal infinity $\mathcal{I}$. As the specific models we use the Plebański–Demiański family of solutions, first presented in [8]. This class of solutions was studied recently in detail by Griffiths and Podolský in [9, 10, 11] and [12].

The text is organized as follows. In section 2 we review necessary facts about the family of spacetimes of type D found by Plebański and Demiański. In section 3 we first introduce geometrical concepts and objects and we set up the notation which will be followed throughout the text. Then we define tetrads and explicit expression which describes the behavior of the field at any conformal infinity. The relevant work [6] is more explicitly reviewed. In section 4 we apply the general theory introduced in section 3 to particular exact model spacetimes of type D described in section 2. First, we study the Kerr–Newman–de Sitter solution. Then we investigate the general Plebański–Demiański family of solutions, containing not only the mass and the charge of the black holes but also rotation, the NUT parameter and non-zero acceleration. We derive the corresponding radiation near de Sitter or anti-de Sitter-like conformal infinity. In the final subsection 9.5 we discuss the results, in particular the dependence of the amplitude of radiation on the parameters of the sources in various subcases. We hope that this can provide a deeper insight into the general theory of radiation.

9 The Plebański–Demiański class of solutions

The complete family of spacetimes of algebraic type D with an aligned electromagnetic field and a possibly non-zero cosmological constant Λ can be represented in a form of the Plebański–Demiański metric [8, 10] and those specific metrics which can be derived from it by certain limiting procedures (because some particular and well-known cases are not included explicitly in the original form of the Plebański–Demiański metric).

In [10] a new form of this metric was described which is more useful for physical interpretation and for identifying different subfamilies. The parameters employed in the new metric have clear physical interpretation and it is possible to classify the complete family in the way that clarifies their physical properties. The new metric explicitly contains two parameters α and ω which describe the acceleration of the sources and the twist of the principal null congruences. These solutions are characterized by two general quartic functions whose coefficients are related to- the physical parameters of the spacetime. The physical meaning of these coefficients was clarified and the relation between the Plebański–Demiański parameter n and the NUT parameter l was identified. These coefficients were traditionally misinterpreted in the general case.

We will study here the most important solutions from the Plebański–Demiański family. In particular, we wish to investigate the specific radiative properties of (possibly) accelerating black holes with charge and rotation in asymptotically Minkowski, de Sitter or anti–de Sitter universe.

First, we summarize basic facts about these solutions. In [10] the signature $(1, -1, -1, -1)$ of the metric has been used. We use here an opposite signature of the metric because we need to follow the notation of the review [3] for further study of the asymptotic behavior of these solutions. The metrics are thus changed as $\mathbf{g}_{ab} \to -\mathbf{g}_{ab}$. We will also make some other minor changes in the spacetimes forms presented in [9, 10, 11, 12].

9.1 Initial form of the metric

The original Plebański–Demiański metric [8] can be written in the form [10]

$$\begin{aligned}\mathbf{g}_{ab} = \frac{1}{(1-\alpha pr)^2}\Bigg[&\frac{-Q}{\omega^2p^2+r^2}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)^2+\frac{P}{\omega^2p^2+r^2}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)^2\\ &+\frac{\omega^2p^2+r^2}{P}\mathrm{d}p^2+\frac{\omega^2p^2+r^2}{Q}\,\mathrm{d}r^2\Bigg],\end{aligned} \tag{9.1}$$

where

$$\begin{aligned}P(p) &= k+2\omega^{-1}np-\epsilon p^2+2\alpha mp^3-\left[\alpha^2(\omega^2k+e^2+g^2)+\omega^2\Lambda/3\right]p^4,\\ Q(r) &= (\omega^2k+e^2+g^2)-2mr+\epsilon r^2-2\alpha\omega^{-1}nr^3-(\alpha^2k+\Lambda/3)r^4,\end{aligned} \tag{9.2}$$

and m, n, e, g, Λ, ϵ, k, α and ω are arbitrary real parameters of which two can be chosen for convenience. Only the parameters Λ, e and g have their traditional physical interpretation in this metric. The large family of type D spacetimes given by the metric (9.1) admits (at least) two commuting Killing vectors ∂_σ and ∂_τ whose orbits are spacelike for $Q > 0$ and timelike for $Q < 0$. Surfaces on which $Q = 0$ are horizons, points at $P = 0$ are poles (axes).

Plebański and Demiański considered in their original work [8] that $\alpha = 1$ and $\omega = 1$. In [10, 11] it was shown that the physical interpretation of the parameters can be determined more easily when α and ω are retained as continuous parameters and ϵ and k are set to convenient values (without changing their signs).

It is thus appropriate to describe the full family of solutions in terms of seven continuous parameters m, n, e, g, Λ, α and ω and two auxiliary ones ϵ and k which can be set conveniently. Their meaning is as following:

- e and g are the electric and the magnetic charges
- Λ is the cosmological constant
- α is the acceleration of the sources
- ω is the twist parameter
- m is related to the mass of the source
- n is the Plebański and Demiański parameter.

In some particular cases ω is directly related to both the angular velocity of sources and the effects of the NUT parameter. The metric is flat when $m = n = 0$, $e = g = 0$ and $\Lambda = 0$ (see equation (9.14) below). The remaining parameters ϵ, k, α and ω may be non-zero even in this flat limit. The well-known solutions such as the Schwarzschild–de Sitter, the Reissner–Nordström, the Kerr metric, the NUT solution or the C-metric, and other type D spaces are also included: the simple transformation (9.16) leads to a form that explicitly includes all these well-known special cases, see [10, 11] for more details and section 9.2 for demonstration of this fact.

9.1.1 Natural null tetrad

In [10] the null tetrad was expressed in the coordinates (τ, σ, p, r) using the signature $(1, -1, -1, -1)$. For our later purposes, we need to have the null tetrad with opposite signature of the metric. Moreover, it is necessary to introduce the null tetrad both for $Q > 0$ and for $Q < 0$ because the sign of this function Q is different near $r \to \infty$, as

$$\begin{aligned} Q < 0 \quad &\text{if} \quad (\alpha^2 k + \Lambda/3) > 0, \\ Q > 0 \quad &\text{if} \quad (\alpha^2 k + \Lambda/3) < 0. \end{aligned} \tag{9.3}$$

The function Q thus may become negative under the square root near the conformal infinity where we wish to study radiative properties and therefore we need to consider also the case $Q < 0$. Concretely, for $\alpha = 0$ the sign of Q depends only on the sign of Λ. After some calculations outlined below we found the following null tetrads in the metric signature $(-1,1,1,1)$:

The null tetrad for $Q<0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{-Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{-Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{1}{\sqrt{-Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{-Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right),\\
\overline{\mathbf{m}}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right).
\end{aligned}
\tag{9.4}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{1-\alpha pr}\left(-\sqrt{\frac{-Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{-2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{-Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{-2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{1-\alpha pr}\left(i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{1-\alpha pr}\left(-i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right).
\end{aligned}
\tag{9.5}
$$

The null tetrad for $Q>0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{Q}}(r^2\partial_\tau-\omega\partial_\sigma)+\sqrt{Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-1}{\sqrt{Q}}(r^2\partial_\tau-\omega\partial_\sigma)-\sqrt{Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right),\\
\overline{\mathbf{m}}^a &= \frac{1-\alpha pr}{\sqrt{2(\omega^2p^2+r^2)}}\left(\frac{-i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{P}\,\partial_p\right).
\end{aligned}
\tag{9.6}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)+\sqrt{\frac{\omega^2p^2+r^2}{2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{1-\alpha pr}\left(\sqrt{\frac{Q}{2(\omega^2p^2+r^2)}}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{1-\alpha pr}\left(i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{1-\alpha pr}\left(-i\sqrt{\frac{P}{2(\omega^2p^2+r^2)}}(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)-\sqrt{\frac{\omega^2p^2+r^2}{2P}}\mathrm{d}p\right).
\end{aligned}
\tag{9.7}
$$

When deriving these null tetrads we began with the null tetrad presented in [10]. It was sufficient to transform only the vectors, the covariant one-forms were easily obtained by lowering the indeces, $\mathbf{k}_a = \mathbf{g}_{ab}\mathbf{k}^b$, etc., using the metric (9.1). We expected the changes to occur only in the signs of the two parts of each vector. We thus added coefficients in front of each part of the vector with possible values ± 1 in $\mathbf{k}_a, \mathbf{l}_a$, and also $\pm i$ in $\mathbf{m}_a, \overline{\mathbf{m}}_a$. The new vectors must satisfy the basic relations for the null tetrad, namely that the scalar products of the tetrad vectors all vanish, except for

$$\mathbf{k}^a\mathbf{l}_a = -1, \tag{9.8}$$

$$\mathbf{m}^a\overline{\mathbf{m}}_a = +1. \tag{9.9}$$

We thus obtained several conditions for the vector coefficients.

For the case when $Q < 0$, the condition (9.8) implies four possible combinations of the coefficients, two of them are equivalent as

$$(-\mathbf{k}^a)(-\mathbf{k}_a) = \mathbf{k}^a\,\mathbf{k}_a, \quad (-\mathbf{l}^a)(-\mathbf{l}_a) = \mathbf{l}^a\,\mathbf{l}_a, \quad \mathbf{k}^a(-\mathbf{l}_a) = (-\mathbf{k}^a)\,\mathbf{l}_a. \tag{9.10}$$

Each two pairs of equivalent vectors contain a combination of future oriented null vectors $\mathbf{k}^a, \mathbf{l}^a$ and a combination of past oriented vectors $\mathbf{k}^a, \mathbf{l}^a$ which differ only in the sign in front of the first part $(r^2\,\partial_\tau - \omega\partial_\sigma)$ of the vectors. The future/past orientation means that plus/minus stands in front of the second part ∂_r of the vector. The sign in front of the first part is thus eligible, of course, only ± 1. It corresponds to two possible orientations of the two null vectors aligned along the light cone. We have chosen the future orientation. The condition (9.9) implies two possible choices of vectors $\mathbf{m}_a, \overline{\mathbf{m}}_a$ which are completely independent of the condition (9.8), so we have chosen one orientation arbitrarily.

For the case $Q > 0$, the condition (9.8) again implies four combinations of the signs, of which two are equivalent but the two pairs contain one vector future oriented and one past oriented. We can not choose two vectors oriented in the same way as in the case $Q < 0$. We have rather chosen the vectors according to their first parts: we took the choice such that the first part of the vector $\mathbf{k}^a$ remains the same as in the case $Q < 0$. The vectors $\mathbf{m}_a, \overline{\mathbf{m}}_a$ were chosen the same as in the case $Q < 0$.

The above null tetrads $\{\mathbf{e}_A\} = \{\mathbf{m}^a, \overline{\mathbf{m}}^a, \mathbf{l}^a, \mathbf{k}^a\}$ given by (9.4) or (9.6) satisfy further relations (see also [13]). The vectors are related to the dual one-forms by the non-diagonal metric $\mathbf{g}_{AB}$ whose components are

$$\mathbf{g}_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad {\mathbf{g}_{AB}}^{-1} \equiv \mathbf{g}^{AB} = \mathbf{g}_{AB}. \tag{9.11}$$

In practice, the related one-forms to our vectors are obtained by the following algorithm:

1. Set up the basis of the vectors and the basis of the dual forms $\boldsymbol{\omega}^A$ as

$$\begin{aligned} \boldsymbol{e}_1 &= \mathbf{m}^a, & \boldsymbol{\omega}^1 &= \mathbf{m}_a, \\ \boldsymbol{e}_2 &= \overline{\mathbf{m}}^a, & \boldsymbol{\omega}^2 &= \overline{\mathbf{m}}_a, \\ \boldsymbol{e}_3 &= \mathbf{l}^a, & \boldsymbol{\omega}^3 &= \mathbf{l}_a, \\ \boldsymbol{e}_4 &= \mathbf{k}^a, & \boldsymbol{\omega}^4 &= \mathbf{k}_a. \end{aligned}$$

2. Insert all vectors into each one-form and find to which vector the one-form is associated, e.g. $\boldsymbol{\omega}^1(\mathbf{e}_1) = 1$. If -1 is obtained, change the sign of the given one-form.

3. Finally, confirm that the relations (9.8) and (9.9) are satisfied.

We have thus identified the one-forms associated to the vectors (9.4) and (9.6) as

$$\begin{aligned} \boldsymbol{\mu} &\equiv \boldsymbol{\omega}^1 = \mathbf{m}_a, \\ \overline{\boldsymbol{\mu}} &\equiv \boldsymbol{\omega}^2 = \overline{\mathbf{m}}_a, \\ \boldsymbol{\lambda} &\equiv -\boldsymbol{\omega}^4 = -\mathbf{k}_a, \\ \boldsymbol{\kappa} &\equiv -\boldsymbol{\omega}^3 = -\mathbf{l}_a, \end{aligned} \tag{9.12}$$

where we introduced a new (temporary) notation for the dual one-forms which satisfy

$$\begin{aligned} \boldsymbol{\mu}(\mathbf{e}_1) &= 1, \\ \overline{\boldsymbol{\mu}}(\mathbf{e}_2) &= 1, \\ \boldsymbol{\kappa}(\mathbf{e}_3) &= 1, \\ \boldsymbol{\lambda}(\mathbf{e}_4) &= 1. \end{aligned} \tag{9.13}$$

Consequently, $\{\mathbf{m}^a, \overline{\mathbf{m}}^a, \mathbf{l}^a, \mathbf{k}^a\}$ is dual to $\{\boldsymbol{\mu}, \overline{\boldsymbol{\mu}}, \boldsymbol{\kappa}, \boldsymbol{\lambda}\}$.

9.1.2 Weyl and Ricci tensors

It can be calculated (for example by using the Maple package *GR tensor*) that the only non-zero component of the Weyl in the tetrad (9.4) and (9.6) is given by

$$\Psi_2 = -(m+in)\left(\frac{1-\alpha pr}{r+i\omega p}\right)^3 + (e^2+g^2)\left(\frac{1-\alpha pr}{r+i\omega p}\right)^3 \frac{1+\alpha pr}{r-i\omega p}. \tag{9.14}$$

This confirms that the spacetimes studied are of algebraic type D and that the tetrad vectors $\mathbf{l}^a$ and $\mathbf{k}^a$ are aligned with the principal null directions of the Weyl tensor. The only non-zero component of the Ricci tensor is

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(1-\alpha pr)^4}{(\omega^2p^2+r^2)^2}. \tag{9.15}$$

These components of the curvature tensor indicate that there is a curvature singularity at $p = r = 0$. This singularity can be considered as the "source" of gravitational field. It is necessary that $P > 0$ to retain a Lorentzian signature in metrics. $P(p)$ is generally a quartic function, so the coordinate p must be restricted to a particular range between appropriate roots. If this range includes $p = 0$, it is necessary that $k > 0$. The condition $P > 0$ thus places restriction on the possible signs of the parameters ε and k. Note also, that there are non-singular solutions in the Plebański–Demiański family when $\omega \neq 0$.

9.2 More general form of the metric

The metric (9.1) does not include the type D non-singular NUT solution. It is necessary to introduce a specific shift in the coordinate p to cover such cases with the NUT parameter. This procedure is also essential to obtain the correct metric for accelerating and rotating black holes. We will start with the metric (9.1) and perform the coordinate transformation

$$p = \frac{l}{\omega} + \frac{a}{\omega}\,\tilde{p}, \qquad \tau = \tilde{t} - \frac{(l+a)^2}{a}\,\tilde{\phi}, \qquad \sigma = -\frac{\omega}{a}\,\tilde{\phi}, \tag{9.16}$$

where a and l are new arbitrary parameters. These solutions have two parameters k and ϵ which can be scaled to any convenient values. In addition, we have the further parameters a and l which can be chosen arbitrarily. In practice, it is convenient to choose a and l to satisfy certain conditions which simplify the form of the metric, and then to re-express n and ω in terms of these parameters. We will show that the parameter a corresponds to a Kerr-like rotation parameter and l corresponds to a NUT parameter. The properties of the solution depend on the character of the function P. It becomes $\tilde{P} = P\frac{\omega^2}{a^2}$ after transformation (9.16). This function is a quartic function and can have up to four distinct roots.

We will consider the case when $\tilde{P}$ has two roots $\tilde{p} = \pm 1$ and $\tilde{p}$ covers the range between these roots. The function $\tilde{P}$ has a simplified form

$$\tilde{P} = (1 - \tilde{p}^2)(a_0 - a_3\tilde{p} - a_4\tilde{p}^2). \tag{9.17}$$

Then the conditions specifying the two parameters ϵ and n in terms of a and l are

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}m - (a^2 + 3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right], \tag{9.18}$$

$$n = \frac{\omega^2 kl}{a^2 - l^2} - \alpha\frac{a^2 - l^2}{\omega}m + (a^2 - l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3}\right]. \tag{9.19}$$

(see [9], [10], [11] for more details). The next equation defines the parameter k for any given value of a_0 as

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right) k = a_0 + 2\alpha\frac{l}{\omega}m - 3\alpha^2\frac{l^2}{\omega^2}(e^2 + g^2) - l^2\Lambda. \tag{9.20}$$

These constraints define the values of k, n and ϵ, the remaining scaling freedom is in the parameter ω which we can set to any convenient value with assumption that a and l do not both vanish. The remaining parameters are thus α, a and l in addition to m, e, g and Λ. We will concentrate on the physically most relevant case for which $\tilde{P}$ has two roots and $a_0 > 0$. The scaling freedom can be used to set $a_0 = 1$.

The coordinate $\tilde{p}$ covers the range between the roots $\tilde{p} = \pm 1$ and we put $\tilde{p} = \cos\vartheta$, where $\vartheta \in [0, \pi]$. The transformation (9.16) from the metric (9.1) to new general metric has thus the form

$$p = \frac{l}{\omega} + \frac{a}{\omega}\cos\vartheta, \qquad \tau = \tilde{t} - \frac{(l + a)^2}{a}\tilde{\phi}, \qquad \sigma = -\frac{\omega}{a}\tilde{\phi}. \tag{9.21}$$

Then the metrics (9.1) becomes

$$\begin{aligned}\mathbf{g}_{ab} = \frac{1}{\Omega^2}\Bigg[&-\frac{Q}{\rho^2}(\mathrm{d}\tilde{t} - (a\sin^2\vartheta + 4l\sin^2\tfrac{\vartheta}{2})\mathrm{d}\tilde{\phi})^2 + \frac{\rho^2}{\hat{P}}\mathrm{d}\vartheta^2 + \frac{\rho^2}{Q}\,\mathrm{d}r^2 \\ &+ \frac{\hat{P}\sin^2\vartheta}{\rho^2}(\mathrm{d}\tilde{t} - (r^2 + (a + l)^2)\mathrm{d}\tilde{\phi})^2\Bigg],\end{aligned} \tag{9.22}$$

where

$$\begin{aligned}\Omega &= 1 - \frac{\alpha}{\omega}(l + a\cos\vartheta)r, \\ \rho^2 &= r^2 + (l + a\cos\vartheta)^2, \\ \hat{P} &\equiv \frac{\tilde{P}}{\sin^2\vartheta} = 1 - a_3\cos\vartheta - a_4\cos^2\vartheta, \\ Q &= (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha\omega^{-1}nr^3 - (\alpha^2 k + \Lambda/3)r^4,\end{aligned} \tag{9.23}$$

and

$$a_3 = 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k + e^2 + g^2) - 4\frac{\Lambda}{3}al, \tag{9.24}$$

$$a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + e^2 + g^2) - \frac{\Lambda}{3}a^2. \tag{9.25}$$

with ϵ, n and k given by (9.18)-(9.20).

This solution represents *the complete family of black hole spacetimes* and contains eight arbitrary parameters $m, e, g, a, l, \alpha, \Lambda$ and ω. The first seven of these parameters can be varied independently and ω can be set to any convenient value if a or l are not both zero. It was shown in [11] that the metric (9.22) represents for $\Lambda = 0$ accelerating and rotating charged black holes with a generally non-zero NUT parameter. The authors found subsequently in [12] that there is analytical extension of the metric which was expressed in terms of the Weyl–Lewis–Papapetrou metric and the boost-rotation form.

This more general form of metric (9.22) contains various special cases which can be obtained explicitly by setting certain parameters to zero. These will be reviewed below in subsections 9.2.1-9.2.3.

By performing the transformation (9.21) from the coordinates (τ, σ, p, r) to $(\tilde{t}, \tilde{\phi}, \vartheta, r)$, the null tetrads (9.4) and (9.5), (9.6) and (9.7), the Weyl tensor and the Ricci tensor will change their forms:

The vectors transform by "inverse" transformation of (9.21),

$$\begin{aligned} \partial_\sigma &= -\frac{a}{\omega}(\partial_{\tilde{\phi}} + \frac{(l+a)^2}{a}\partial_{\tilde{t}}), \\ \partial_p &= -\frac{\omega}{a}\frac{1}{\sin\vartheta}\partial_\vartheta, \\ \partial_\tau &= \partial_{\tilde{t}}, \end{aligned} \tag{9.26}$$

and the one-forms transform by "straight" differentiation of (9.21)

$$\begin{aligned} \mathrm{d}p &= -\frac{a}{\omega}\sin\vartheta\mathrm{d}\vartheta, \\ \mathrm{d}\sigma &= -\frac{\omega}{a}\mathrm{d}\tilde{\phi}, \\ \mathrm{d}\tau &= \mathrm{d}\tilde{t} - \frac{(l+a)^2}{a}\mathrm{d}\tilde{\phi}. \end{aligned} \tag{9.27}$$

When we substitute these relations into (9.4) and (9.5), we obtain *the null tetrad vectors for the case $Q < 0$:*

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{-Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{-Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{1}{\sqrt{-Q}}[(a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{-Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}]) + \sqrt{\hat{P}}\,\partial_\vartheta\right),\\
\overline{\mathbf{m}}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right).
\end{aligned}
\tag{9.28}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Omega}\left(-\sqrt{\frac{-Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] - \frac{\rho}{\sqrt{-2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{-Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t} - (4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\mathrm{d}\tilde{\phi}] - \frac{\rho}{\sqrt{-2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{\Omega}\left(i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2+(l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{\Omega}\left(-i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t} - (r^2+(l+a)^2)\mathrm{d}\tilde{\phi}] + \frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right).
\end{aligned}
\tag{9.29}
$$

When we substitute these relations into (9.6) and (9.7), we get *the null tetrad for* $Q > 0$:
The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] + \sqrt{Q}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-1}{\sqrt{Q}}[a\partial_{\tilde{\phi}} + (r^2+(l+a)^2)\partial_{\tilde{t}}] - \sqrt{Q}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right),\\
\overline{\mathbf{m}}^a &= \frac{\Omega}{\sqrt{2}\rho}\left(\frac{-i}{\sqrt{\hat{P}}\sin\vartheta}[(4l\sin^2\frac{\vartheta}{2} + a\sin^2\vartheta)\partial_{\tilde{t}} + \partial_{\tilde{\phi}}] + \sqrt{\hat{P}}\,\partial_\vartheta\right).
\end{aligned}
\tag{9.30}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t}-(4l\sin^2\frac{\vartheta}{2}+a\sin^2\vartheta)\mathrm{d}\tilde{\phi}]+\frac{\rho}{\sqrt{2Q}}\mathrm{d}r\right),\\
\mathbf{l}_a &= \frac{1}{\Omega}\left(\sqrt{\frac{Q}{2}}\frac{1}{\rho}[\mathrm{d}\tilde{t}-(4l\sin^2\frac{\vartheta}{2}+a\sin^2\vartheta)\mathrm{d}\tilde{\phi}]-\frac{\rho}{\sqrt{2Q}}\mathrm{d}r\right),\\
\mathbf{m}_a &= \frac{1}{\Omega}\left(i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t}-(r^2+(l+a)^2)\mathrm{d}\tilde{\phi}]+\frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{\Omega}\left(-i\sqrt{\frac{\hat{P}}{2}}\frac{\sin\vartheta}{\rho}[a\mathrm{d}\tilde{t}-(r^2+(l+a)^2)\mathrm{d}\tilde{\phi}]+\frac{\rho}{\sqrt{2\hat{P}}}\mathrm{d}\vartheta\right).
\end{aligned}
\tag{9.31}
$$

The only non-zero component of the Weyl tensor in these tetrads is given by

$$
\Psi_2=\left[-(m+in)+(e^2+g^2)\left(\frac{1+\frac{\alpha}{\omega}r(l+a\cos\vartheta)}{r-i(l+a\cos\vartheta)}\right)\right]\left(\frac{1-\frac{\alpha}{\omega}r(l+a\cos\vartheta)}{r+i(l+a\cos\vartheta)}\right)^3.
\tag{9.32}
$$

The only non-zero component of the Ricci tensor is

$$
\Phi_{11}=\frac{1}{2}(e^2+g^2)\frac{(1-\frac{\alpha}{\omega}r(l+a\cos\vartheta))^4}{(r^2+(l+a\cos\vartheta)^2)^2}.
\tag{9.33}
$$

9.2.1 Kerr–Newman–NUT–de Sitter spacetime ($\alpha=0$)

We obtain this particular case when we set $\alpha=0$. The constraint (9.20) becomes $\omega^2k=(1-l^2\Lambda)(a^2-l^2)$, the relations (9.18) and (9.19) become

$$
\epsilon=1-(\tfrac{1}{3}a^2+2l^2)\Lambda,\quad n=l+\tfrac{1}{3}(a^2-4l^2)l\Lambda.
\tag{9.34}
$$

The metric is the same as (9.22) with

$$
\begin{aligned}
&\Omega=1\\
&\rho^2=r^2+(l+a\cos\vartheta)^2\\
&\hat{P}=1+\tfrac{4}{3}\Lambda al\cos\vartheta+\tfrac{1}{3}\Lambda a^2\cos^2\vartheta\\
&Q=(a^2-l^2+e^2+g^2)-2mr+r^2-\Lambda\left[(a^2-l^2)l^2+(\tfrac{1}{3}a^2+2l^2)r^2+\tfrac{1}{3}r^4\right].
\end{aligned}
$$

This solution represents a *non-accelerating black hole* with mass m, electric and magnetic charges e,g, rotational parameter a and a NUT parameter l in Minkowski, de Sitter or anti–de Sitter background. It reduces to well-known forms when $l=0$ or $a=0$ or $\Lambda=0$. However, it is necessary to distinguish the two cases in which $|a|$ is greater or less than $|l|$. When $a^2\geq l^2,\ k\geq 0$, the metric has Kerr like ring singularity at $r=0$. This case represents a Kerr–Newman–de Sitter solution, it means a charged black hole with a small NUT parameter. This solution will be discussed properly in the next section. Alternatively, when $a^2 < l^2$, $k < 0$, the metric is singularity free. This case is best described as a charged NUT–de Sitter solution with a small Kerr-like rotation. Although these cases have identical metric forms, their singularity and global structures differ substantially.

9.2.2 Accelerating Kerr–Newman–de Sitter black holes ($l = 0$)

This is the case where the NUT parameter vanishes but α is arbitrary. Now the equation (9.20) implies that $\omega^2 k = a^2$. We use the remaining scaling freedom in ω to set $k = 1$ so then $\omega = a$ (the Kerr rotation parameter) and from the constraints (9.18)-(9.20) we obtain

$$\epsilon = 1 - \alpha^2(a^2 + e^2 + g^2) - \tfrac{1}{3}\Lambda a^2, \quad k = 1, \quad n = -\alpha a m.$$

Interestingly, while the NUT parameter vanishes, the Plebański–Demiański parameter n is not zero. The metric (9.22) becomes

$$\begin{aligned}\mathbf{g}_{ab} = \frac{1}{\Omega^2}\Bigg[&\frac{-Q}{\rho^2}(\mathrm{d}\tilde{t} - a\sin^2\vartheta \mathrm{d}\tilde{\phi})^2 + \frac{\hat{P}\sin^2\vartheta}{\rho^2}(a\mathrm{d}\tilde{t} - (r^2+a^2)\mathrm{d}\tilde{\phi})^2 \\ &+ \frac{\rho^2}{\hat{P}}\mathrm{d}\vartheta^2 + \frac{\rho^2}{Q}\,\mathrm{d}r^2\Bigg],\end{aligned} \tag{9.35}$$

where

$$\begin{aligned}\Omega &= 1 - \alpha r\cos\vartheta, \\ \rho^2 &= r^2 + a^2\cos^2\vartheta, \\ \hat{P} &= 1 - 2\alpha m\cos\vartheta + [\alpha^2(a^2+e^2+g^2) + \tfrac{1}{3}\Lambda a^2]\cos^2\vartheta, \\ Q &= (\omega^2 k + e^2 + g^2) - 2mr + r^2 - \frac{1}{3}\Lambda r^2(r^2+a^2).\end{aligned}$$

The only non-zero components of the curvature tensor are now given by

$$\Psi_2 = \left[-m(1 - i\alpha a) + (e^2+g^2)\frac{1+\alpha r\cos\vartheta}{r - ia\cos\vartheta}\right]\left(\frac{1-\alpha r\cos\vartheta}{r + ia\cos\vartheta}\right)^3, \tag{9.36}$$

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{(1-\alpha r\cos\vartheta)^4}{(r^2 + a^2\cos^2\vartheta)^2}. \tag{9.37}$$

This metric (9.35) clearly exhibits the singularity and horizon structure of an *accelerating charged and rotating black hole* in Minkowski, de Sitter or anti-de Sitter background. It represents the spacetime from the singularity through the inner and outer black hole horizons and out to the acceleration horizon. Nevertheless, it does not cover the complete analytic extension inside the black hole horizon. In [12] it was shown for the case $\Lambda = 0$ that the complete spacetime contains two causally separated charged and rotating black holes which accelerate away from each other in opposite spatial directions. Special subcase of (9.35) is *the charged C-metric with a cosmological constant*. We get this when we put $a = 0$, and the metric (9.35) reduces to the simple diagonal form

$$\mathbf{g}_{ab} = \frac{1}{(1-\alpha r\cos\vartheta)^2}\left(-\frac{Q}{r^2}\mathrm{d}\tilde{t}^2 + \frac{r^2}{Q}\mathrm{d}r^2 + \hat{P}r^2\sin^2\vartheta\mathrm{d}\tilde{\phi}^2 + \frac{r^2}{\hat{P}}\mathrm{d}\vartheta^2\right), \qquad (9.38)$$

where

$$\hat{P} = 1 - 2\alpha m\cos\vartheta + \alpha^2(e^2+g^2)\cos^2\vartheta,$$
$$Q = (e^2+g^2-2mr+r^2)(1-\alpha^2 r^2) - \tfrac{1}{3}\Lambda r^4.$$

For $\Lambda = 0$, it describes a pair of black holes of mass m and electric and magnetic charges e and g which accelerate towards infinity under the action of forces represented by a conical singularity, where α is the acceleration. The acceleration horizon is $r = \alpha^{-1}$. The location of other horizons depends on e, g, m and Λ.

9.2.3 Kerr–Newman–de Sitter spacetime ($\alpha = l = 0$)

This is obviously the $l = 0$ subcase of the spacetime discussed in subsection 9.2.1. It can be written in the standard form of the Kerr–Newman–de Sitter solution in Boyer–Lindquist coordinates by a simple rescaling transformation

$$\tilde{t} = t\,\Xi^{-1}, \quad \tilde{\phi} = \phi\,\Xi^{-1}, \qquad (9.39)$$

where $\Xi = 1 + \frac{1}{3}\Lambda a^2$. This transformation leads us to

$$\begin{aligned}\mathbf{g}_{ab} = \frac{-\Delta_r}{\Xi^2\rho^2}\left[\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\phi\right]^2 &+ \frac{\Delta_\vartheta\sin^2\vartheta}{\Xi^2\rho^2}\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2 \\ &+ \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2,\end{aligned} \qquad (9.40)$$

where

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\vartheta, \\ \Delta_r \equiv Q &= (r^2+a^2)(1-\tfrac{1}{3}\Lambda r^2) - 2mr + (e^2+g^2), \\ \Delta_\vartheta \equiv \hat{P} &= 1 + \tfrac{1}{3}\Lambda a^2\cos^2\vartheta,\end{aligned} \qquad (9.41)$$

In fact, there is no need to introduce the constant rescaling Ξ in t and ϕ. But it is included so that the metric has well-behaved axis at $\vartheta = 0$ and $\vartheta = \pi$ with $\phi \in [0, 2\pi)$.

Notice that there exists a direct transformation from the initial metric (9.1) to the metric (9.40): by inserting (9.39) into (9.16) we get

$$\begin{aligned}p &= \cos\vartheta, \\ \tau &= (t - a\phi)\,\Xi^{-1}, \\ \sigma &= -\phi\,\Xi^{-1}.\end{aligned} \qquad (9.42)$$

(Here we assume $\alpha = 0$, $l = 0$ and we set $a = \omega$, $\epsilon = 1 - \frac{1}{3}\Lambda a^2$, $k = 1$, $n = 0$).

Now we can directly rewrite the null tetrads (9.4) and (9.6), as well as the Weyl tensor and the Ricci tensor into coordinates of the Kerr–Newman–de Sitter solution (9.40). The vectors transform by inverse transformation of (9.42),

$$\begin{aligned} \partial_\sigma &= -\Xi(\partial_\phi + a\partial_t), \\ \partial_p &= -\frac{1}{\sin\vartheta}\partial_\vartheta, \\ \partial_\tau &= \Xi\,\partial_t, \end{aligned} \tag{9.43}$$

while the one-forms transform by

$$\begin{aligned} \mathrm{d}p &= -\sin\vartheta\,\mathrm{d}\vartheta, \\ \mathrm{d}\sigma &= -\Xi^{-1}\mathrm{d}\phi, \\ \mathrm{d}\tau &= \Xi^{-1}(\mathrm{d}t - a\mathrm{d}\phi). \end{aligned} \tag{9.44}$$

Substituting these relations into (9.4) and (9.5), we finally obtain *the null tetrad vectors for the case* $\Lambda > 0$*:*

The vectors are

$$\begin{aligned} \mathbf{k}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{-\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{-\Delta_r}\,\partial_r\right), \\ \mathbf{l}^a &= \frac{1}{\sqrt{2}\rho}\left(+\frac{\Xi}{\sqrt{-\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{-\Delta_r}\,\partial_r\right), \\ \mathbf{m}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta - i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right), \\ \overline{\mathbf{m}}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta + i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right), \end{aligned} \tag{9.45}$$

and the corresponding one-forms

$$\begin{aligned} \mathbf{k}_a &= -\frac{1}{\Xi\rho}\sqrt{\frac{-\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{-2\Delta_r}}\,\mathrm{d}r, \\ \mathbf{l}_a &= +\frac{1}{\Xi\rho}\sqrt{\frac{-\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{-2\Delta_r}}\,\mathrm{d}r, \\ \mathbf{m}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta + \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi), \\ \overline{\mathbf{m}}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta - \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi). \end{aligned} \tag{9.46}$$

Substituting these relations (9.43), (9.44) into (9.6) and (9.7), we analogously obtain *the null tetrad vectors for the case* $\Lambda < 0$*:*

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] + \sqrt{\Delta_r}\,\partial_r\right),\\
\mathbf{l}^a &= \frac{1}{\sqrt{2}\rho}\left(-\frac{\Xi}{\sqrt{\Delta_r}}[(r^2+a^2)\,\partial_t + a\,\partial_\phi] - \sqrt{\Delta_r}\,\partial_r\right),\\
\mathbf{m}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta - i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t +\ \partial_\phi)\right),\\
\overline{\mathbf{m}}^a &= \frac{1}{\sqrt{2}\rho}\left(\sqrt{\Delta_\vartheta}\,\partial_\vartheta + i\frac{\Xi}{\sqrt{\Delta_\vartheta}\sin\vartheta}(a\sin^2\vartheta\,\partial_t + \partial_\phi)\right),
\end{aligned}
\tag{9.47}
$$

and the corresponding one-forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\Xi\rho}\sqrt{\frac{\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) + \frac{\rho}{\sqrt{2\Delta_r}}\,\mathrm{d}r,\\
\mathbf{l}_a &= \frac{1}{\Xi\rho}\sqrt{\frac{\Delta_r}{2}}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi) - \frac{\rho}{\sqrt{2\Delta_r}}\,\mathrm{d}r,\\
\mathbf{m}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta + \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi),\\
\overline{\mathbf{m}}_a &= \frac{\rho}{\sqrt{2\Delta_\vartheta}}\,\mathrm{d}\vartheta - \frac{i}{\Xi\rho}\sqrt{\frac{\Delta_\vartheta}{2}}\sin\vartheta\,(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi).
\end{aligned}
\tag{9.48}
$$

The only non-zero component of the Weyl tensor (9.14) is now

$$
\Psi_2 = -\frac{m}{(r+ia\cos\vartheta)^3} + \frac{e^2+g^2}{(r+ia\cos\vartheta)^3(r-ia\cos\vartheta)},
\tag{9.49}
$$

and the Ricci tensor (9.15) becomes

$$
\Phi_{11} = \frac{1}{2}\frac{e^2+g^2}{(r^2+a^2\cos^2\vartheta)}.
\tag{9.50}
$$

9.3 Alternative form of the general metric

For our later purposes it is also convenient to introduce a new coordinate q which is simply related to r by

$$
q = -\frac{1}{r}.
\tag{9.51}
$$

We will express the metric (9.22) in coordinates (p, τ, σ, q). We will use the inverse transformations of (9.21), its differentials and forms (9.26), (9.27). The metric (9.22) then becomes

$$
\mathbf{g}_{ab} = \frac{1}{\mathbf{\Omega}^2}\left[\frac{-\mathcal{Q}}{\varrho^2}(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma)^2 + \frac{\varrho^2}{P}\mathrm{d}p^2 + \frac{\varrho^2}{\mathcal{Q}}\mathrm{d}q^2 + \frac{P}{\varrho^2}(\omega q^2\mathrm{d}\tau + \mathrm{d}\sigma)^2\right],
\tag{9.52}
$$

where

$$
\begin{aligned}
\mathbf{\Omega} \equiv& q\,\Omega = -(q+\alpha p),\\
\varrho^2 =& q^2\,\rho^2 = 1 + q^2\omega^2 p^2,\\
P =& \frac{a^2}{\omega^2}(1-\frac{\omega^2}{a^2}(\cos\vartheta-\frac{l}{\omega})^2)(1-a_3\frac{\omega}{a}(\cos\vartheta-\frac{l}{\omega})-a_4\frac{\omega^2}{a^2}(\cos\vartheta-\frac{l}{\omega})^2),\\
\mathcal{Q}(q) \equiv& q^4\,Q = -(\alpha^2 k+\Lambda/3)+2\alpha\omega^{-1}nq+\epsilon q^2+2mq^3+(\omega^2 k+e^2+g^2)q^4,
\end{aligned}
\tag{9.53}
$$

and the coefficients are as in (9.24), (9.25)

$$
a_3 = 2\alpha\frac{a}{\omega}m - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k+e^2+g^2) - 4\frac{\Lambda}{3}al, \tag{9.54}
$$

$$
a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k+e^2+g^2) - \frac{\Lambda}{3}a^2. \tag{9.55}
$$

with ϵ, n and k again given by (9.18)-(9.20).

We obtain the null tetrads by performing the inverse transformation of (9.21) in (9.56) and (9.57), (9.58) and (9.59). Then the vectors and their dual forms are expressed in the coordinates (p, τ, σ, q).

The null tetrad for $Q < 0$:

The vectors are

$$
\begin{aligned}
\mathbf{k}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{-\mathcal{Q}}}(\partial_\tau-\omega q^2\partial_\sigma)+\sqrt{-\mathcal{Q}}\,\partial_q\right),\\
\mathbf{l}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau-\omega q^2\partial_\sigma)+\sqrt{-\mathcal{Q}}\,\partial_q\right),\\
\mathbf{m}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{\mathcal{P}}\,\partial_p\right),\\
\overline{\mathbf{m}}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-i}{\sqrt{P}}(\omega p^2\partial_\tau+\partial_\sigma)-\sqrt{\mathcal{P}}\,\partial_p\right).
\end{aligned}
\tag{9.56}
$$

The corresponding one–forms are

$$
\begin{aligned}
\mathbf{k}_a &= \frac{1}{\mathbf{\Omega}}\left(-\sqrt{\frac{-\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\frac{\varrho}{\sqrt{-2\mathcal{Q}}}\mathrm{d}q\right),\\
\mathbf{l}_a &= \frac{1}{\mathbf{\Omega}}\left(+\sqrt{\frac{-\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)-\frac{\varrho}{\sqrt{-2\mathcal{Q}}}\mathrm{d}q\right),\\
\mathbf{m}_a &= \frac{1}{\mathbf{\Omega}}\left(+i\sqrt{\frac{P}{2}}\frac{1}{\varrho}(\omega q^2\mathrm{d}\tau+\mathrm{d}\sigma)-\frac{\varrho}{\sqrt{2P}}\mathrm{d}p\right),\\
\overline{\mathbf{m}}_a &= \frac{1}{\mathbf{\Omega}}\left(-i\sqrt{\frac{P}{2}}\frac{1}{\varrho}(\omega q^2\mathrm{d}\tau+\mathrm{d}\sigma)-\frac{\varrho}{\sqrt{2P}}\mathrm{d}p\right).
\end{aligned}
\tag{9.57}
$$

The null tetrad for $Q > 0$:
The vectors are

$$\begin{aligned}
\mathbf{k}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \sqrt{\mathcal{Q}}\,\partial_q\right), \\
\mathbf{l}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{-1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) - \sqrt{\mathcal{Q}}\,\partial_q)\right), \\
\mathbf{m}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau + \partial_\sigma) - \sqrt{P}\,\partial_p\right), \\
\overline{\mathbf{m}}^a &= \frac{\mathbf{\Omega}}{\sqrt{2}\varrho}\left(-\frac{i}{\sqrt{P}}(\omega p^2\partial_\tau + \partial_\sigma) - \sqrt{P}\,\partial_p\right).
\end{aligned} \tag{9.58}$$

The corresponding one–forms are

$$\begin{aligned}
\mathbf{k}_a &= \frac{1}{\mathbf{\Omega}}\left(\sqrt{\frac{\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma) + \frac{\varrho}{\sqrt{2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{l}_a &= \frac{1}{\mathbf{\Omega}}\left(\sqrt{\frac{\mathcal{Q}}{2}}\frac{1}{\varrho}(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma) - \frac{\varrho}{\sqrt{2\mathcal{Q}}}\mathrm{d}q\right), \\
\mathbf{m}_a &= \frac{1}{\mathbf{\Omega}}\left(+i\sqrt{\frac{P}{2}}\frac{1}{\varrho}(\omega q^2\mathrm{d}\tau + \mathrm{d}\sigma) - \frac{\varrho}{\sqrt{2P}}\mathrm{d}p\right), \\
\overline{\mathbf{m}}_a &= \frac{1}{\mathbf{\Omega}}\left(-i\sqrt{\frac{P}{2}}\frac{1}{\varrho}(\omega q^2\mathrm{d}\tau + \mathrm{d}\sigma) - \frac{\varrho}{\sqrt{2P}}\mathrm{d}p\right).
\end{aligned} \tag{9.59}$$

The non-zero component of the Weyl in these tetrads is given by

$$\Psi_2 = \left[(m + in) + (e^2 + g^2)\left(\frac{q - \alpha p}{1 + iq\omega p}\right)\right]\left(\frac{q + \alpha p}{1 - iq\omega p}\right)^3. \tag{9.60}$$

The non-zero component of the Ricci tensor is

$$\Phi_{11} = \frac{1}{2}(e^2 + g^2)\frac{(q + \alpha p)^4}{(1 + q^2\omega^2p^2)^2}. \tag{9.61}$$

As we will demonstrate, this alternative form of the metric (2.52) is more useful for investigation of the directional structure of radiation near conformal infinity.

9.4 Radiation in the complete family of the Plebański–Demiański black hole spacetimes

Now we will generalize the above results to the $\Lambda \neq 0$ case of *accelerating* charged black holes with rotation a and NUT parameter l. We will use the alternative form (2.52) of the general metric for our investigation. Let us recall that

$$\mathbf{g}_{ab} = \frac{1}{\mathbf{\Omega}^2}\left[\frac{-\mathcal{Q}}{\varrho^2}(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma)^2 + \frac{\varrho^2}{P}\mathrm{d}p^2 + \frac{\varrho^2}{\mathcal{Q}}\mathrm{d}q^2 + \frac{P}{\varrho^2}(\omega q^2\mathrm{d}\tau + \mathrm{d}\sigma)^2\right], \tag{9.62}$$

where

$$\begin{aligned}
\mathbf{\Omega} &= -(q+\alpha p),\\
\varrho^2 &= 1+q^2\omega^2p^2,\\
P &= \frac{a^2}{\omega^2}(1-\frac{\omega^2}{a^2}(\cos\vartheta-\frac{l}{\omega})^2)(1-a_3\frac{\omega}{a}(\cos\vartheta-\frac{l}{\omega})-a_4\frac{\omega^2}{a^2}(\cos\vartheta-\frac{l}{\omega})^2),\\
\mathcal{Q}(q) &= -(\alpha^2k+\Lambda/3)+2\alpha\omega^{-1}nq+\epsilon q^2+2mq^3+(\omega^2k+e^2+g^2)q^4,
\end{aligned} \tag{9.63}$$

α is the acceleration parameter, the coefficients a_3 and a_4 are given by (9.54) and (9.55), and the parameters ϵ, n are defined by the constraints (9.18), (9.19)

$$\epsilon = \frac{\omega^2 k}{a^2-l^2}+4\alpha\frac{l}{\omega}m-(a^2+3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2k+e^2+g^2)+\frac{\Lambda}{3}\right], \tag{9.64}$$

$$n = \frac{\omega^2 kl}{a^2-l^2}-\alpha\frac{a^2-l^2}{\omega}m+(a^2-l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2k+e^2+g^2)+\frac{\Lambda}{3}\right]. \tag{9.65}$$

The parameter k for $a_0=1$ is

$$\left(\frac{\omega^2}{a^2-l^2}+3\alpha^2l^2\right)k = 1+2\alpha\frac{l}{\omega}m-3\alpha^2\frac{l^2}{\omega^2}(e^2+g^2)-l^2\Lambda. \tag{9.66}$$

These constraints define the values of k, n and ϵ, the remaining scaling freedom is in the parameter ω which we can set to any convenient value with assumption that a and l do not both vanish (otherwise $\omega=0$).

We observe that $\mathbf{\Omega}^{-2}$ is factorized from the metric (9.62). The remaining part of the metric in the square brackets is the conformal metric $\tilde{\mathbf{g}}_{ab}$. The conformal factor is in the sense of the relation (3.1) between the conformal metric $\tilde{\mathbf{g}}_{ab}$ and the physical one $\mathbf{g}_{ab}$ given exactly as

$$\mathbf{\Omega} = -\left(q+\alpha p\right). \tag{9.67}$$

The conformal factor must be positive, so we assume $q+\alpha p<0$. Obviously, when we perform a limit $q\to 0$ of the conformal factor, there remains the term αp with the acceleration parameter which is non-trivial. This metric is thus convenient from the point of view of the conformal structure and it is more useful for our analyses.

The conformal infinity is localized on $\mathbf{\Omega}=0$, and from (9.67) it gives an important equation

$$q = -\alpha p. \tag{9.68}$$

This relation enables us to evaluate our results directly on $\mathcal{I}$.

We easily obtain a gradient of $\mathbf{\Omega}$ by differentiating (9.67) as

$$\mathrm{d}\mathbf{\Omega} = -(\mathrm{d}q+\alpha\mathrm{d}p), \tag{9.69}$$

which, evaluated on $\Omega=$ const., gives the relation

$$\mathrm{d}q = -\alpha\mathrm{d}p. \tag{9.70}$$

For further purposes, it is convenient to investigate the general expression

$$\mathcal{Q}+\alpha^2P=-\tfrac{\Lambda}{3}(1+\alpha^2\omega^2p^4)+\mathbf{\Omega}\left[(\omega^2k+e^2+g^2)(q-\alpha p)(q^2+\alpha^2p^2)\right.$$
$$\left.+2m(q^2-\alpha qp+\alpha^2p^2)+\epsilon(q-\alpha p)+2\alpha n\omega^{-1}\right]. \tag{9.71}$$

The expression (9.71) has two parts, one with the parameter Λ and the second part with a factorized conformal factor. The second part evaluated on $\mathcal{I}$ gives zero. The expression (4.52) evaluated directly on $\mathcal{I}$, by setting $\mathbf{\Omega}=0$, gives

$$\mathcal{Q}_{\mathcal{I}}+\alpha^2P=-\frac{\Lambda}{3}\left(1+\alpha^2\omega^2p^4\right). \tag{9.72}$$

This expression will be very useful in evaluating our results directly on $\mathcal{I}$.

Now we will examine the conformal metric $\tilde{\mathbf{g}}_{ab}$ on $\mathcal{I}$. When we substitute (9.68), (9.70) into $\tilde{\mathbf{g}}_{ab}$ from (9.62) and (9.63), we obtain

$$\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}=\left[\frac{-\mathcal{Q}_{\mathcal{I}}}{\varrho_{\mathcal{I}}^2}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)^2+\varrho_{\mathcal{I}}^2\frac{\mathcal{Q}_{\mathcal{I}}+\alpha^2P}{P\mathcal{Q}_{\mathcal{I}}}\mathrm{d}p^2+\frac{P}{\varrho_{\mathcal{I}}^2}\frac{a}{\omega}(\alpha^2p^2\omega\mathrm{d}\tau+\mathrm{d}\sigma)^2\right], \tag{9.73}$$

where the expressions for ϱ and $\mathcal{Q}$ changed to

$$\varrho_{\mathcal{I}}=1+\alpha^2p^2\omega^2 \tag{9.74}$$

and

$$\mathcal{Q}_{\mathcal{I}}=-(\alpha^2k+\tfrac{\Lambda}{3})-2\tfrac{\alpha^2}{\omega}np+\epsilon\alpha^2p^2-2m\alpha^3p^3+(\omega^2k+e^2+g^2)\alpha^4p^4, \tag{9.75}$$

where n, ϵ and k are the constraints (9.64), (9.65) and (9.66). The coordinate q is not present in (9.73) because we made the substitution (9.68) and therefore only the term proportional to $\mathrm{d}p^2$ appears in the conformal metric $\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}$. Furthermore, when we use the expression (9.72), it is possible to evaluate the conformal metric (9.73) on $\mathcal{I}$ more explicitly as

$$\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}=\left[\frac{-\mathcal{Q}_{\mathcal{I}}}{\varrho|_{\mathcal{I}}^2}(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)^2+\varrho|_{\mathcal{I}}^2\frac{-\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}{P\mathcal{Q}_{\mathcal{I}}}\mathrm{d}p^2+\frac{P}{\varrho|_{\mathcal{I}}^2}\frac{a}{\omega}(\alpha^2p^2\omega\mathrm{d}\tau+\mathrm{d}\sigma)^2\right]. \tag{9.76}$$

We observe from the last expression that $\tilde{\mathbf{g}}_{ab}|_{\mathcal{I}}$ is regular on $\mathcal{I}$ for $\sigma=\pm1$ ($\Lambda\neq0$). The case $\Lambda=0$ has to be studied in another way, like the Kerr–Newman–de Sitter solution in section 4.1.

We also calculated the inverse form of the physical metric (9.62) (using the Maple software)

$$\mathbf{g}^{ab}\partial_a\,\partial_b=\mathbf{\Omega}^2\left[\frac{\mathcal{Q}}{\varrho^2}\partial_q\partial_q+\frac{P}{\varrho^2}\partial_p\partial_p-\frac{\varrho^2}{\mathcal{Q}\mathcal{W}}[\partial_\tau-\omega q^2\partial_\sigma]^2+\frac{\varrho^2}{P\mathcal{W}}[\omega p^2\partial_\tau+\partial_\sigma]^2\right], \tag{9.77}$$

where we denoted
$$\mathcal{W} = [1 + \omega^2 q^2 p^2]^2 = \varrho^2. \tag{9.78}$$
The trace of metric (9.62) and (9.77) is $\mathbf{g}_{ab}\mathbf{g}^{ab} = 4$. This confirms that (9.77) is really the inverse metric of (9.62).

We are now able to calculate the function $\tilde{N}$ and the conformal normal $\tilde{\mathbf{n}}^a$ to $\mathcal{I}$, and σ from the definition (3.3). The function $\tilde{N}$ is, for the cases $\sigma = \pm 1$ and for every $\Omega = \text{const}$, given by
$$\tilde{N} = \frac{\varrho}{\sqrt{|\mathcal{Q} + \alpha^2 P|}}. \tag{9.79}$$
Then the conformal normal $\tilde{\mathbf{n}}^a$ to $\mathcal{I}$ and σ are
$$\tilde{\mathbf{n}}^a = -\frac{1}{\sqrt{|\mathcal{Q} + \alpha^2 P|}} \left(\frac{\mathcal{Q}}{\varrho}\partial_q + \frac{\alpha P}{\varrho}\partial_p \right), \tag{9.80}$$
$$\sigma = \frac{\mathcal{Q} + \alpha^2 P}{|\mathcal{Q} + \alpha^2 P|}. \tag{9.81}$$
Clearly, we observe that if $\mathcal{Q}+\alpha^2 P > 0$ then $\sigma = 1$, and also conversely $\mathcal{Q}+\alpha^2 P < 0$ for $\sigma = -1$. Obviously, we can just explicitly evaluate the expression (9.81) on $\mathcal{I}$ using (9.72). It gives
$$\sigma|_{\mathcal{I}} = \frac{-\frac{\Lambda}{3}(1 + \alpha^2\omega^2 p^4)}{|-\frac{\Lambda}{3}(1 + \alpha^2\omega^2 p^4)|} = \frac{-\Lambda}{|-\Lambda|} = -\text{sign}\Lambda. \tag{9.82}$$
We have thus confirmed the relation (3.5), $\sigma = -\text{sign}\,\Lambda$ which tells us that the character of $\mathcal{I}$ is related to the sign of cosmological constant.

The physical normal vector is according to (3.7) defined as $\mathbf{n}^a = \boldsymbol{\Omega}\tilde{\mathbf{n}}^a$, thus for $\sigma = \pm 1$
$$\mathbf{n}^a = -\frac{\boldsymbol{\Omega}}{\sqrt{|\mathcal{Q} + \alpha^2 P|}} (\frac{\mathcal{Q}}{\varrho}\partial_q + \frac{\alpha P}{\varrho}\partial_p). \tag{9.83}$$

In the next sections we will derive explicit form of the radiation in the complete family of Plebański–Demiański black holes spacetimes. We will discuss separately the cases $\Lambda > 0$ and $\Lambda < 0$.

9.4.1 Radiation in the general metric with spacelike $\mathcal{I}$

We will investigate the radiative properties of the general metric (9.62). The separate cases can be divided according to the sign of the cosmological constant as in the Kerr–Newman spacetimes. The cases $\Lambda > 0$ and $\Lambda < 0$ occur for $\mathcal{Q} < 0$ and $\mathcal{Q} > 0$. The sign of the term $\mathcal{Q}$ specifies which null tetrad from the section (9.3) is convenient to use.

First, we will study the case when $\Lambda > 0$ with the tetrad chosen for $\mathcal{Q} < 0$. The algebraically special null tetrad is the tetrad (2.56) and (2.57) for the case

$\mathcal{Q} < 0$ of the general metric. We can easily identify the algebraically special orthonormal tetrad using the definition (3.12) from the vectors (2.56) as

$$
\begin{aligned}
\mathbf{t}_\mathrm{s} &= \frac{\boldsymbol{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\,\partial_q,\\
\mathbf{q}_\mathrm{s} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{-1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma),\\
\mathbf{r}_\mathrm{s} &= -\frac{\boldsymbol{\Omega}}{\varrho}\sqrt{P}\partial_p,\\
\mathbf{s}_\mathrm{s} &= -\frac{\boldsymbol{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned}
\tag{9.84}
$$

We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.47), as

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\,\mathbf{t}_\mathrm{s} - \tan\theta_\mathrm{s}\,\mathbf{r}_\mathrm{s},\\
\mathbf{q}_\mathrm{o} &= \mathbf{q}_\mathrm{s},\\
\mathbf{r}_\mathrm{o} &= \cos^{-1}\theta_\mathrm{s}\,\mathbf{r}_\mathrm{s} - \tan\theta_\mathrm{s}\,\mathbf{t}_\mathrm{s},\\
\mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}.
\end{aligned}
\tag{9.85}
$$

Substituting (9.84) into (9.85), we obtain the reference tetrad for spacelike $\mathcal{I}$

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho}\left(\cos^{-1}\theta_\mathrm{s}\sqrt{-\mathcal{Q}}\partial_q + \tan\theta_\mathrm{s}\sqrt{P}\partial_p\right),\\
\mathbf{q}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho}\frac{-1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\\
\mathbf{r}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho}\left(-\cos^{-1}\theta_\mathrm{s}\sqrt{P}\partial_p + \tan\theta_\mathrm{s}\sqrt{-\mathcal{Q}}\partial_q\right),\\
\mathbf{s}_\mathrm{o} &= -\frac{\boldsymbol{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned}
\tag{9.86}
$$

The reference tetrad (9.86) must satisfy the adjustment condition (3.19) for $\sigma = -1$ which is explicitly $\mathbf{n} = \epsilon_\mathrm{o}\mathbf{t}_\mathrm{o}$, i.e.

$$
\mathbf{n} = \epsilon_\mathrm{o}\boldsymbol{\Omega}\left(\cos^{-1}\theta_\mathrm{s}\,\frac{\sqrt{-\mathcal{Q}}}{\varrho}\partial_q + \tan\theta_\mathrm{s}\,\frac{\sqrt{P}}{\varrho}\partial_p\right).
\tag{9.87}
$$

On the other hand, the normal vector $\mathbf{n}$ is given by (9.83) for $\sigma = -1$,

$$
\mathbf{n} = -\frac{\boldsymbol{\Omega}}{\sqrt{-\mathcal{Q} - \alpha^2 P}}\left(\frac{\mathcal{Q}}{\varrho}\partial_q + \frac{\alpha P}{\varrho}\partial_p\right).
\tag{9.88}
$$

The normal vectors (9.87) and (9.88) have to be equal. By comparing them we observe that it is necessary to set and then evaluate on $\mathcal{I}$ the following expressions (which we have already done by the substitution (9.68)),

$$\cos\theta_{\rm s} = \sqrt{1+\alpha^2 \tfrac{P}{\mathcal{Q}_{\mathcal{I}}}}, \tag{9.89}$$

$$\tan\theta_{\rm s} = -\frac{\alpha\sqrt{P}}{\sqrt{-\mathcal{Q}_{\mathcal{I}} - \alpha^2 P}}, \tag{9.90}$$

$$\sin\theta_{\rm s} = -\frac{\alpha\sqrt{P}}{\sqrt{-\mathcal{Q}_{\mathcal{I}}}}, \tag{9.91}$$

where $\mathcal{Q}_{\mathcal{I}}(p)$ is the expression (9.75), $\mathcal{P}(p)$ is given by (9.63), and we also observe that $\epsilon_{\rm o} = 1$. In other words, the normal vector and the reference tetrad are outgoing with respect to the future spacelike conformal infinity $\mathcal{I}^+$. We evaluated the above expressions on $\mathcal{I}$ because we are interested in the form of $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ on $\mathcal{I}$. The parameter $\theta_{\rm s}$ *encodes the orientation of the algebraically special tetrad with respect to the spacelike* $\mathcal{I}$. The expressions $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ and $\tan\theta_{\rm s}$ are functions of only one coordinate p on the conformal infinity $\mathcal{I}$.

Now we calculate the radiative component of the gravitational field. The only non-zero component of the Weyl tensor (following the procedure described in [6]) with respect to the algebraically special tetrad is (9.60), let us recall it again,

$$\Psi_2^{\rm s} = \left[(m+in) + (e^2+g^2)\left(\frac{q-\alpha p}{1+iq\omega p}\right)\right]\left(\frac{q+\alpha p}{1-iq\omega p}\right)^3. \tag{9.92}$$

The coefficient $\Psi^{\rm s}_{2*}$ is obtained from (3.51) for $s=2$ as

$$\Psi_2^{\rm s} \approx \Psi^{\rm s}_{2*}\,\eta^{-3}, \tag{9.93}$$

where $\eta \approx \epsilon\,\Omega^{-1}$ and $\Omega = -(q+\alpha p)$ so that $\eta \approx \epsilon\,(-q-\alpha p)^{-1}$. The leading factor of the gravitational field is thus

$$\Psi^{\rm s}_{2*} \approx \eta\Psi_2^{\rm s} = \epsilon(-q-\alpha p)^{-3}\Psi_2^{\rm s}, \tag{9.94}$$

for $q \to -\alpha p$ when $\Omega \to 0$. After substitution of (9.92) into (9.94) we get

$$\Psi^{\rm s}_{2*} \approx \frac{\epsilon}{(1-iq\omega p)^3}\left[(m+in) + (e^2+g^2)\frac{q-\alpha p}{1+iq\omega p}\right]. \tag{9.95}$$

The asymptotic directional structure has a standard form for spacelike $\mathcal{I}$ described by (3.53), namely

$$|\Psi_4^{\rm i}| \approx \frac{3}{2}\frac{1}{|\eta|}\frac{|\Psi^{\rm s}_{2*}|}{\cos^2\theta_{\rm s}}\mathcal{A}(\theta,\phi,\theta_{\rm s}). \tag{9.96}$$

where we have conveniently denoted the angular directional dependence as

$$\mathcal{A}(\theta,\phi,\theta_{\rm s}) = (\sin\theta + \sin\theta_{\rm s}\cos\phi)^2 + \sin^2\theta_{\rm s}\cos^2\theta\sin^2\phi. \tag{9.97}$$

The asymptotic directional structure is determined by $\Psi^{\rm s}_{2*}$ evaluated on $\mathcal{I}$ and by $\cos\theta_{\rm s}$, $\sin\theta_{\rm s}$ which are given by (9.89) and (9.91), both evaluated on $\mathcal{I}$. The parameter $\sin\theta_{\rm s}$ can be rewritten from (9.91) using (9.72) as

$$\sin\theta_{\rm s} = -\frac{\alpha\sqrt{P}}{\sqrt{\alpha^2 P + \frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)}}, \tag{9.98}$$

and we can also rewrite $\cos\theta_{\rm s}$ from (9.89) and (9.72) as

$$\cos^2\theta_{\rm s} = \frac{\frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)}{\alpha^2 P + \frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)}, \tag{9.99}$$

where the function P is given by (9.63). We are mainly interested in the concrete form of $\Psi^{\rm s}_{2*}$ evaluated on $\mathcal{I}$. This normalization component (9.95) can be rewritten as

$$|\Psi^{\rm s}_{2*}| = \frac{\sqrt{\mathcal{D}}}{(1+q^2\omega^2 p^2)^{3/2}}, \tag{9.100}$$

where

$$\mathcal{D} = m^2 + n^2 + 2(e^2+g^2)\frac{(q-\alpha p)(m-nq\omega p)}{1+q^2\omega^2p^2} + (e^2+g^2)^2\frac{(q-\alpha p)^2}{1+q^2\omega^2p^2}, \tag{9.101}$$

and the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65). The particular form of $|\Psi^{\rm s}_{2*}|$ on $\mathcal{I}$ can be obtained by substituting (9.68) into the previous expression (9.101) as

$$|\Psi^{\rm s}_{2*}|_{\mathcal{I}} = \frac{\sqrt{\mathcal{D}_{\mathcal{I}}}}{(1+\alpha^2\omega^2 p^4)^{3/2}}, \tag{9.102}$$

where

$$\mathcal{D}_{\mathcal{I}} = m^2 + n^2 - 4\alpha(e^2+g^2)p\frac{m+n\alpha\omega p^2}{1+\alpha^2\omega^2p^4} + 4\alpha^2(e^2+g^2)^2\frac{p^2}{1+\alpha^2\omega^2p^4}. \tag{9.103}$$

We observe again that $|\Psi^{\rm s}_{2*}|_{\mathcal{I}}$ depends only on the coordinate p. Then it is useful to denote the factor of the radiation as

$$B(p) \equiv \frac{|\Psi^{\rm s}_{2*}|_{\mathcal{I}}}{\cos^2\theta_{\rm s}}. \tag{9.104}$$

The final form of the asymptotic directional structure of radiation is thus

$$|\Psi^{\rm i}_4| \approx \frac{3}{2}\frac{1}{|\eta|}B(p)\mathcal{A}(\theta,\phi,\theta_{\rm s}), \tag{9.105}$$

where $B(p)$ can be rewritten by (9.99), (9.102) and (9.103) as

$$B(p) = \frac{\alpha^2 P + \frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)}{\frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)^{5/2}} \sqrt{\mathcal{D}_{\mathcal{I}}}, \tag{9.106}$$

with

$$\mathcal{D}_{\mathcal{I}} = m^2 + n^2 - 4\alpha(e^2+g^2)p\frac{m+n\alpha\omega p^2}{1+\alpha^2\omega^2 p^4} + 4\alpha^2(e^2+g^2)^2\frac{p^2}{1+\alpha^2\omega^2 p^4}, \tag{9.107}$$

and the function P given by (9.63). The expressions (9.105), (9.106) and (9.107) give the explicit formula for the radiative component of gravitational field in the complete Plebański–Demiański black hole spacetimes in the de Sitter universe for $\mathcal{Q} < 0$. The other possibility $\mathcal{Q} > 0$... The null direction along which the field is measured is parametrized by angles θ, ϕ in expression $\mathcal{A}$ in (9.97). The field itself is characterized by the normalization component $|\Psi^{\mathrm{s}}_{2*}|_{\mathcal{I}}$ or $B(p)$ and the parameter θ_{s} fixes the orientation of the algebraically special directions with respect to a spacelike infinity $\mathcal{I}$. Two PNDs are both oriented outside the manifold on future spacelike $\mathcal{I}^+$. There are two directions in which the radiation vanish as which are spatially opposite to PNDs.

The result is consistent with [4] where the C-metric was investigated. The difference is in the sign of $\sin\theta_{\mathrm{s}}$ that it is probably caused by the choice of the reference tetrad. The other minor difference is again in the normalization of the normal component of the vector $\mathbf{k}_{\mathrm{i}}$ which is defined slightly differently in [4] and in the [3]. The result for $\alpha = 0$, $l = 0$ reduces to our previous result for Kerr–Newman–de Sitter presented in the section 4.1.1.

The amplitude of radiation $B(p)$ is represented by expression which depends on a single coordinate p and on the parameters m, α, ω, a, l, Λ, e, g. The coordinate p specifies one point on the conformal infinity $\mathcal{I}$ while the parameters characterize the sources (black holes) which generate the radiation. The dependence of the amplitude $B(p)$ on the parameters of the black holes is discussed in section 9.5. Surprisingly, we will observe that the amplitude $B(p)$ is quite similar for spacelike $\mathcal{I}$ and also for the cases with timelike $\mathcal{I}$, except of one special case. Thus all these cases can be investigated together.

9.4.2 Radiation in the general metric with timelike $\mathcal{I}$ $\mathcal{Q} > 0$

In this section we will subsequently discuss four possible orientations of PNDs on timelike $\mathcal{I}$ for $\Lambda < 0$ in the tetrad for $\mathcal{Q} > 0$.

We start with the case when *one PND is outgoing and one is ingoing with respect to $\mathcal{I}$*, see the section 3.6.3. The algebraically special null tetrad is represented by (2.58) and (2.59) for the case $\mathcal{Q} > 0$ of the general metric (9.62). We can easily identify the algebraically special orthonormal tetrad using the definition

(3.12) from the vectors (2.58)

$$
\begin{aligned}
\mathbf{t}_\mathrm{s} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2 \partial_\sigma), \\
\mathbf{q}_\mathrm{s} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q, \\
\mathbf{r}_\mathrm{s} &= -\frac{\mathbf{\Omega}}{\varrho}\sqrt{P}\partial_p, \\
\mathbf{s}_\mathrm{s} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2 \partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned}
\tag{9.108}
$$

This algebraically orthonormal tetrad will be used troughout this section for all cases which may appear on timelike $\mathcal{I}$ for $\Lambda < 0$ ($\mathcal{Q} > 0$).

We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.60),

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= \mathbf{t}_\mathrm{s}, \\
\mathbf{q}_\mathrm{o} &= -\tanh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \cosh^{-1}\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s}, \\
\mathbf{r}_\mathrm{o} &= \cosh^{-1}\psi_\mathrm{s}\,\mathbf{r}_\mathrm{s} + \tanh\psi_\mathrm{s}\,\mathbf{q}_\mathrm{s}, \\
\mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}.
\end{aligned}
\tag{9.109}
$$

We find the reference tetrad by substituting (9.108) into (9.109) as

$$
\begin{aligned}
\mathbf{t}_\mathrm{o} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2 \partial_\sigma), \\
\mathbf{q}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(+\tanh\psi_\mathrm{s}\sqrt{P}\partial_p + \cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q\right), \\
\mathbf{r}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\cosh^{-1}\psi_\mathrm{s}\sqrt{P}\partial_p + \tanh\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q\right), \\
\mathbf{s}_\mathrm{o} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2 \partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned}
\tag{9.110}
$$

The reference tetrad must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_\mathrm{o}\mathbf{q}_\mathrm{o}$, i.e.

$$
\mathbf{n} = -\epsilon_\mathrm{o}\frac{\mathbf{\Omega}}{\varrho}\left(+\tanh\psi_\mathrm{s}\sqrt{P}\partial_p + \cosh^{-1}\psi_\mathrm{s}\sqrt{\mathcal{Q}}\partial_q\right). \tag{9.111}
$$

On the other hand, the normal vector $\mathbf{n}$ is given by (9.83) for $\sigma = +1$,

$$
\mathbf{n} = -\frac{\mathbf{\Omega}}{\sqrt{\mathcal{Q} + \alpha^2 P}}\left(\frac{\mathcal{Q}}{\varrho}\partial_q + \alpha\frac{P}{\varrho}\partial_p\right). \tag{9.112}
$$

The normal vectors (9.111) and (9.112) have to be equal. By comparing them we observe that it is necessary to set and evaluate on $\mathcal{I}$ the expressions as

$$\cosh\psi_{\rm s} =\sqrt{1+\alpha^2\tfrac{P}{\mathcal{Q}_{\mathcal{I}}}}, \tag{9.113}$$

$$\tanh\psi_{\rm s} =\frac{\alpha\sqrt{P}}{\sqrt{\mathcal{Q}_{\mathcal{I}}+\alpha^2P}}, \tag{9.114}$$

$$\sinh\psi_{\rm s} =\frac{\alpha\sqrt{P}}{\sqrt{\mathcal{Q}_{\mathcal{I}}}}, \tag{9.115}$$

where $\mathcal{Q}_{\mathcal{I}}$ is the expression (9.75), P is (9.63), and we also observe that $\epsilon_{\rm o}=+1$. It means, the normal vector is ingoing and one PND $\mathbf{k}_{\rm s}$ is outgoing while $\mathbf{l}_{\rm s}$ is ingoing with respect to the timelike conformal infinity $\mathcal{I}$.

Again, we evaluated the above expressions on $\mathcal{I}$ because then the parameter $\psi_{\rm s}$ encodes the orientation of the algebraically special tetrad with respect to $\mathcal{I}$. The expressions $\cos\psi_{\rm s}$, $\sin\psi_{\rm s}$ and $\tan\psi_{\rm s}$ are functions of coordinate p only.

We are able to calculate the radiative component of the gravitational field. The procedure is almost the same as in the previous spacelike case, the normalization factor is (9.95) and its evaluation on $\mathcal{I}$ is (9.102). The asymptotic directional structure of radiation has a form (3.64), namely

$$|\Psi_4^{\rm i}|\approx\frac{3}{2}\frac{1}{|\eta|}\frac{|\Psi_{2*}^{\rm s}|}{\cosh^2\psi_{\rm s}}\mathcal{A}_1(\psi,\phi,\psi_{\rm s}) \tag{9.116}$$

and it is determined by $\Psi_{2*}^{\rm s}$ evaluated on $\mathcal{I}$, and by $\cosh\psi_{\rm s}$, $\sinh\psi_{\rm s}$ which are given by (9.113) and (9.115), both evaluated on $\mathcal{I}$, and the directional angular dependence is

$$\mathcal{A}_1(\psi,\phi,\psi_{\rm s})=(\sinh\psi+\epsilon\sinh\psi_{\rm s}\cos\phi)^2+\sinh^2\psi_{\rm s}\cosh^2\psi\sin^2\phi,$$

where we substituted $\epsilon_{\rm o}=+1$.
The term $\sinh\psi_{\rm s}$ can be rewritten from (9.115) by (9.72) as

$$\sinh\psi_{\rm s}=\frac{\alpha\sqrt{P}}{\sqrt{-\alpha^2P-\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}}, \tag{9.117}$$

and we can also rewrite $\cosh\psi_{\rm s}$ from (9.113) by (9.72) as

$$\cosh^2\psi_{\rm s}=\frac{\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}{\alpha^2P+\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}, \tag{9.118}$$

where the function P is given by(9.63). If we denote the amplitude of the radiation as

$$B(p)=\frac{|\Psi_{2*}^{\rm s}|_{\mathcal{I}}}{\cosh^2\psi_{\rm s}}, \tag{9.119}$$

the asymptotic directional structure has the form

$$|\Psi_4^{\rm i}| \approx \frac{3}{2}\frac{1}{|\eta|}B(p)\mathcal{A}_1(\psi,\phi,\psi_{\rm s}) \tag{9.120}$$

where $B(p)$ can be rewritten by (9.102),(9.118) and (9.107) as

$$B(p) = \frac{\alpha^2 P + \frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)}{\frac{\Lambda}{3}(1+\alpha^2\omega^2 p^4)^{5/2}}\sqrt{\mathcal{D}_{\mathcal{I}}} \tag{9.121}$$

and

$$\mathcal{D}_{\mathcal{I}} = m^2 + n^2 - 4\alpha(e^2+g^2)p\frac{m+n\alpha\omega p^2}{1+\alpha^2\omega^2 p^4} + 4\alpha^2(e^2+g^2)^2\frac{p^2}{1+\alpha^2\omega^2 p^4}, \tag{9.122}$$

where the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65).

We obtained the explicit formula for the radiative component of gravitational field of the black hole spacetimes in the anti-de Sitter background for $Q > 0$. The null direction is parametrized by ψ, ϕ in expression (9.120). The field itself is characterized by the normalization component $|\Psi^{\rm s}_{2*}|_{\rm s}$ or $B(p)$ and the parameter $\psi_{\rm s}$ fixes the orientation of the algebraically special directions with respect to a timelike infinity $\mathcal{I}$.

The PNDs are oriented on $\mathcal{I}$ such that one PND $\mathbf{k}$ is outgoing and the other PND $\mathbf{l}$ is ingoing with respect to $\mathcal{I}$. There exists just one ingoing direction of vanishing radiation which is a mirror reflection of PND $\mathbf{k}$ and one outgoing direction of mirrored reflection of PND $\mathbf{l}$. The result is fully consistent with [5] where the C-metric was investigated for $\alpha < \sqrt{-\Lambda/3}$. For this range of acceleration, the C-metric represents a single accelerating black hole.

We observe that the amplitude $B(p)$ is identical with the amplitude from previous section for spacelike $\mathcal{I}$, see equations (9.106), (9.107). The dependence of the amplitude $B(p)$ will be investigated in the section 9.5 together with other cases.

In the following we will discuss the remaining possible cases.

First, we first consider the case 3.6.4 where *both PNDs are oriented to be outgoing or both ingoing.* The identification of the algebraically special orthonormal tetrad from its definition (3.12) for the vectors of the tetrad (2.58) is the same as in the previous case (9.108). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.68) as

$$\begin{aligned}
\mathbf{t}_{\rm o} &= +\sinh^{-1}\psi_{\rm s}\,\mathbf{r}_{\rm s} + \coth\psi_{\rm s}\mathbf{t}_{\rm s},\\
\mathbf{q}_{\rm o} &= -\sinh^{-1}\psi_{\rm s}\,\mathbf{t}_{\rm s} - \coth\psi_{\rm s}\,\mathbf{r}_{\rm s},\\
\mathbf{r}_{\rm o} &= \mathbf{q}_{\rm s},\\
\mathbf{s}_{\rm o} &= \mathbf{s}_{\rm s}.
\end{aligned} \tag{9.123}$$

When we substitute (9.108) into (9.123), we obtain the reference tetrad

$$
\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\sinh^{-1}\psi_{\mathrm{s}}\sqrt{P}\partial_p - \coth\psi_{\mathrm{s}}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right), \\
\mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\sinh^{-1}\psi_{\mathrm{s}}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \coth\psi_{\mathrm{s}}\sqrt{P}\partial_p\right), \\
\mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q \\
\mathbf{s}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned}
\tag{9.124}
$$

This reference tetrad must satisfy the adjustment condition (3.19) for $\sigma = +1$ which is explicitly $\mathbf{n} = -\epsilon_{\mathrm{o}}\mathbf{q}_{\mathrm{o}}$ as

$$
\mathbf{n} = -\epsilon_{\mathrm{o}}\frac{\mathbf{\Omega}}{\varrho}\left(\sinh^{-1}\psi_{\mathrm{s}}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \coth\psi_{\mathrm{s}}\sqrt{P}\partial_p\right). \tag{9.125}
$$

The normal vector $\mathbf{n}$ calculated directly from the metric is given by (9.112). These two vectors have to be equal again. We have to choose the parameter ψ_{s} in (9.125) to obtain the vector $\mathbf{q}_{\mathrm{o}}$ proportional to the normal $\mathbf{n}$. However, we observe that the component ∂_q of the normal vector $\mathbf{n}$ (9.112) is not present in the vector (9.125). This reference tetrad thus does *not* satisfy the adjustment condition, so that this case does not appear as a possible asymptotic behavior of the gravitational field of the studied family of black hole spacetime in the anti de Sitter universe.

We can also calculate the projection of $\mathbf{k}_{\mathrm{s}}$ into $\mathcal{I}$, to check this result. Beginning from (3.25), using (9.124) and (3.67), we obtain $\mathbf{k}_{\mathrm{s}}\cdot\mathbf{q}_{\mathrm{o}} = \frac{1}{\sqrt{2}}\sinh^{-1}\psi_{\mathrm{s}}$, and the projection is thus

$$
\begin{aligned}
\mathbf{t}_{\mathrm{s}} =& \frac{\mathbf{\Omega}}{\varrho}\Big(-\sinh\psi_{\mathrm{s}}(1+\sinh^{-2}\psi_{\mathrm{s}})\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) \\
&+ \sinh\psi_{\mathrm{s}}\sqrt{\mathcal{Q}}\partial_q - \coth\psi_{\mathrm{s}}\sqrt{P}\partial_p\Big).
\end{aligned}
\tag{9.126}
$$

Again, because the vector $\mathbf{t}_{\mathrm{s}}$ is a projection of $\mathbf{k}_{\mathrm{s}}$ onto $\mathcal{I}$ and $\mathbf{n}$ is normal vector to $\mathcal{I}$, the scalar product must be $\mathbf{t}_{\mathrm{s}}\cdot\mathbf{n} = 0$. The scalar product has a form

$$
\mathbf{t}_{\mathrm{s}}\cdot\mathbf{n} = \frac{-\sinh\psi_{\mathrm{s}}}{\sqrt{\mathcal{Q}+\alpha^2 P}}\left(\sqrt{\mathcal{Q}} + \frac{\sqrt{P}\alpha}{\cosh\psi_{\mathrm{s}}}\right). \tag{9.127}
$$

The scalar product is zero only when $\sinh\psi_{\mathrm{s}} = 0$ or when $\cosh\psi_{\mathrm{s}} = -\frac{\sqrt{P}\alpha}{\sqrt{\mathcal{Q}}}$. When we substitute these conditions into $\mathbf{q}_{\mathrm{o}}$, the first condition implies that the first part of $\mathbf{q}_{\mathrm{o}}$ diverges, the second condition implies that the second part of the vector $\mathbf{q}_{\mathrm{o}}$ is proportional to the normal vector $\mathbf{n}$ (9.112), but still the first part of $\mathbf{q}_{\mathrm{o}}$ is not. Therefore, these two conditions which satisfy $\mathbf{t}_{\mathrm{s}} \cdot \mathbf{n} = 0$ do not give consistent results.

Now we will investigate the case described in section 3.6.5, *the case when one PND is tangent to $\mathcal{I}$*. The identification of the algebraically special orthonormal tetrad is (9.108). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.77). We obtain

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \tfrac{3}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}} - \mathbf{r}_{\mathrm{s}}, \\
\mathbf{q}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \tfrac{1}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}}, \\
\mathbf{r}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \mathbf{t}_{\mathrm{s}} - \mathbf{q}_{\mathrm{s}}, \\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.
\end{aligned} \tag{9.128}$$

Substituting (9.108) into (9.128), we obtain the reference tetrad

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\frac{3}{2}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q + \sqrt{P}\partial_p\right), \\
\mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\sqrt{P}\partial_p + \frac{1}{2}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q\right), \\
\mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\sqrt{P}\partial_p + \frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) - \sqrt{\mathcal{Q}}\,\partial_q\right), \\
\mathbf{s}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned} \tag{9.129}$$

This reference tetrad must satisfy the adjustment condition (3.19) When we compare the vector $\mathbf{q}_{\mathrm{o}}$ with such normal $\mathbf{n}$ (9.112), we observe that the vector $\mathbf{q}_{\mathrm{o}}$ has components ∂_q and ∂_p pointing in the same direction as the normal $\mathbf{n}$, but $\mathbf{q}_{\mathrm{o}}$ has one another component $\frac{1}{2}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)$ which cannot disappear. This reference tetrad thus again does *not* satisfy the adjustment condition. We can also calculate the projection of $\mathbf{k}_{\mathrm{s}}$ into $\mathcal{I}$. Using (3.25), (9.129) and (3.76), we obtain $\mathbf{k}_{\mathrm{s}} \cdot \mathbf{q}_{\mathrm{o}} = \frac{1}{\sqrt{2}}$, and the the projection is

$$\mathbf{t}_{\mathrm{s}} = \frac{\mathbf{\Omega}}{\varrho}\left(\frac{1}{2}\sqrt{\mathcal{Q}}\,\partial_q + \sqrt{P}\partial_p - \frac{3}{2}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right). \tag{9.130}$$

Again, the scalar product must be $\mathbf{t}_{\mathrm{s}} \cdot \mathbf{n} = 0$ and thus this condition is *not* satisfied: we obtain $\mathbf{t}_{\mathrm{s}} \cdot \mathbf{n} = -(\sqrt{P}\alpha + \frac{1}{2}\sqrt{\mathcal{Q}})/(\sqrt{\mathcal{Q} + \alpha^2 P})$. When this vanishes, the situation is unphysical.
This case thus again does not represent the asymptotic behavior of the gravitational field of the complete family of black hole solutions in anti-de Sitter universe.

The last possible case is described in the section 3.6.6, when *two PNDs are tangent to $\mathcal{I}$*. We only write the inverse relation between the reference tetrad and the algebraically special tetrad (3.86) as

$$\begin{aligned} \mathbf{t}_{\mathrm{o}} &= \mathbf{t}_{\mathrm{s}}, \\ \mathbf{q}_{\mathrm{o}} &= -\mathbf{r}_{\mathrm{s}}, \\ \mathbf{r}_{\mathrm{o}} &= \mathbf{q}_{\mathrm{s}}, \\ \mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}, \end{aligned} \tag{9.131}$$

and after substituting from (9.108) into (9.131) we get

$$\begin{aligned} \mathbf{t}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma), \\ \mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{P}\partial_p, \\ \mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{\mathcal{Q}}\,\partial_q, \\ \mathbf{s}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}. \end{aligned} \tag{9.132}$$

The vector $\mathbf{q}_{\mathrm{o}}$ does not point into the both directions ∂_q and ∂_p as the normal vector $\mathbf{n}$. But when we set $\mathcal{Q} = 0$, the normal vector (9.112) becomes (the function $P > 0$ is positive)

$$\mathbf{n} = \frac{\mathbf{\Omega}}{\varrho}\sqrt{P}\partial_p. \tag{9.133}$$

and the vector $\mathbf{q}_{\mathrm{o}}$ is proportional to (9.133). The adjustment condition is thus *satisfied on horizons* where $\mathcal{Q} = 0$, and we observe that $\epsilon_{\mathrm{o}} = +1$. But we investigate the radiation at infinity, so that the adjustment condition is satisfied on roots of $\mathcal{Q}_{\mathcal{I}}$ given by (9.75). The directional structure is given by (3.91), where the amplitude of the radiation will be fully determined by $|\Psi^{\mathrm{s}}_{2*}|_{\mathcal{I}}$. When we substitute the roots of $\mathcal{Q}_{\mathcal{I}}$ into $|\Psi^{\mathrm{s}}_{2*}|_{\mathcal{I}}$, it becomes constant, of course, depending on the parameters of the sources.

This special case thus also occurs as a possible directional structure of radiation of general metric. It is a new feature, which does not occur in the C–metric and need further investigation.

To summarize the asymptotic directional structure of radiation near the timelike conformal infinity $\mathcal{I}$ for the general metric (9.62) in anti-de Sitter spacetime with small acceleration: it is characterized by the directional structure (9.121) corresponding to the situation when one PND $\mathbf{k}$ is ingoing and the other PND $\mathbf{l}$ is outgoing with respect to $\mathcal{I}$, which is fully consistent with the results presented in [5] for the C–metric, and by the structure where both PNDs are tangent to $\mathcal{I}$ on roots of $\mathcal{Q}_{\mathcal{I}}$ that is not present in the C–metric and is completely new. Again, there is a small difference in the normalization of the normal component of the vector $\mathbf{k}_{\mathrm{i}}$. We also observe that this section for timelike $\mathcal{I}$ of the complete family of black hole spacetimes for small acceleration is quite similar to the section 4.1.2 of Kerr–Newman–anti-de Sitter spacetime. This section is thus a generalization of the section on the Kerr–Newman–anti-de Sitter solution which includes acceleration.

9.4.3 Radiation in the general metric with timelike $\mathcal{I}$ $\mathcal{Q} < 0$

Again, we will gradually discuss all four possible orientations of PNDs on timelike $\mathcal{I}$ for $\Lambda < 0$ in tetrad where $\mathcal{Q} < 0$. This situation does not appear in the directional structure of the Kerr–Newman–de Sitter spacetime.

First, we start with the case when *one PND is outgoing and one is ingoing with respect to* $\mathcal{I}$, see section 3.6.3. We have previously identified the algebraically special orthonormal tetrad (9.84) and the normal vector $\mathbf{n}$ for $\mathcal{Q} < 0$, (9.88) in the section 9.4.1 for spacelike $\mathcal{I}$. This tetrad and the normal vector are the same for all cases in this section. We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.60), we obtain

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \mathbf{t}_{\mathrm{s}}, \\
\mathbf{q}_{\mathrm{o}} &= -\tanh\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{s}} + \cosh^{-1}\psi_{\mathrm{s}}\,\mathbf{q}_{\mathrm{s}}, \\
\mathbf{r}_{\mathrm{o}} &= \cosh^{-1}\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{s}} + \tanh\psi_{\mathrm{s}}\,\mathbf{q}_{\mathrm{s}}, \\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.
\end{aligned} \tag{9.134}$$

Substituting (9.84) into (9.134), we find the reference tetrad

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\partial_q, \\
\mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\tanh\psi_{\mathrm{s}}\sqrt{P}\partial_p - \cosh^{-1}\psi_{\mathrm{s}}\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right), \\
\mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\cosh^{-1}\psi_{\mathrm{s}}\sqrt{P}\partial_p - \tanh\psi_{\mathrm{s}}\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right), \\
\mathbf{s}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned} \tag{9.135}$$

The reference tetrad must satisfy the adjustment condition (3.19) which is explicitly $\mathbf{n} = -\epsilon_{\mathrm{o}}\mathbf{q}_{\mathrm{o}}$, i.e.

$$\mathbf{n} = -\epsilon_{\mathrm{o}}\frac{\mathbf{\Omega}}{\varrho}\left(\tanh\psi_{\mathrm{s}}\sqrt{P}\partial_p - \cosh^{-1}\psi_{\mathrm{s}}\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right). \tag{9.136}$$

The normal vector obtained from the metric (9.112) and (9.136) have to be equal. By comparing them we observe that the component ∂_q of the normal vector $\mathbf{n}$ does not appear in the normal from reference tetrad (9.136). This reference tetrad thus does *not* satisfy the adjustment condition, this case does not appear as a possible asymptotic behavior of the gravitational field of the complete family of black hole spacetime in anti-de Sitter universe.

We can also calculate the projection of $\mathbf{k}_\mathrm{s}$ into $\mathcal{I}$, to prove this result. Beginning from (3.25), using (9.135) and (3.58), we obtain $\mathbf{k}_\mathrm{s} \cdot \mathbf{q}_\mathrm{o} = \frac{1}{\sqrt{2}} \cosh^{-1} \psi_\mathrm{s}$, and the projection then is

$$\begin{aligned} \mathbf{t}_\mathrm{s} =& \frac{\boldsymbol{\Omega}}{\varrho} \Big(\cosh \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q - \tanh \psi_\mathrm{s} \sqrt{P} \partial_p \\ & + (\frac{1}{\cosh \psi_\mathrm{s}} - \cosh \psi_\mathrm{s}) \frac{1}{\sqrt{-\mathcal{Q}}} (\partial_\tau - \omega q^2 \partial_\sigma)) \Big) . \end{aligned} \tag{9.137}$$

Again, the vector $\mathbf{t}_\mathrm{s}$ is a projection of $\mathbf{k}_\mathrm{s}$ onto $\mathcal{I}$ and $\mathbf{n}$ is normal vector $\mathbf{n}$ (9.88) to $\mathcal{I}$, the scalar product must be $\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = 0$. When we calculate the scalar product,

$$\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = -\frac{1}{\sqrt{\mathcal{Q} + \alpha^2 P}} \left(\cosh \psi_\mathrm{s} \sqrt{-\mathcal{Q}} + \tanh \psi_\mathrm{s} \alpha \sqrt{P} \right) , \tag{9.138}$$

we observe that when $\mathbf{t}_\mathrm{s} \cdot \mathbf{n} = 0$ is satisfied, the implied expression does not give useful results. This case again does not represent the asymptotic behavior of the gravitational field of the complete family of black hole solutions in anti-de Sitter background.

Now we will investigate the case 3.6.4, in which *both PNDs are oriented to be outgoing or both ingoing.* The algebraically special orthonormal tetrad is (9.84). The reference tetrad expressed in terms of the algebraically special tetrad is the inverse of relations (3.68),

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= + \sinh^{-1} \psi_\mathrm{s} \, \mathbf{r}_\mathrm{s} + \coth \psi_\mathrm{s} \mathbf{t}_\mathrm{s}, \\ \mathbf{q}_\mathrm{o} &= - \sinh^{-1} \psi_\mathrm{s} \, \mathbf{t}_\mathrm{s} - \coth \psi_\mathrm{s} \, \mathbf{r}_\mathrm{s}, \\ \mathbf{r}_\mathrm{o} &= \mathbf{q}_\mathrm{s}, \\ \mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}. \end{aligned} \tag{9.139}$$

When we substitute (9.84) into (9.139), we obtain the reference tetrad in terms of the coordinate tetrad as

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \left(- \sinh^{-1} \psi_\mathrm{s} \sqrt{P} \partial_p + \coth \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q \right) , \\ \mathbf{q}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \left(- \sinh^{-1} \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q + \coth \psi_\mathrm{s} \sqrt{P} \partial_p \right) , \\ \mathbf{r}_\mathrm{o} &= \frac{\boldsymbol{\Omega}}{\varrho} \frac{-1}{\sqrt{-\mathcal{Q}}} (\partial_\tau - \omega q^2 \partial_\sigma), \\ \mathbf{s}_\mathrm{o} &= -\frac{\boldsymbol{\Omega}}{\varrho} \frac{[\omega p^2 \partial_\tau + \partial_\sigma]}{\sqrt{P}}. \end{aligned} \tag{9.140}$$

This reference tetrad must satisfy the adjustment condition, so $\mathbf{n} = -\epsilon_0 \mathbf{q}_0$,

$$\mathbf{n} = -\epsilon_\mathrm{o} \frac{\boldsymbol{\Omega}}{\varrho} \left(- \sinh^{-1} \psi_\mathrm{s} \sqrt{-\mathcal{Q}} \partial_q + \coth \psi_\mathrm{s} \sqrt{P} \partial_p \right) . \tag{9.141}$$

By comparing (9.141) with (9.112), we observe that it is necessary to set and evaluate on $\mathcal{I}$ the following expressions

$$\cosh\psi_{\rm s} =\frac{\alpha\sqrt{P}}{\sqrt{-\mathcal{Q}_{\mathcal{I}}}}, \tag{9.142}$$

$$\coth\psi_{\rm s} =\frac{\alpha\sqrt{P}}{\sqrt{\mathcal{Q}_{\mathcal{I}}+\alpha^2 P}}, \tag{9.143}$$

$$\sinh\psi_{\rm s} =\sqrt{\frac{\mathcal{Q}_{\mathcal{I}}+\alpha^2 P}{-\mathcal{Q}_{\mathcal{I}}}}, \tag{9.144}$$

where $\mathcal{Q}_{\mathcal{I}}$ is the expression (9.75), P is (9.63) and we also observe that $\epsilon_{\rm o}=+1$. It means that the normal vector is outgoing and both PNDs are outgoing with respect to the timelike conformal infinity $\mathcal{I}$. Again, we evaluated the above expressions on $\mathcal{I}$ and they are dependent only on the coordinate p.

We may calculate the radiative component of the gravitational field. The procedure is almost the same as in the previous spacelike case, the normalization factor is (9.95) and its evaluation on $\mathcal{I}$ is (9.102). The asymptotic directional structure has a form (3.73),

$$|\Psi_4^{\rm i}| \approx \frac{1}{|\eta|}\frac{3}{2}\frac{|\Psi_{2*}^{\rm s}|}{\sinh^2\psi_{\rm s}}\mathcal{A}_2(\psi,\phi,\psi_{\rm s}). \tag{9.145}$$

Where we denoted the angular dependence as

$$\mathcal{A}_2(\psi,\phi,\psi_{\rm s}) = (\cosh\psi_{\rm s}+\epsilon\cosh\psi)^2+\sinh^2\psi_{\rm s}\sinh^2\psi\sin^2\phi,$$

and we substituted $\epsilon_{\rm o}=+1$. Then $\sinh\psi_{\rm s}$ can be rewritten from (9.144) using (9.72) as

$$\sinh^2\psi_{\rm s} = -\frac{\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}{\alpha^2 P+\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)^{5/2}} \tag{9.146}$$

and we can also rewrite $\cosh\psi_{\rm s}$ from (9.142) as

$$\cosh\psi_{\rm s} = \frac{\sqrt{P}\alpha}{\sqrt{\alpha^2 P+\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}}, \tag{9.147}$$

where the function P is given by (9.63).
We also denote the amplitude of the radiation as

$$B(p) = \frac{|\Psi_{2*}^{\rm s}|_{\mathcal{I}}}{\sinh^2\psi_{\rm s}}. \tag{9.148}$$

Consequently, the asymptotic directional structure has a form

$$|\Psi_4^{\rm i}| \approx \frac{3}{2}\frac{1}{|\eta|}B(p)\mathcal{A}_2(\psi,\phi,\psi_{\rm s}), \tag{9.149}$$

where $B(p)$ can be rewritten by (9.102), (9.147) and (9.107) as

$$B(p) = -\frac{\alpha^2 P + \frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)}{\frac{\Lambda}{3}(1+\alpha^2\omega^2p^4)^{5/2}}\sqrt{\mathcal{D}_{\mathcal{I}}} \tag{9.150}$$

with

$$\mathcal{D}_{\mathcal{I}} = m^2 + n^2 - 4\alpha(e^2+g^2)p\frac{m+n\alpha\omega p^2}{1+\alpha^2\omega^2p^4} + 4\alpha^2(e^2+g^2)^2\frac{p^2}{1+\alpha^2\omega^2p^4}, \tag{9.151}$$

where the parameters n, k and ϵ are given by the constraints (9.64), (9.66) and (9.65).

We obtained the explicit formula for the radiative component of gravitational field of the accelerated black holes in the anti-de Sitter universe for $\mathcal{Q}<0$. Both PNDs are oriented outwards with respect to $\mathcal{I}$. There is no *outgoing* direction along which the radiation vanishes because mirrored reflections of both PNDs are ingoing. But there are two *ingoing* vanishing-radiation directions given by the mirrored reflections of both PNDs. This result is consistent with [5] where the C-metric was investigated for $\alpha > \sqrt{-\Lambda/3}$.

We observe that the amplitude $B(p)$ given by (9.149) is *identical* up to a sign with the magnitudes from previous sections for spacelike $\mathcal{I}$ and for timelike $\mathcal{I}$ with small acceleration, see equations (9.106), (9.107) and (9.121), (9.122). The dependence of the amplitude $B(p)$ will be investigated in section 9.5.

We now investigate the next case described in section 3.6.5, *the case when one PND is tangent to* $\mathcal{I}$. The identification of the algebraically special orthonormal tetrad is (9.84). We express the reference tetrad in terms of the algebraically special tetrad by inverting the relations (3.77). We obtain

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \tfrac{3}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}} - \mathbf{r}_{\mathrm{s}},\\
\mathbf{q}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \tfrac{1}{2}\mathbf{t}_{\mathrm{s}} + \tfrac{1}{2}\mathbf{q}_{\mathrm{s}},\\
\mathbf{r}_{\mathrm{o}} &= \mathbf{r}_{\mathrm{s}} - \mathbf{t}_{\mathrm{s}} - \mathbf{q}_{\mathrm{s}},\\
\mathbf{s}_{\mathrm{o}} &= \mathbf{s}_{\mathrm{s}}.
\end{aligned} \tag{9.152}$$

Substituting (9.84) into (9.152), we obtain the reference tetrad

$$\begin{aligned}
\mathbf{t}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(\frac{3}{2}\sqrt{-\mathcal{Q}}\,\partial_q - \frac{1}{2}\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) + \sqrt{P}\partial_p\right),\\
\mathbf{q}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\sqrt{P}\partial_p - \frac{1}{2}\sqrt{-\mathcal{Q}}\,\partial_q - \frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma)\right),\\
\mathbf{r}_{\mathrm{o}} &= \frac{\mathbf{\Omega}}{\varrho}\left(-\sqrt{P}\partial_p + \frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma) - \sqrt{-\mathcal{Q}}\,\partial_q\right),\\
\mathbf{s}_{\mathrm{o}} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}.
\end{aligned} \tag{9.153}$$

This reference tetrad must satisfy the adjustment condition (3.19). When we compare the vector $\mathbf{q}_{\mathrm{o}}$ with the normal $\mathbf{n}$ (9.112), we observe that the vector

$\mathbf{q}_\mathrm{o}$ has components ∂_q and ∂_p pointing in the same direction as the normal $\mathbf{n}$, but $\mathbf{q}_\mathrm{o}$ has another component which cannot disappear. This reference tetrad thus again does *not* satisfy the adjustment condition. We can also calculate the projection of $\mathbf{k}_\mathrm{s}$ into $\mathcal{I}$ as in the section for small acceleration and its discussion is quite similar. This case again does not represent the asymptotic behaviour of the gravitational field of black hole solutions in anti-de Sitter.

The last possible case is described in the section 3.6.6 when *two PNDs are tangent to* $\mathcal{I}$. We write the inverse relation between the reference tetrad and the algebraically special tetrad (3.86) as

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \mathbf{t}_\mathrm{s}, \\ \mathbf{q}_\mathrm{o} &= -\mathbf{r}_\mathrm{s}, \\ \mathbf{r}_\mathrm{o} &= \mathbf{q}_\mathrm{s}, \\ \mathbf{s}_\mathrm{o} &= \mathbf{s}_\mathrm{s}, \end{aligned} \tag{9.154}$$

and after substituting from (9.84) into (9.154), we get

$$\begin{aligned} \mathbf{t}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{-\mathcal{Q}}\,\partial_q, \\ \mathbf{q}_\mathrm{o} &= \frac{\mathbf{\Omega}}{\varrho}\sqrt{P}\partial_p, \\ \mathbf{r}_\mathrm{o} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{1}{\sqrt{-\mathcal{Q}}}(\partial_\tau - \omega q^2\partial_\sigma), \\ \mathbf{s}_\mathrm{o} &= -\frac{\mathbf{\Omega}}{\varrho}\frac{[\omega p^2\partial_\tau + \partial_\sigma]}{\sqrt{P}}. \end{aligned} \tag{9.155}$$

As in the previous section, this reference tetrad satisfy the adjustment condition $\mathbf{n} = -\epsilon_\mathrm{o}\mathbf{q}_o$ for $\sigma = +1$ because the vector $\mathbf{q}_\mathrm{o}$ points into the ∂_p direction as the normal for special case when $\mathcal{Q}_\mathcal{I} = 0$. We will not discuss this special case in this work. This degenerate case thus also occur as a possible asymptotic directional structure of the complete family of black hole spacetimes in anti-de Sitter space.

To sum up: the asymptotic directional structure of radiation near the time-like conformal infinity $\mathcal{I}$ for the general metric (9.62) for $\Lambda < 0$ and $\mathcal{Q} < 0$ is characterized by the case (9.149) corresponding to the situation when *both PNDs* are oriented outward with respect to $\mathcal{I}$. The other possible exceptional structure is when both PNDs are tangent to $\mathcal{I}$.

This result is not yet completely consistent with the results presented in [5] for the C-metric. The C-metric has two possible directional structures, first when one PND $\mathbf{k}$ is outgoing and the other one PND $\mathbf{l}$ is ingoing, second when two PNDs are oriented outwards with respect to $\mathcal{I}$. The conformal infinity in C–metric is divided into domains with different structures of PNDs, on the boundaries the PNDs are tangent to $\mathcal{I}$. The C–metric represents two accelerating black holes for $\alpha > \sqrt{-\Lambda/3}$ which are causally separated.

The first case does not occur in the general metric studied here for large acceleration. Most probably, the coordinates which we are using here do *not cover completely* the manifold of the general metric near $\mathcal{I}$. The possible structure of both PNDs tangent to $\mathcal{I}$ on horizons of $\mathcal{Q}$ on $\mathcal{I}$ also does not appear in the C–metric. More work is necessary to investigate this particular feature.

9.5 Discussion of the amplitude $B(p)$ of radiation

In previous sections we found that the amplitude of radiation $B(p)$ has an identical form for spacelike $\mathcal{I}$ as (9.106), (9.107) with $\mathcal{Q} < 0$, and for timelike $\mathcal{I}$ as (9.121), (9.122) with $\mathcal{Q} > 0$. The amplitude $B(p)$ for timelike $\mathcal{I}$ given by (9.150), (9.151) with $\mathcal{Q} < 0$ is similar up to a sign.

The amplitude $B(p)$ is very similar for all these cases because it is generally given by the same field component $|\Psi^{\mathrm{s}}_{2*}|_{\mathcal{I}}$ and by $\cos\theta_{\mathrm{s}}$ (9.99) for spacelike $\mathcal{I}$, or $\cosh\psi_{\mathrm{s}}$ (9.118) for timelike $\mathcal{Q} > 0$) which have, quite surprisingly, the same form. The term $\sinh\psi_{\mathrm{s}}$ (9.146) for timelike $\mathcal{I}$ ($\mathcal{Q} < 0$) is the same up to a sign. To have just one expression for amplitude $B(p)$ for all cases mentioned above. For convenience, we introduce *parameter* b which distinguishes these cases. The parameter is $b = 1$ applies for spacelike $\mathcal{I}$ and timelike $\mathcal{I}$ for $\mathcal{Q} > 0$, while $b = -1$ for timelike $\mathcal{I}$ $\mathcal{Q} < 0$.

The amplitude $B(p)$ depends on a single coordinate p and on the physical parameters α, m, a, l, e, g, Λ (and ω). We will now investigate the shape of the function $B(p)$ and the influence of these parameters. The conformal infinities are distinguished by different ranges of parameters α and Λ and also by the parameter b.

It is also very convenient to do transformation

$$p = \cos\vartheta, \tag{9.156}$$

where $\vartheta \in [0, \pi]$.

First, we will study the physically most interesting case of accelerating Kerr–Newman–de Sitter black holes (see section 9.2.2): we set the NUT parameter $l = 0$. This implies $\omega = a$ ($k = 1$). The parameter n is then $n = -\alpha a m$, and when we put these parameters into (9.150) and (9.151), we get the amplitude $B(\vartheta)$ in the form

$$\begin{aligned} B(\vartheta) &= b\frac{(1+\alpha^2a^2\cos^4\vartheta) + \frac{3}{\Lambda}\alpha^2 P}{(1+\alpha^2a^2\cos^4\vartheta)^{5/2}} \\ &\times \sqrt{m^2(1+\alpha^2a^2) - 4m\alpha(e^2+g^2)\tfrac{(1-\alpha^2a^2\cos^2\vartheta)\cos\vartheta}{1+\alpha^2a^2\cos^4\vartheta} + 4(e^2+g^2)^2\tfrac{\alpha^2\cos^2\vartheta}{1+\alpha^2a^2\cos^4\vartheta}}, \end{aligned} \tag{9.157}$$

where

$$P = \sin^2\vartheta\big(1 - 2\alpha m\cos\vartheta + \left[\alpha^2(a^2+e^2+g^2) + \tfrac{\Lambda}{3}a^2\right]\cos^2\vartheta\big) \tag{9.158}$$

which must be positive to retain the Lorentzian signature ($P > 0$).

Notice that for $\vartheta = \frac{\pi}{2}$ one gets simply

$$B(\vartheta = \tfrac{\pi}{2}) = b\,m(1 + \tfrac{3}{\Lambda}\alpha^2)\sqrt{1 + \alpha^2 a^2}. \tag{9.159}$$

For bigger rotational parameter a, the radiation is thus stronger when ($\alpha \neq 0$).

In particular, to visualize the results we have chosen the values of parameters of the Kerr–Newman black holes as $m = 1.1$, $e = g = 0.5$. The parameter a then can not be chosen arbitrarily. It is necessary to respect the location of horizon of the black hole. The function Q has a form (see section (9.2.2))

$$Q = (\omega^2 k + e^2 + g^2) - 2mr + r^2 - \frac{1}{3}\Lambda r^2(r^2 + a^2). \tag{9.160}$$

The horizons are determined by $Q = 0$ and the above function is cubic equation for r and it is not easy to solve. If we consider horizons for $\Lambda = 0$ the function (9.160) factorizes as

$$Q = \left((a^2 + e^2 + g^2) - 2mr + r^2\right)(1 - \alpha^2 r^2). \tag{9.161}$$

The inner/outer horizons of the black hole occurs for $\Lambda = 0$ on

$$r_\pm = m \pm \sqrt{m^2 - a^2 - e^2 - g^2} \tag{9.162}$$

and the acceleration horizon is located on $r = 1/\alpha$. To avoid a naked singularity case, we thus have to satisfy the condition

$$m^2 > a^2 + e^2 + g^2. \tag{9.163}$$

which follows from (9.162). According to this relation we set $a \in [0, 0.7]$. Let us remark that we have used the coordinate $r = -1/q$ instead of q, but these conclusions are obviously valid in general. The parameters must also satisfy following: $\alpha > 0$, $e > 0, g > 0$, $a^2 > e^2, a^2 > g^2$.

We realized that we have problems with negative values of $B(\vartheta)$ for timelike $\mathcal{I}$ $\mathcal{Q} < 0$, so I tried the next methods.

First, I tried to look numerically for roots of $\mathcal{Q}$ and then draw graphs for values of parameters, which give four roots of $\mathcal{Q}$. But it did not work, because when I found four roots of $\mathcal{Q}$, the $B(\vartheta)$ went to negative values (the reason is that condition $P > 0$ was not satisfied).

So, I tried something more easier. I started to fix gradually the parameters, first m, e, g, a, Λ and then look for values of α to make the graph $B(\vartheta)$ positive. I used the values of parameters from the first part of this text. The ranges of α really correspond to the limit from the beginning of this text (9.3). The graphs in the first version (first part of this text) of my diploma thesis are correct and I found that I am not able to make the graphs for timelike $\mathcal{I}$ $\mathcal{Q} < 0$ fully positive and this is also valid for the limit of C–metric.

9.5.1 Discussion of $B(\vartheta)$ for C-metric

Secondly, we will discuss the important special case namely, the non–rotating charged C–metric with cosmological constant. It can be obtained from the previous amplitude $B(\vartheta)$ given by (9.157) and (9.158) when we simply set $a = 0$. Then we obtain the following expression for the amplitude of radiation,

$$B(\vartheta) = b\,(1 + \tfrac{3}{\Lambda}\alpha^2 P)(m - 2\alpha(e^2 + g^2)\cos\vartheta), \tag{9.164}$$

where

$$P = \sin^2\vartheta(1 - 2\alpha m\cos\vartheta + \alpha^2(e^2 + g^2)\cos^2\vartheta). \tag{9.165}$$

and subsequently,

$$m - 2\alpha(e^2 + g^2)\cos\vartheta > 0 \quad \text{and} \quad P > 0. \tag{9.166}$$

We first look at the correspondence with papers about C-metric. We will compare our result (9.164) with papers [4] ($\Lambda > 0$) and [5] ($\Lambda < 0$).

For $\Lambda > 0$, our result (9.164) fully agrees with Eq. 5.19 [4], because there exists transformation between the coordinates used in [4] and coordinates used in our work, that

$$-x_\infty = \xi = p = \cos\vartheta, \tag{9.167}$$

and also it is necessary to rescale the affine parameter η, which is fixed by γ, like

$$(\sqrt{2}a_\Lambda)^2\gamma = 1. \tag{9.168}$$

After we use (9.167) and (9.168), we observe that our result agrees with the paper [4]. (We also observe, that expressions for $\sin\theta_s$ and $\cos\theta_s$ from [4] agree with our expressions).

For $\Lambda < 0$, we are able to compare our result (9.164) only with the Eq. 4.21 from [5]. We use the transformation (9.167) and we have to rescale the affine parameter η by

$$\frac{|\Lambda|}{4\gamma} = \frac{3}{2}, \tag{9.169}$$

to observe that the prefactors coincide. Unfortunately, we are not able to compare explicit expressions for There are not explicit expressions for $\sinh\psi_s$ and $\cosh\psi_s$ which are not presented in the paper [5] for all possibilities of orientation of PNDs.

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