

Electromagnetic field in higher-dimensional black-hole spacetimes

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A special test electromagnetic field in the spacetime of the higher-dimensional generally rotating NUT–(anti-)de Sitter black hole is found. It is adjusted to the hidden symmetries of the background represented by the principal Killing-Yano tensor. Such an electromagnetic field generalizes the field of charged black hole in four dimensions. In higher dimensions, however, the gravitational backreaction of such a field cannot be consistently solved.

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I. INTRODUCTION

In recent years people have become interested in higher-dimensional spacetimes with motivation, among others, coming from string theories. One of the important classes of exact solutions of the Einstein equations are the spacetimes representing black holes. In past years various generalizations of black-hole solutions to the higher-dimensional gravity have been discovered. Starting with nonrotating and rotating black holes [1,2] and following with the metric describing the rotating black hole with the cosmological constant in five dimensions [3], the generally rotating black hole in arbitrary dimension with the cosmological constant was found in [4,5]. In 2006 this solution was rewritten [6] in much more convenient coordinates which allowed one to add Newman-Unti-Tamburino (NUT) parameters and compute the curvature tensors explicitly [7].

Recently, the properties of this solution have been discussed in various papers. It has been shown that the spacetime possesses hidden symmetries which can be described by the principal Killing-Yano tensor [8], the properties of which have been thoroughly discussed in [9]. One of the main consequences of these hidden symmetries is a possibility to find a complete set of integration constants for a geodesic motion which are in involution [9–11]. This result is related to the separability of the Hamilton-Jacobi equation which, together with the separability of the Klein-Gordon equation, was demonstrated in [12].

It seems that a generalization of the black-hole solution to include an electromagnetic field is not straightforward. Some partial results are known [13–18], however a solution representing a rotating black hole with electromagnetic field parametrized by full number of black-hole parameters and electric/magnetic charges is not known.

In this work we present the test electromagnetic field on the background described by the metric with the structure of black-hole solution [6], however, we do not enforce the specific form of the metric function X_μ which is needed to satisfy the vacuum Einstein equations. The electromag-

netic field is described by $n = [D/2]$ electric and magnetic charges and it is adjusted to the explicit as well as hidden symmetries of the spacetime. In even dimensions our field is equivalent to the harmonic forms independently found in [19], our results supply the proof that these forms are harmonic in all even dimensions.

We also show that our field is a generalization of the electromagnetic field known from the Carter [20] and Plebański–Demiański [21] form of the black-hole solution in $D = 4$. In this special case, however, the metric function can be modified in such a way that the metric and the electromagnetic field satisfy the full coupled Einstein–Maxwell equations. Unfortunately, the same is not true in the higher-dimensional case.

The plan of the paper is the following. We discuss separately even and odd spacetime dimensions—although our procedure is very similar in both cases, the resulting expressions are slightly different and it would be artificial to try to “squeeze” them into the same equations. We discuss the case of even dimensions in detail and then sketch the corresponding results for the odd dimensions. First we review the metric and the symmetries of the black-hole spacetime, next we find the test electromagnetic field “adjusted” to the symmetries of the spacetime. For even dimensions we also discuss the case of the physical dimension $D = 4$, in the odd dimension case we study the Chern–Simons modification of the electromagnetic field. Finally, we discuss a possibility to generalize our fields to a solution of the coupled Einstein–Maxwell equations. The results are recapitulated in the Sec. IV and the paper is concluded by the technical appendix.

II. EVEN DIMENSIONS, $D = 2n$ **A. The metric**

We start with a simpler case of even spacetime dimension $D = 2n$. The spacetime of generally rotating black hole with NUT charges can be described by the metric

$$g = \sum_{\mu=1}^n \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right]. \quad (2.1)$$

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Here x_μ , $\mu = 1, \dots, n$, correspond to radial¹ and latitudinal directions, while ψ_j , $j = 0, \dots, n-1$, to temporal and longitudinal directions. The functions U_μ and $A_\mu^{(k)}$ [together with U and $A^{(k)}$ used below] are defined as

$$U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (x_\nu^2 - x_\mu^2), \quad U = \prod_{\substack{\mu,\nu=1 \\ \mu < \nu}}^n (x_\mu^2 - x_\nu^2), \quad (2.2)$$

$$A_\mu^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k=1 \\ \nu_1 < \dots < \nu_k, \nu_1 \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2, \quad A^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k=1 \\ \nu_1 < \dots < \nu_k}} x_{\nu_1}^2 \dots x_{\nu_k}^2. \quad (2.3)$$

We call the remaining functions X_μ the metric functions.

In the following we consider a broader class of spacetimes than merely the black-hole spacetime, namely, we just assume that the metric functions X_μ depend arbitrarily on a *single* coordinate, $X_\mu = X_\mu(x_\mu)$. For the black-hole spacetime they acquire a specific form [see (2.10) below] determined by the vacuum Einstein equations.

The metric can be diagonalized

$$\mathbf{g} = \sum_{\mu=1}^n \left(\frac{U_\mu}{X_\mu} \boldsymbol{\epsilon}^\mu \boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu} \boldsymbol{\epsilon}^{\hat{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}} \right) = \sum_{a=1}^D \mathbf{e}^a \mathbf{e}^a \quad (2.4)$$

introducing the unnormalized frame $\boldsymbol{\epsilon}^a$ and normalized frame \mathbf{e}^a of 1-forms

$$\begin{aligned} \boldsymbol{\epsilon}^\mu &= dx_\mu, & \boldsymbol{\epsilon}^{\hat{\mu}} &= \left[\frac{U_\mu}{X_\mu} \right]^{1/2} \boldsymbol{\epsilon}^\mu, \\ \boldsymbol{\epsilon}^{\hat{\mu}} &= \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k, & \boldsymbol{\epsilon}^{\hat{\mu}} &= \left[\frac{X_\mu}{U_\mu} \right]^{1/2} \boldsymbol{\epsilon}^{\hat{\mu}}. \end{aligned} \quad (2.5)$$

The dual frames of vectors are given by

$$\begin{aligned} \boldsymbol{\epsilon}_\mu &= \boldsymbol{\partial}_{x_\mu}, & \mathbf{e}_\mu &= \left[\frac{X_\mu}{U_\mu} \right]^{1/2} \boldsymbol{\epsilon}_\mu, \\ \boldsymbol{\epsilon}_{\hat{\mu}} &= \sum_{k=0}^{n-1} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}, & \mathbf{e}_{\hat{\mu}} &= \left[\frac{U_\mu}{X_\mu} \right]^{1/2} \boldsymbol{\epsilon}_{\hat{\mu}}. \end{aligned} \quad (2.6)$$

Here we use the convention $\hat{\mu} = \mu + n$ and Greek indices run from 1 to n . The inverse relations can be easily obtained with help of the relations (A1) in the appendix.

Let us note that the definition of unnormalized 1-forms $\boldsymbol{\epsilon}^a$ and vectors $\boldsymbol{\epsilon}_a$ does not depend on the metric functions X_μ .

It was shown in [7] that the Ricci curvature for such a metric is

$$\mathbf{Ric} = - \sum_{\mu=1}^n r_\mu (\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}), \quad (2.7)$$

where

$$\begin{aligned} r_\mu &= \frac{1}{2} \frac{X_\mu''}{U_\mu} + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{U_\nu} \frac{x_\nu X_\nu' - x_\mu X_\mu'}{x_\nu^2 - x_\mu^2} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{U_\nu} \frac{X_\nu - X_\mu}{x_\nu^2 - x_\mu^2} \\ &= \frac{\partial}{\partial x_\mu^2} \left[\sum_{\nu=1}^n \frac{x_\nu^2 (x_\nu^{-1} X_\nu)_{,\nu}}{U_\nu} \right]. \end{aligned} \quad (2.8)$$

The scalar curvature simplifies to

$$\mathcal{R} = - \sum_{\nu=1}^n \frac{X_\nu''}{U_\nu}. \quad (2.9)$$

Here, the primes denote the differentiation with respect to the single argument of the metric function $X_\mu' = X_{\mu,\mu}$.

The requirement that the metric (2.1) satisfies the Einstein equations with the cosmological constant implies that the metric functions have the form

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^n c_k x_\mu^{2k}, \quad (2.10)$$

see [6,7] or (A6) in the appendix. The constants c_k and b_μ are then related to the cosmological constants, angular momenta, mass, and NUT charges, see, e.g., [6] for details. For $b_\mu = 0$ we obtain [7] the constant curvature spacetime with the scalar curvature $\mathcal{R} = -2n(2n-1)c_{2n}$.

The spacetime with the metric (2.1) possesses the explicit symmetries given by the Killing vectors $\boldsymbol{\partial}_{\psi_j}$ and hidden symmetries which are related to the principal Killing-Yano tensor discovered in [8] and discussed in detail in [9]. The principal Killing-Yano tensor is dual to the rank-2 closed conformal Killing-Yano tensor which has a very simple form in the frames $\boldsymbol{\epsilon}^a$ and \mathbf{e}^a :

$$\mathbf{h} = \sum_{\mu=1}^n x_\mu \boldsymbol{\epsilon}^\mu \wedge \boldsymbol{\epsilon}^{\hat{\mu}} = \sum_{\mu=1}^n x_\mu \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\mu}}. \quad (2.11)$$

It has been demonstrated in [9] that it is possible to generate a series of higher-rank Killing-Yano tensors and a series of rank-2 Killing tensors from \mathbf{h} . The conformal Killing-Yano tensor also identifies eigenspaces spanned on the pairs $\{\boldsymbol{e}_\mu, \boldsymbol{e}_{\hat{\mu}}\}$ and coordinates x_μ as corresponding eigenvalues.

B. Algebraically special test electromagnetic field

Now we turn to the task to find an algebraically special test electromagnetic field on the background given by the metric (2.1). By *algebraically special* we mean that the Maxwell tensor \mathbf{F} shares the explicit symmetry of the metric (it is independent of ψ_j 's) and it is aligned with the hidden symmetry of the spacetime, namely, it has the same eigenspaces as the principal conformal

¹The radial coordinate (and some other related quantities) are actually rescaled by the imaginary unit i in order to put the metric to a more symmetric and compact form—cf., e.g., [6].

Killing-Yano tensor \mathbf{h} . We thus require

$$\mathbf{F} = \sum_{\mu=1}^n f_{\mu} \boldsymbol{\epsilon}^{\mu} \wedge \boldsymbol{\epsilon}^{\hat{\mu}}, \quad f_{\mu} = f_{\mu}(x_1, \dots, x_n). \quad (2.12)$$

The Maxwell tensor is generated by the vector potential, $\mathbf{F} = d\mathbf{A}$. We assume the vector potential

$$\mathbf{A} = \sum_{\mu=1}^n (A_{\mu} \boldsymbol{\epsilon}^{\mu} + A_{\hat{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}}) \quad (2.13)$$

with the components A_{μ} and $A_{\hat{\mu}}$ independent of ψ_j . Comparing $d\mathbf{A}$ with (2.12) we find that A_{μ} terms are gauge trivial and $A_{\hat{\mu}}$ must satisfy $((x_{\nu}^2 - x_{\mu}^2)A_{\hat{\mu}})_{,\nu} = 0$, from which follows that the vector potential can be written as

$$\mathbf{A} = \sum_{\mu=1}^n \frac{g_{\mu} x_{\mu}}{U_{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}}, \quad (2.14)$$

where g_{μ} are functions of a *single variable* only, $g_{\mu} = g_{\mu}(x_{\mu})$. Evaluating the Maxwell tensor we get the components f_{μ} :

$$f_{\mu} = \frac{g_{\mu}}{U_{\mu}} + \frac{x_{\mu} g'_{\mu}}{U_{\mu}} + 2x_{\mu} \sum_{\nu \neq \mu}^n \frac{1}{U_{\nu}} \frac{x_{\nu} g_{\nu} - x_{\mu} g_{\mu}}{x_{\nu}^2 - x_{\mu}^2}. \quad (2.15)$$

Alternatively, we could apply directly the first Maxwell equation $d\mathbf{F} = 0$ to the Maxwell tensor (2.12). With help of identity (A5) we find that f_{μ} are generated by an auxiliary potential ϕ ,

$$f_{\mu} = \phi_{,\mu}, \quad (2.16)$$

which satisfies the equation

$$\phi_{,\mu\nu} = 2 \frac{x_{\nu} \phi_{,\mu} - x_{\mu} \phi_{,\nu}}{x_{\mu}^2 - x_{\nu}^2} \quad \text{for } \mu \neq \nu. \quad (2.17)$$

The field (2.14) found above is generated by the potential

$$\phi = \sum_{\nu=1}^n \frac{g_{\nu} x_{\nu}}{U_{\nu}}. \quad (2.18)$$

Next we proceed to calculate the source \mathbf{J} of the electromagnetic field using the second Maxwell equation $\mathbf{J} = -\nabla \cdot \mathbf{F}$. Expressing the Maxwell tensor in coordinates $\{x_{\mu}, \psi_j\}$, using $F^{na}{}_{;n} = g^{-1/2} \sum_{\nu} (g^{1/2} F^{\nu a})_{,\nu}$, the fact that the determinant g of the metric in these coordinates is $g = U^2$, identities (A1) and (A2), and relations (2.12) and (2.16) we obtain

$$\mathbf{J} = \sum_{\mu=1}^n j_{\mu} \boldsymbol{\epsilon}^{\hat{\mu}}, \quad (2.19)$$

with

$$j_{\mu} = -2 \frac{\partial}{\partial x_{\mu}^2} \left[\phi - x_{\mu}^2 \sum_{\nu=1}^n x_{\nu}^{-1} \phi_{,\nu} \right]. \quad (2.20)$$

Substituting (2.18) we finally obtain

$$j_{\mu} = 2 \frac{\partial}{\partial x_{\mu}^2} \left[\sum_{\nu=1}^n \frac{x_{\nu}^2 g'_{\nu}}{U_{\nu}} \right]. \quad (2.21)$$

It is worth mentioning that the expression for the source \mathbf{J} does not contain any reference to the metric functions X_{μ} .

We are interested in the electromagnetic field without sources, so we require $\mathbf{J} = 0$. Integrating (2.21) we find that the sum in the square brackets has to be a constant. However, this sum has a special form discussed in the appendix. Using (A6) we find that g'_{μ} are given by a single polynomial of the $(n-1)$ th order in variable x_{μ}^2 . Integrating once more we find

$$g_{\mu} x_{\mu} = e_{\mu} x_{\mu} + \sum_{k=0}^{n-1} a_k (-x_{\mu}^2)^{n-1-k}. \quad (2.22)$$

Substituting into the vector potential (2.14) or the scalar potential (2.18) we find with help of the relations (A1) that the terms containing the constants a_k are gauge trivial (they contribute by $\sum_{k=0}^{n-1} a_k d\psi_k$ into \mathbf{A} or by just the constant a_0 into ϕ) and they can be ignored.

We thus have found that the algebraically special electromagnetic field [i.e., the field of the form (2.12)] satisfies the Maxwell equations on the background described by the metric (2.1) if and only if it is generated by the vector potential

$$\mathbf{A} = \sum_{\mu=1}^n \frac{e_{\mu} x_{\mu}}{U_{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}}. \quad (2.23)$$

The components f_{μ} of the Maxwell tensor are easily determined by (2.16) from the auxiliary potential

$$\phi = \sum_{\mu=1}^n \frac{e_{\mu} x_{\mu}}{U_{\mu}}, \quad (2.24)$$

and they are

$$f_{\mu} = \frac{e_{\mu}}{U_{\mu}} + 2x_{\mu} \sum_{\nu \neq \mu}^n \frac{1}{U_{\nu}} \frac{x_{\nu} e_{\nu} - x_{\mu} e_{\mu}}{x_{\nu}^2 - x_{\mu}^2}. \quad (2.25)$$

Here, e_{μ} are constants which can be related using the Gauss and Stokes theorems to the electric and magnetic charges of the field.

If we set all charges except one, say e_{ν} , to zero, the Maxwell tensor \mathbf{F} corresponds to the harmonic form $\mathbf{G}_{(2)}^{(\nu)}$ recently found and verified for particular cases in [19].

The surprising property of our field is that it satisfies the Maxwell equations independently of a specific form of the metric functions X_{μ} . Moreover, as we will see in (2.30), the stress-energy tensor corresponding to the field (2.12) has the form consistent with the structure of the Ricci (and the Einstein) tensor (2.7). These facts open a possibility that we could solve the full Einstein–Maxwell equations: mod-

ifying the metric functions X_μ we could construct the spacetime in which the stress-energy tensor \mathbf{T} would be a source for the Einstein equations, and the electromagnetic field would remain the solution of the Maxwell equations.

First, we will show that this goal can be achieved in $D = 4$ dimensions. Unfortunately, next we will demonstrate that this procedure does not work in higher dimensions.

C. Case $D = 4$

The metric (2.1) is a generalization of the $D = 4$ black-hole solution in the form found by Carter [20] and elaborated by Plebański and Demiański [21]. The full class of Plebański–Demiański solutions includes accelerated rotating NUT and electromagnetically charged black holes. The meaning of all parameters of the solutions was recently discussed in the series of papers [22–26]. We use the charged Plebański–Demiański metric with the acceleration set to zero²:

$$\mathbf{g} = -\frac{Q}{r^2 + p^2}(dt + p^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 + \frac{P}{r^2 + p^2}(dt - r^2 d\sigma)^2, \quad (2.26)$$

where

$$\begin{aligned} Q &= e^2 - 2mr + k + \varepsilon r^2 + \lambda r^4 \\ &= \mathcal{X}(-r^2) - 2mr + e^2, \\ P &= -g^2 + 2np + k - \varepsilon p^2 + \lambda p^4 \\ &= \mathcal{X}(p^2) + 2np - g^2. \end{aligned} \quad (2.27)$$

The electromagnetic field is given by

$$\mathbf{A} = -\frac{1}{r^2 + p^2}(er(dt + p^2 d\sigma) + gp(dt - r^2 d\sigma)). \quad (2.28)$$

Here t, r, p, σ are temporal, radial, latitudinal, and longitudinal coordinates, $m, n, \varepsilon, \lambda$ are parameters related to mass, NUT charge, angular momentum, and cosmological constant, and e, g are electric and magnetic charges.

This metric and the electromagnetic field satisfy coupled Einstein–Maxwell equations. For $e, g = 0$ the metric (2.26) satisfies the vacuum Einstein equations, however, even in this case the electromagnetic field (2.28) (with nonzero charges) satisfies the Maxwell equations and it is thus a valid test electromagnetic field on the given vacuum background.

Now, we can easily identify the Plebański–Demiański \mathbf{g} and \mathbf{A} with our general metric (2.1) and field (2.23) using the dictionary

²See, e.g., Eqs. (5) and (6) of [24] with acceleration $\alpha = 0$. Here we also used the gauge freedom to set $\omega = 1$ and redefined k .

$$\begin{aligned} \psi_0 &= t, & x_1 &= ir, & X_1 &= Q, \\ U_1 &= r^2 + p^2, & e_1 &= ie, \\ \psi_1 &= \sigma, & x_2 &= p, & X_2 &= P, \\ U_2 &= -r^2 - p^2, & e_2 &= g. \end{aligned} \quad (2.29)$$

Clearly, if the charges e, g are missing in the metric functions Q and P , these functions corresponds exactly to X_1 and X_2 given by (2.10) and the electromagnetic field (2.28) corresponds to the test field given by (2.23). However, it is possible to modify P and Q by adding e^2 and $-g^2$ respectively (i.e., changing the vacuum X_μ 's by adding $-e_\mu^2$), and we obtain the metric and the electromagnetic field satisfying the coupled Einstein–Maxwell equations.

D. Einstein–Maxwell equations in even $D \neq 4$

In a generic dimension we first evaluate the stress-energy tensor \mathbf{T} of the electromagnetic field (2.12). A straightforward calculation leads to

$$8\pi\mathbf{T} = \sum_{\mu=1}^n (2f_\mu^2 - f^2)(e^\mu e^\mu + e^\mu e^\mu), \quad (2.30)$$

with the trace

$$8\pi T = 2(2 - n)f^2, \quad (2.31)$$

where the function f^2 is defined as

$$f^2 = \sum_{\nu=1}^n f_\nu^2. \quad (2.32)$$

We explicitly see that the trace of the stress energy is nonvanishing for $D \neq 4$ which is related to the fact that the electromagnetic field is not conformally invariant in a general dimension.

Now we would like to solve the Einstein equations $\mathbf{Ric} - \frac{1}{2}\mathcal{R}\mathbf{g} + \Lambda\mathbf{g} = 8\pi\mathbf{T}$. The trace gives the condition

$$\mathcal{R} = 2\frac{D}{D-2}\Lambda + 2\frac{D-4}{D-2}f^2. \quad (2.33)$$

However, the scalar curvature has the form (2.9) and it immediately follows that

$$\frac{\partial^{2n-2}}{\partial x_\mu^{2n-2}}(U_\mu \mathcal{R}) = -X_\mu^{[2n]}, \quad (2.34)$$

which is a function of x_μ only. Applying this to the right-hand-side of (2.33) we obtain the condition

$$\frac{\partial^{2n-2}}{\partial x_\mu^{2n-2}}(U_\mu f^2) \quad \text{must be a function of } x_\mu \text{ only.} \quad (2.35)$$

It was checked by *Mathematica* that this conditions does not hold for the electromagnetic field given by (2.25), at least for the lowest nontrivial values of n . It seems that the

main problem is that \mathcal{R} behaves as $\sum h_\mu/U_\mu$ while f^2 as a square of such sums.

We thus may conclude that in a generic even dimension the electromagnetic field of the form (2.12) and (2.25) cannot couple to the metric given by (2.1).

III. ODD DIMENSIONS, $D = 2n + 1$

A. The metric

Let us briefly review modifications which appear in the odd dimensional case. We have an additional coordinate ψ_n which labels an ‘‘unpaired’’ angular direction. The metric contains an additional term,

$$\mathbf{g} = \sum_{\mu=1}^n \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right] + \frac{c}{A^{(n)}} \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2. \quad (3.1)$$

Here c is a conventional constant.³ We can again introduce the frames of 1-forms given by (2.5) completed with 1-forms which we label by the index $\hat{0} \equiv 2n + 1$

$$\boldsymbol{\epsilon}^{\hat{0}} = \sum_{k=0}^n A^{(k)} d\psi_k, \quad \mathbf{e}^{\hat{0}} = \left[\frac{c}{A^{(n)}} \right]^{1/2} \boldsymbol{\epsilon}^{\hat{0}}. \quad (3.2)$$

The dual frames then become

$$\boldsymbol{\epsilon}_{\hat{\mu}} = \sum_{k=0}^n \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}, \quad \mathbf{e}_{\hat{\mu}} = \left[\frac{U_\mu}{X_\mu} \right]^{1/2} \boldsymbol{\epsilon}_{\hat{\mu}},$$

$$\boldsymbol{\epsilon}_{\hat{0}} = \frac{1}{A^{(n)}} \boldsymbol{\partial}_{\psi_n}, \quad \mathbf{e}_{\hat{0}} = \left[\frac{A^{(n)}}{c} \right]^{1/2} \boldsymbol{\epsilon}_{\hat{0}}. \quad (3.3)$$

The metric diagonalizes

$$\mathbf{g} = \sum_{\mu=1}^n \left(\frac{U_\mu}{X_\mu} \boldsymbol{\epsilon}^\mu \boldsymbol{\epsilon}_\mu + \frac{X_\mu}{U_\mu} \boldsymbol{\epsilon}^{\hat{\mu}} \boldsymbol{\epsilon}_{\hat{\mu}} \right) + \frac{c}{A^{(n)}} \boldsymbol{\epsilon}^{\hat{0}} \boldsymbol{\epsilon}_{\hat{0}} = \sum_{a=1}^D \mathbf{e}^a \mathbf{e}_a, \quad (3.4)$$

as well as the Ricci tensor [7]

$$\mathbf{Ric} = - \sum_{\mu=1}^n r_\mu (\mathbf{e}^\mu \mathbf{e}_\mu + \mathbf{e}^{\hat{\mu}} \mathbf{e}_{\hat{\mu}}) - r_0 \mathbf{e}^{\hat{0}} \mathbf{e}_{\hat{0}}. \quad (3.5)$$

Here

³The constant c could be eliminated by a appropriate rescaling of the coordinates and other parameters.

$$r_\mu = \frac{1}{2} \frac{\bar{X}''_\mu}{U_\mu} + \frac{1}{2x_\mu} \frac{\bar{X}'_\mu}{U_\mu} + \sum_{\nu \neq \mu}^n \frac{1}{U_\nu} \frac{x_\nu \bar{X}'_\nu - x_\mu \bar{X}'_\mu}{x_\nu^2 - x_\mu^2}$$

$$= \frac{\partial}{\partial x_\mu^2} \left[\sum_{\nu=1}^n \frac{x_\nu \bar{X}'_\nu}{U_\nu} \right], \quad (3.6)$$

$$r_0 = \sum_{\nu=1}^n \frac{\bar{X}'_\nu}{x_\nu U_\nu},$$

and we used shifted metric functions

$$\bar{X}_\mu = X_\mu + \frac{c}{x_\mu^2}. \quad (3.7)$$

The scalar curvature becomes

$$\mathcal{R} = - \sum_{\nu=1}^n \frac{\bar{X}''_\nu}{U_\nu} - 2 \sum_{\nu=1}^n \frac{1}{x_\nu} \frac{\bar{X}'_\nu}{U_\nu}. \quad (3.8)$$

Finally, the vacuum Einstein equations require

$$X_\mu = - \frac{c}{x_\mu^2} + b_\mu + \sum_{k=0}^{n-1} c_k x_\mu^{2k}. \quad (3.9)$$

The spacetime has $n + 1$ Killing vectors $\boldsymbol{\partial}_{\psi_j}$ and there exists the principal Killing–Yano tensor dual to the conformal Killing–Yano tensor which is given again by (2.11).

B. Algebraically special test electromagnetic field

As in the even dimensions we look for the electromagnetic field with the structure given by (2.12). One could ask if in the odd dimension could not be this ansatz extended by an additional term related to the ‘‘unpaired’’ direction, i.e., by the term of the form $f_0 dx \wedge \boldsymbol{\epsilon}^{\hat{0}}$ with functions f_0 and x independent of the coordinates ψ_j . However, we do not consider such terms since they lead to some unwanted consequences (e.g., they lead to a nondiagonal stress-energy tensor).

It follows from the first Maxwell equations that the vector potential can be written as

$$\mathbf{A} = \frac{g_0}{A^{(n)}} \boldsymbol{\epsilon}^{\hat{0}} + \sum_{\mu=1}^n \frac{g_\mu}{U_\mu} \boldsymbol{\epsilon}^{\hat{\mu}}, \quad (3.10)$$

where g_μ is a function of x_μ only and g_0 is a constant. The components of the Maxwell tensor can be again generated by (2.16) from the auxiliary potential

$$\phi = \frac{g_0}{A^{(n)}} + \sum_{\nu=1}^n \frac{g_\nu}{U_\nu}, \quad (3.11)$$

and in terms of g 's they take the form

$$f_\mu = \frac{g'_\mu}{U_\mu} + 2x_\mu \sum_{\nu=1}^n \frac{1}{U_\nu} \frac{g_\nu - g_\mu}{x_\nu^2 - x_\mu^2} - \frac{2g_0}{x_\mu A^{(n)}}. \quad (3.12)$$

The electromagnetic current becomes

$$\mathbf{J} = -2 \sum_{\mu=1}^n \frac{\partial}{\partial x_\mu^2} \left[x_\mu^2 \sum_{\nu=1}^n x_\nu^{-1} \phi_\nu \right] \boldsymbol{\epsilon}_{\hat{\mu}} + 2 \sum_{\nu=1}^n \frac{1}{x_\mu} \phi_{,\nu} \boldsymbol{\epsilon}_{\hat{0}}. \quad (3.13)$$

Solving the second Maxwell equation without sources (i.e., $\mathbf{J} = 0$) we obtain

$$g_\mu = e_\mu - \frac{g_0}{x_\mu^2} + \sum_{k=0}^{n-2} a_k (-x_\mu^2)^{n-1-k}. \quad (3.14)$$

The last two terms are gauge trivial and the vector potential can thus be written as

$$\mathbf{A} = \sum_{\mu=1}^n \frac{e_\mu}{U_\mu} \boldsymbol{\epsilon}_{\hat{\mu}}, \quad (3.15)$$

and the auxiliary potential ϕ as

$$\phi = \sum_{\nu=1}^n \frac{e_\nu}{U_\nu}. \quad (3.16)$$

Finally, for the components of the Maxwell tensor we find

$$f_\mu = 2x_\mu \sum_{\nu=1}^n \frac{1}{U_\nu} \frac{e_\nu - e_\mu}{x_\nu^2 - x_\mu^2}. \quad (3.17)$$

C. The Chern–Simons term

In an odd number of dimensions we can consider Chern–Simons modification of the electromagnetic field. Since the Chern–Simons term in the action does not refer to the metric, it does not change the Einstein equation and the stress-energy tensor of the electromagnetic field. However, it modifies the divergence $-\nabla \cdot \mathbf{F}$ in the second Maxwell equation by adding a nonlinear term \mathbf{J}_{ChS} proportional (with a constant coefficient) to the Hodge dual of the n th wedge power of \mathbf{F} ,

$$\mathbf{J}_{\text{ChS}} \propto * \left(\underbrace{\mathbf{F} \wedge \cdots \wedge \mathbf{F}}_{n \text{ times}} \right). \quad (3.18)$$

Since the Levi-Civita tensor ε used in the Hodge dual is given by the product $e^1 \wedge \cdots \wedge e^{2n+1}$ we find that the Chern–Simons term \mathbf{J}_{ChS} for the field (2.12) is aligned with the “unpaired” direction

$$\mathbf{J}_{\text{ChS}} \propto \left[\prod_{\mu=1}^n f_\mu \right] \boldsymbol{e}_{\hat{0}}. \quad (3.19)$$

We thus have to solve the Maxwell equation $\mathbf{J} = \mathbf{J}_{\text{ChS}}$ with \mathbf{J} given by (3.13). The condition that \mathbf{J} vanishes in $\boldsymbol{e}_{\hat{\mu}}$, $\mu = 1, \dots, n$, directions is only slightly weaker than that we solved in the previous section. We find that the vector potential \mathbf{A} , the auxiliary potential ϕ , and the components of the electromagnetic field are

$$\begin{aligned} \mathbf{A} &= \sum_{\mu=1}^n \frac{e_\mu + a \log x_\mu^2}{U_\mu} \boldsymbol{\epsilon}_{\hat{\mu}}, \\ \phi &= \sum_{\nu=1}^n \frac{e_\nu + a \log x_\nu^2}{U_\nu}, \\ f_\mu &= 2x_\mu \sum_{\nu=1}^n \frac{1}{U_\nu} \frac{e_\nu - e_\mu}{x_\nu^2 - x_\mu^2} \\ &\quad + 2a \left[\frac{1}{x_\mu} + x_\mu \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{U_\nu} \frac{\log x_\nu^2 - \log x_\mu^2}{x_\nu^2 - x_\mu^2} \right], \end{aligned} \quad (3.20)$$

where a is a constant parameter. Moreover, the source term \mathbf{J} must be

$$\mathbf{J} = \frac{4a}{A^{(n)}} \boldsymbol{e}_{\hat{0}}. \quad (3.21)$$

Although we did not prove it rigorously it seems evident that this component cannot be equal to (3.19) with f_μ given by (3.20). We thus conclude that the Chern–Simons modification of the algebraically special electromagnetic field is not possible.

D. Einstein–Maxwell equations in odd D

The stress-energy tensor for the field (2.12) in odd dimensions is also diagonal (with the additional component $8\pi T_{\hat{0}\hat{0}} = -f^2$). The condition for the scalar curvature following from the Einstein equations is again given by (2.33), and by the same argument as in the even dimensions [now with help of (3.8)] we conclude that the algebraically special electromagnetic field (2.12) and (3.17) cannot couple to the metric given by (3.1).

IV. SUMMARY

We have found explicitly the test electromagnetic field on the background of the high-dimensional rotating NUT charged black hole [6]. The field solves the Maxwell equations even in a broader class of spacetimes since the specific form of the metric functions X_μ is not needed. The common feature of these spacetimes is the presence of $[D/2]$ Killing vectors and the existence of the principal Killing–Yano tensor [9]. The constructed electromagnetic field is adjusted to this structure—it shares the explicit symmetries and it has the same eigenspaces as the principal conformal Killing–Yano tensor.

The electromagnetic field depends on $n = [D/2]$ constants e_μ related to the global electric and magnetic charges. It generalizes the field known on the background of the Plebański–Demiański spacetime in $D = 4$ dimensions. In this case the metric functions can be modified in such a way that the field and the metric solve the full Einstein–Maxwell equations. Unfortunately, an analogous modification is not possible in a generic dimension.

Finally, we have shown that the Chern–Simons generalization in an odd dimension is also not permitted for the electromagnetic field of this form.

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APPENDIX

The functions $A_\mu^{(k)}$ and U_μ defined in (2.2) satisfy the following useful identities:

$$\begin{aligned} \sum_{\mu=1}^n A_\mu^{(i)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} &= \delta_j^i, & \sum_{j=0}^{n-1} A_\mu^{(j)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} &= \delta_\mu^\nu, \\ \sum_{\mu=1}^n A_\mu^{(i)} \frac{(-x_\mu^2)^n}{U_\mu} &= -A^{(j+1)}, & \sum_{\mu=1}^n \frac{A_\mu^{(j)}}{x_\mu^2 U_\mu} &= \frac{A^{(j)}}{A^{(n)}}, \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} \sum_{j=0}^n (n-j) A_\mu^{(j)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} &= 1, \\ \sum_{j=0}^{n-1} (n-1-j) A_\mu^{(j)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} &= \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{x_\mu^2}{x_\mu^2 - x_\nu^2}, \\ \sum_{j=0}^{n-1} (n-1-j) A_\nu^{(j)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} &= \frac{x_\mu^2}{x_\mu^2 - x_\nu^2} \quad \text{for } \mu \neq \nu, \end{aligned} \quad (\text{A2})$$

and

$$\frac{1}{U_\mu} \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{x_\nu^2 - x_\mu^2} = - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{U_\nu} \frac{1}{x_\nu^2 - x_\mu^2}. \quad (\text{A3})$$

We list also external derivatives of the function $1/U_\mu$ and of the 1-form $\epsilon^{\hat{\mu}}$ which have been used repeatedly in the computations:

$$d \frac{1}{U_\mu} = - \frac{1}{U_\mu} \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{2}{x_\nu^2 - x_\mu^2} (x_\nu \epsilon^\nu - x_\mu \epsilon^\mu), \quad (\text{A4})$$

$$d \epsilon^{\hat{\mu}} = \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{2}{x_\nu^2 - x_\mu^2} (x_\nu \epsilon^\nu \wedge \epsilon^{\hat{\mu}} - x_\nu \epsilon^\nu \wedge \epsilon^{\hat{\nu}}). \quad (\text{A5})$$

Finally, let us formulate explicitly an important lemma (which has been already used implicitly in [7]) concerning properties of the sums $\sum_\nu h_\nu / U_\nu$. Let us consider the equation

$$\sum_{\mu=1}^n \frac{h_\mu}{U_\mu} = 0, \quad (\text{A6a})$$

where h_μ are functions of a single variable x_μ only. Then these functions are given by a single polynomial

$$h_\mu = \sum_{k=0}^{n-2} c_k x_\mu^{2k} \quad (\text{A6b})$$

with arbitrary coefficients c_k , $k = 0, \dots, n-2$.

The fact that the polynomial functions (A6b) solve Eq. (A6a) follows from (A1). The opposite implication is less trivial—first one has to show that h_μ must be polynomials of the order $n-2$ (by differentiating repeatedly $U_\mu \sum_\nu h_\nu / U_\nu$). Next, a more intricate task is to prove that the polynomials for different μ are the same. It can be achieved by an induction in n .

The solution of the functional Eq. (A6a) with a non-trivial right-hand-side is then given by the sum of a particular solution with the homogeneous solution (A6b). We mention three important particular solutions for simple right-hand-side terms:

$$\begin{aligned} \sum_{\mu=1}^n \frac{(-x_\mu^2)^{n-1}}{U_\mu} &= 1, \\ \sum_{\mu=1}^n \frac{(-x_\mu^2)^n}{U_\mu} &= -A^{(1)}, \\ \sum_{\mu=1}^n \frac{1}{x_\mu^2 U_\mu} &= \frac{1}{A^{(n)}}, \end{aligned} \quad (\text{A6c})$$

all following from the relations (A1).

The property (A6) has been used, for example, to derive the specific forms (2.10) and (3.9) of the metric functions X_μ starting from the condition $\mathcal{R} = \text{constant}$ with the scalar curvature given by (2.9) or (3.8). It has been used also to find the components (2.22) and (3.14) of the vector potential.

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