Selected Solutions of Einstein's Field Equations: Their Role in General Relativity and Astrophysics

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1 Introduction and a Few Excursions

The primary purpose of all physical theory is rooted in reality, and most relativists pretend to be physicists. We may often be members of departments of mathematics and our work oriented towards the mathematical aspects of Einstein's theory, but even those of us who hold a permanent position on "scri", are primarily looking there for gravitational waves. Of course, the builder of this theory and its field equations was *the* physicist. Jürgen Ehlers has always been very much interested in the conceptual and axiomatic foundations of physical theories and their rigorous, mathematically elegant formulation; but he has also developed and emphasized the importance of such areas of relativity as kinetic theory, the mechanics of continuous media, thermodynamics and, more recently, gravitational lensing. Feynman expressed his view on the relation of physics to mathematics as follows [1]:

"The physicist is always interested in the special case; he is never interested in the general case. He is talking about something; he is not talking abstractly about anything. He wants to discuss the gravity law in three dimensions; he never wants the arbitrary force case in n dimensions. So a certain amount of reducing is necessary, because the mathematicians have prepared these things for a wide range of problems. This is very useful, and later on it always turns out that the poor physicist has to come back and say, 'Excuse me, when you wanted to tell me about four dimensions...' " Of course, this is Feynman, and from 1965...

However, physicists are still rightly impressed by special explicit formulae. Explicit solutions enable us to discriminate more easily between a "physical" and "pathological" feature. Where are there singularities? What is their character? How do test particles and fields behave in given background spacetimes? What are their global structures? Is a solution stable and, in some sense, generic? Clearly, such questions have been asked not only within general relativity.

By studying a *special* explicit solution one acquires an intuition which, in turn, stimulates further questions relevant to more *general* situations. Consider, for example, charged black holes as described by the Reissner–Nordström solution. We have learned that in their interior a Cauchy horizon

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exists and that the singularities are timelike. We shall discuss this in greater detail in Sect. 3.1. The singularities can be seen by, and thus exert an influence on, an observer travelling in their neighborhood. However, will this violation of the (strong) cosmic censorship persist when the black hole is perturbed by weak ("linear") or even strong ("nonlinear") perturbations? We shall see that, remarkably, this question can also be studied by explicit exact special model solutions. Still more surprisingly, perhaps, a similar question can be addressed and analyzed by means of explicit solutions describing completely diverse situations – the collisions of plane waves. As we shall note in Sect. 8.3, such collisions may develop Cauchy horizons and subsequent timelike singularities. The theory of black holes and the theory of colliding waves have intriguing structural similarities which, first of all, stem from the circumstance that in both cases there exist two symmetries, i.e. two Killing fields. What, however, about more general situations? This is a natural question inspired by the explicit solutions. Then "the poor physicists have to come back" to a mathematician, or today alternatively, to a numerical relativist, and hope that somehow they will firmly learn whether the cosmic censorship is the "truth", or that it has been a very inspirational, but in general false conjecture. However, even *after* the formulation of a conjecture about a general situation inspired by particular exact solutions, *newly* discovered exact solutions can play an important role in verifying, clarifying, modifying, or ruling out the conjecture. And also "old" solutions may turn out to act as asymptotic states of general classes of models, and so become still more significant.

Exact explicit solutions have played a crucial role in the development of many areas of physics and astrophysics. Later on in this Introduction we will take note of some general features which are specific to the solutions of Einstein's equations. Before that, however, for illustration and comparison we shall indicate briefly with a few examples what influence exact explicit solutions have had in other physical theories. Our next introductory excursion, in Sect. 1.2, describes in some detail the (especially early) history of Einstein's route to the gravitational field equations for which his short stay in Prague was of great significance. The role of Ernst Mach (who spent 28 years in Prague before Einstein) in the construction of the first modern cosmological model, the Einstein static universe, is also touched upon. Section 1.3 is devoted to a few remarks on some old and new impacts of the other simplest "cosmological" solutions of Einstein's equations – the Minkowski, the de Sitter, and the anti de Sitter spacetimes. Some specific features of solutions in Einstein's theory, such as the observability and interpretation of metrics, the role of general covariance, the problem of the equivalence of two metrics, and of geometrical characterization of solutions are mentioned in Sect. 1.4. Finally, in the last (sub)sections of the "Introduction" we give some reasons why we consider our choice of solutions to be "a natural selection", and we briefly outline the main body of the article.

1.1 A Word on the Role of Explicit Solutions in Other Parts of Physics and Astrophysics

Even in a linear theory like Maxwell's electrodynamics one needs a good sample, a useful kit, of exact fields like the homogeneous field, the Coulomb field, the dipole, the quadrupole and other simple solutions, in order to gain a physical intuition and understanding of the theory. Similarly, of course, with the linearized theory of gravity. Going over to the Schrödinger equation of standard quantum mechanics, again a linear theory, consider what we have learned from simple, explicitly soluble problems like the linear and the three-dimensional harmonic oscillator, or particles in potential wells of various shapes. We have acquired, for example, a transparent insight into such basic quantum phenomena as the existence of minimum energy states whose energy is not zero, and their associated wave functions which have a certain spatial extent, in contrast to classical mechanics. The three-dimensional problems have taught us, among other things, about the degeneracy of the energy levels. The case of the harmonic oscillator is, of course, very exceptional since Hamiltonians of the same type appear in all problems involving quantized oscillations. One encounters them in quantum electrodynamics, quantum field theory, and likewise in the theory of molecular and crystalline vibrations. It is thus perhaps not so surprising that the Hamiltonian and the wave functions of the harmonic oscillator arise even in the minisuperspace models associated with the Hartle–Hawking no-boundary proposal for the wave function of the universe [2], and in the minisuperspace model of homogeneous spherically symmetric dust filled universes [3].

In *nonlinear* problems explicit solutions play still a greater role since to gain an intuition of nonlinear phenomena is hard. Landau and Lifshitz in their Fluid Mechanics (Volume 6 of their course) devote a whole section to the exact solutions of the nonlinear Navier–Stokes equations for a viscous fluid (including Landau's own solution for a jet emerging from the end of a narrow tube into an infinite space filled with fluid).

Although Poisson's equation for the gravitational potential in the classical theory of gravity is linear, the combined system of equations describing both the field and its *fluid* sources (not rigid bodies, these are simple!) characterized by Euler's equations and an equation of state are nonlinear. In classical astrophysical fluid dynamics perhaps the most distinct and fortunate example of the role of explicit solutions is given by the exact descriptions of ellipsoidal, uniform density masses of self-gravitating fluids. These "ellipsoidal figures of equilibrium" [4] include the familiar Maclaurin spheroids and triaxial Jacobi ellipsoids, which are characterized by rigid rotation, and a wider class discovered by Dedekind and Riemann, in which a motion of uniform vorticity exists, even in a frame in which the ellipsoidal surface is at rest. The solutions representing the rotating ellipsoids did not only play an inspirational role in developing basic concepts of the theory of rigidly rotating stars, but quite unexpectedly in the study of inviscid, differentially rotating polytropes. These closely resemble Maclaurin spheroids, although they do not maintain rigid rotation. As noted in the well-known monograph on rotating stars [5], "the classical work on uniformly rotating, homogeneous spheroids has a range of validity much greater than was usually anticipated". It also influenced galactic dynamics [6]: the existence of Jacobi ellipsoids suggested that a rapidly rotating galaxy may not remain axisymmetric, and the Riemann ellipsoids demonstrated that there is a distinction between the rate at which the matter in a triaxial rotating body streams and the rate at which the figure of the body rotates. Since rotating incompressible ellipsoids adequately illustrate the general feature of rotating axisymmetric bodies, they are also used in the studies of double stars whose components are close to each other. The disturbances caused by a neighbouring component are treated as first order perturbations. Relativistic effects on the rotating incompressible ellipsoids have been investigated in the post-Newtonian approximation by various authors, recently with a motivation to understand the coalescence of binary neutron stars near their innermost stable circular orbit (see [7] for the latest work and a number of references).

As for the last subject, which has a more direct connection with exact explicit solutions of Einstein's equations, we want to say a few words about integrable systems and their soliton solutions. Soliton theory has been one of the most interesting developments in the past decades both in physics and mathematics, and gravity has played a role both in its birth and recent developments. It has been known from the end of the last century that the celebrated Korteweg–de Vries nonlinear evolution equation, which governs one dimensional surface gravity waves propagating in a shallow channel of water, admits solitary wave solutions. However, it was not until Zabusky and Kruskal (the Kruskal of Sect. 2.4 below) did extensive *numerical* studies of this equation in 1965 that the remarkable properties of the solitary waves were discovered: the nonlinear solitary waves, named solitons by Zabusky and Kruskal, can interact and then continue, preserving their shapes and velocities. This discovery has stimulated extensive studies of other nonlinear equations, the inverse scattering methods of their solution, the proof of the existence of an infinite number of conservation laws associated with such equations, and the construction of explicit solutions (see [8] for a recent comprehensive treatise). Various other nonlinear equations, similar to the sine-Gordon equation or the nonlinear Schrödinger equation, arising for example in plasma physics, solid state physics, and nonlinear optics, have also been successfully tackled by these methods. At the end of the 1970s several authors discovered that Einstein's vacuum equations for axisymmetric stationary systems can be solved by means of the inverse scattering methods, and it soon became clear that one can employ them also in situations when both Killing vectors are spacelike (producing, for example, soliton-type cosmological gravitational waves). Dieter Maison, one of the pioneers in applying these techniques in general relativity, describes the subject thoroughly in this volume. We shall briefly meet the soliton methods when we discuss the uniformly rotating disk solution of Neugebauer and Meinel (Sect. 6.3), colliding plane waves (Sect. 8.3), and inhomogeneous cosmological models (Sect. 12.2). Our aim, however, is to understand the meaning of solutions, rather than generation techniques of finding them. From this viewpoint it is perhaps first worth noting the interplay between numerical and analytic studies of the soliton solutions – hopefully, a good example of an interaction for numerical and mathematical relativists. However, the explicit solutions of integrable models have played important roles in various other contexts. The most interesting *multi*-dimensional integrable equations are the four-dimensional self-dual Yang–Mills equations arising in field theory. Their solutions, discovered by R. Ward using twistor theory, on one hand stimulated Donaldson's most remarkable work on inequivalent differential structures on four-manifolds. On the other hand, Ward indicated that many of the known integrable systems can be obtained by dimensional reduction from the self-dual Yang–Mills equations. Very recently this view has been substantiated in the monograph by Mason and Woodhouse [9]. The words by which these authors finely express the significance of exact solutions in integrable systems can be equally well used for solutions of Einstein's equations: "they combine tractability with nonlinearity, so they make it possible to explore nonlinear phenomena while working with explicit solutions".[1]

1.2 Einstein's Field Equations

Since Jürgen Ehlers has always been, among other things, interested in the history of science, he will hopefully tolerate a few remarks on the early history of Einstein's equations to which not much attention has been paid in the literature. It was during his stay in Prague in 1911 and 1912 that Einstein's intensive interest in quantum theory diminished, and his systematic effort in constructing a relativistic theory of gravitation began. In his first "Prague theory of gravity" he assumed that gravity can be described by a single function – the local velocity of light. This assumption led to insurmountable difficulties. However, Einstein learned much in Prague on his way to general relativity [11]: he understood the local significance of the principle of equivalence; he realized that the equations describing the gravitational field must be nonlinear and have a form invariant with respect to a larger group

[1] In 1998, in the discussion after his Prague lecture on the present role of physics in mathematics, Prof. Michael Atiyah expressed a similar view that even with more powerful supercomputers and with a growing body of general mathematical results on the existence and uniqueness of solutions of differential equations, the exact, explicit solutions of nonlinear equations will not cease to play a significant role. (As it is well known, Sir Michael Atiyah has made fundamental contributions to various branches of mathematics and mathematical physics, among others, to the theory of solitons, instantons, and to the twistor theory of Sir Roger Penrose, with whom he has been interacting "under the same roof" in Oxford for 17 years [10].)

of transformations than the Lorentz group; and he found that "*spacetime coordinates lose their simple physical meaning*", i.e. they do not determine directly the distances between spacetime points.[2] In his "Autobiographical Notes" Einstein says: "Why were seven years ... required for the construction of general relativity? The main reason lies in the fact that it is not easy to free oneself from the idea that coordinates must have an immediate metrical meaning"... Either from Georg Pick while still in Prague, or from Marcel Grossmann during the autumn of 1912 after his return to Zurich (cf. [11]), Einstein learned that an appropriate mathematical formalism for his new theory of gravity was available in the work of Riemann, Ricci, and Levi–Civita. Several months after his departure from Prague and his collaboration with Grossmann, Einstein had general relativity almost in hand. Their work [13] was already based on the generally invariant line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{I}$$

in which the spacetime metric tensor $g_{\mu\nu}(x^\rho), \mu, \nu, \rho = 0, 1, 2, 3$, plays a dual role: on the one hand it determines the spacetime geometry, on the other it represents the (ten components of the) gravitational potential and is thus a dynamical variable. The disparity between geometry and physics, criticized notably by Ernst Mach,[3] had thus been removed. When searching for the field equations for the metric tensor, Einstein and Grossmann *had* already realized that a natural candidate for generally covariant field equations would be the equations relating – in present-day terminology – the Ricci tensor and the energy-momentum tensor of matter. However, they erroneously concluded that such equations would not yield the Poisson equation of Newton's theory of gravitation as a first approximation for weak gravitational fields (see both §5 in the "Physical part" in [13] written by Einstein and §4, below equation (46), in the "Mathematical part" by M. Grossmann). Einstein then rejected the general covariance. In a subsequent paper with Grossmann [14], they supported this mis-step by a well-known "hole" meta-argument and obtained (in today's terminology) four gauge conditions such that the field equations were covariant only with respect to transformations of coordinates permitted by the gauge conditions. We refer to, for example, [15] for more detailed information on the further developments leading to the final version of the field equations. Let us only summarize that in late 1915 Einstein first readopted the generally covariant field equations from 1913, in which the Ricci tensor $R_{\mu\nu}$ was, up to the gravitational coupling constant, equal to the energy-momentum tensor $T_{\mu\nu}$ (paper submitted to the Prussian Academy on

[2] At that time Einstein's view on the future theory of gravity are best summarized in his reply to M. Abraham [12], written just before departure from Prague.

[3] Mach spent 28 years as Professor of Experimental Physics in Prague, until 1895, when he took the History and Theory of Inductive Natural Sciences chair in Vienna.

November 4). From his *vacuum field equations*

$$R_{\mu\nu} = 0, \tag{II}$$

where $R_{\mu\nu}$ depends nonlinearly on $g_{\alpha\beta}$ and its first derivatives, and linearly on its second derivatives, he was able to explain the anomalous part of the perihelion precession of Mercury – in the note presented to the Academy on November 18. And finally, in the paper [16] submitted on November 25 (published on December 2, 1915), the final version of the *gravitational field equations*, or *Einstein's field equations* appeared:[4]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{III}$$

where the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$. Newton's gravitational constant G and the velocity of light c are the (only) fundamental constants appearing in the theory. If not stated otherwise, in this article we use the geometrized units in which $G = c = 1$, and the same conventions as in [18] and [19].

Now it is well known that Einstein further generalized his field equations by adding a cosmological term $+\Lambda g_{\mu\nu}$ on the left side of the field equations (III). The cosmological constant Λ appeared first in Einstein's work "Cosmological considerations in the General Theory of Relativity" [20] submitted on February 8, 1917 and published on February 15, 1917, which contained the *closed static model of the Universe* (the Einstein static universe) – an exact solution of equations (III) with $\Lambda > 0$ and an energy-momentum tensor of incoherent matter ("dust"). This solution marked the birth of modern cosmology.

We do not wish to embark upon the question of the role that Mach's principle played in Einstein's thinking when constructing general relativity, or upon the intriguing issues relating to aspects of Mach's principle in present-day relativity and cosmology[5] – a problem which in any event would far exceed the scope of this article. Although it would not be inappropriate to

[4] David Hilbert submitted his paper on these field equations five days before Einstein, though it was published only on March 31, 1916. Recent analysis [17] of archival materials has revealed that Hilbert made significant changes in the proofs. The originally submitted version of his paper contained the theory which is not generally covariant, and the paper did not include equations (III).

[5] It was primarily Einstein's recognition of the role of Mach's ideas in his route towards general relativity, and in his christening them by the name "Mach's principle" (though Schlick used this term in a vague sense three years before Einstein), that makes Mach's Principle influential even today. After the 1988 Prague conference on Ernst Mach and his influence on the development of physics [21], the 1993 conference devoted exclusively to Mach's principle was held in Tübingen, from which a remarkably thorough volume was prepared [22], covering all aspects of Mach's principle and recording carefully all discussion. The clarity of ideas and insights of Jürgen Ehlers contributed much to both conferences and their proceedings. For a brief more recent survey of various aspects of Mach's

include it here since exact solutions (such as Gödel's universe or Ozsváth's and Schücking's closed model) have played a prominent role in this context. However, it should be at least stated that Einstein originally invented the idea of a closed space in order to eliminate boundary conditions at spatial infinity. The boundary conditions "flat at infinity" bring with them an inertial frame unrelated to the mass-energy content of the space, and Einstein, in accordance with Mach's views, believed that merely mass-energy can influence inertia. Field equations (III) are not inconsistent with this idea, but they admit as the simplest solution an empty flat *Minkowski space* ($T_{\mu\nu} = 0,\ \ g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\ (-1, +1, +1, +1)$), so some restrictive boundary conditions are essential if the idea is to be maintained. Hence, Einstein introduced the cosmological constant Λ, hoping that with this space will always be closed, and the boundary conditions eliminated. But it was also in 1917 when *de Sitter* discovered the *solution* [25] of the vacuum field equations (II) with added cosmological term ($\Lambda > 0$) which demonstrated that a nonvanishing Λ does not necessarily imply a nonvanishing mass-energy content of the universe.

1.3 "Just So" Notes on the Simplest Solutions: The Minkowski, de Sitter, and Anti-de Sitter Spacetimes

Our brief intermezzo on the cosmological constant brought up three explicit simple exact solutions of Einstein's field equations – the Minkowski, Einstein, and de Sitter spacetimes. To these also belongs the anti de Sitter spacetime, corresponding to a negative Λ. The de Sitter spacetime has the topology $R^1 \times S^3$ (with R^1 corresponding to the time) and is best represented geometrically as the 4-dimensional hyperboloid $-v^2 + w^2 + x^2 + y^2 + z^2 = (3/\Lambda)$ in 5-dimensional flat space with metric $ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2$. The anti de Sitter spacetime has the topology $S^1 \times R^3$, and can be visualized as the 4-dimensional hyperboloid $-U^2 - V^2 + X^2 + Y^2 + Z^2 = (-3/\Lambda)$, $\Lambda < 0$, in flat 5-dimensional space with metric $ds^2 = -dU^2 - dV^2 + dX^2 + dY^2 + dZ^2$. As is usual (cf. e.g. [26,27]), we mean by "anti de Sitter spacetime" the universal covering space which contains no closed timelike lines; this is obtained by unwrapping the circle S^1.

These spacetimes will not be discussed in the following sections. Occasionally, for instance, in Sects. 5 and 10, we shall consider spacetimes which become asymptotically de Sitter. However, since these solutions have played a crucial role in many issues in general relativity and cosmology, and most recently, they have become important prerequisites on the stage of the theoretical physics of the "new age", including string theory and string cosmology, we shall make a few comments on these solutions here, and give some references to recent literature.

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principle in general relativity, see the introductory section in the work [23], in which Mach's principle is analyzed in the context of perturbed Robertson–Walker universes. Most recently, Mach's principle seems to enter even into M theory [24].

The basic geometrical properties of these spaces are analyzed in the Battelle Recontres lectures by Penrose [27], and in the monograph by Hawking and Ellis [26], where also references to older literature can be found. The important role of the de Sitter solution in the theory of the expanding universe is finely described in the book by Peebles [28], and in much greater detail in the proceedings of the Bologna 1988 meeting on the history of modern cosmology [29].

The Minkowski, de Sitter and anti de Sitter spacetimes are the simplest solutions in the sense that their *metrics* are *of constant* (zero, positive, and negative) *curvature*. They admit the same number (ten) of independent Killing vectors, but the interpretations of corresponding symmetries differ for each spacetime. Together with the Einstein static universe, they all are conformally flat, and can be represented as portions of the Einstein static universe [26,27]. However, their *conformal structure* is globally different. In Minkowski spacetime one can go to infinity along timelike geodesics and arrive to the future (or past) *timelike infinity* i^+ (or i^-); along null geodesics one reaches the future (past) *null infinity* $\mathcal{J}^+(\mathcal{J}^-)$; and spacelike geodesics lead to *spatial infinity* i^0. Minkowski spacetime can be compactified and mapped onto a finite region by an appropriate conformal rescaling of the metric. One thus obtains the well-known Penrose diagram in which the three types of infinities are mapped onto the boundaries of the compactified spacetime – see for example the boundaries on the "right side" in the Penrose diagram of the Schwarzschild—Kruskal spacetime in Fig. 3, Sect. 2.4, or the Penrose compactified diagram of boost-rotation symmetric spacetimes in Fig. 13, Sect. 11. (The details of the conformal rescaling of the metric and resulting diagrams are given in [26,27] and in standard textbooks, for example [18,19,30].) In the de Sitter spacetime there are only past and future conformal infinities $\mathcal{J}^-, \mathcal{J}^+$, both being *spacelike* (cf. the Penrose diagram of the "cosmological" Robinson–Trautman solutions in Fig. 11, Sect. 10); the conformal infinity in anti de Sitter spacetime is *timelike*.

These three spacetimes of constant curvature offer many basic insights which have played a most important role elsewhere in relativity. To give just a few examples (see e.g. [26,27]): both the particle (cosmological) horizons and the event horizons for geodesic observers are well illustrated in the de Sitter spacetime; the Cauchy horizons in the anti de Sitter space; and the simplest acceleration horizons in Minkowski space (hypersurfaces $t^2 = z^2$ in Fig. 12, Sect. 11). With the de Sitter spacetime one learns (by considering different cuts through the 4-dimensional hyperboloid) that the concept of an "open" or "closed" universe depends upon the choice of a spacelike slice through the spacetime. There is perhaps no simpler way to understand that Einstein's field equations are of local nature, and that the spacetime topology is thus not given a priori, than by considering the following construction in Minkowski spacetime. Take the region given in the usual coordinates by $|x| \leq 1, |y| \leq 1, |z| \leq 1$, remove the rest and identify pairs of boundary points of the form $(t, 1, y, z)$ and $(t, -1, y, z)$, and similarly for y and z. In this way the spatial sections are identified to obtain a 3-torus – a flat but closed manifold.[6]

The *spacetimes of constant curvature* have been resurrected as basic *arenas of new physical theories* since their first appearance. After the role of the de Sitter universe decreased with the refutation of the steady-state cosmology, it has inflated again enormously in connection with the theory of early quasi-exponential phase of expansion of the universe, due to the false-vacuum state of a hypothetical scalar (inflaton) field(s) (see e.g. [28]). We shall mention the de Sitter space as the asymptotic state of cosmological models with a nonvanishing Λ (so verifying the "cosmic no-hair conjecture") in Sect. 10 on Robinson–Trautman spacetimes. Motivated by its importance in inflationary cosmologies, several new useful papers reviewing the properties of de Sitter spacetime have appeared [33,34]; they also contain many references to older literature. For the most recent work on the quantum structure of de Sitter space, see [35].

In the last two years, anti de Sitter spacetime has come to the fore in light of Maldacena's conjecture [36] relating string theory in (asymptotically) anti de Sitter space to a non-gravitational conformal field theory on the boundary at spatial infinity, which is timelike as mentioned above (see, e.g. [37], where among others, in the Appendix various coordinate systems describing anti de Sitter spaces in arbitrary dimensions are discussed).

Amazingly, the Minkowski spacetime has recently entered the active new area of so called pre-big bang string cosmology [38]. String theory is here applied to the problem of the big bang. The idea is to start from a simple Minkowski space (as an "asymptotic past triviality") and to show that it is in an unstable false-vacuum state, which leads to a long *pre*-big bangian inflationary phase. This, at later times, should provide a hot big bang. Although such a scenario has been criticized on various grounds, it has attractive features, and most importantly, can be probed through its observable relics [38].

Since it is hard to forecast how the roles of these three spacetimes of constant curvature will develop in new and exciting theories in the next millennium, let us better conclude our "just so" notes by stating three "stable" results of complicated, rigorous mathematical analyses of (the classical) Einstein's equations.

In their recent treatise [39], Christodoulou and Klainerman prove that any smooth, asymptotically flat initial data set which is "near flat, Minkowski data" leads to a unique, smooth and geodesically complete solution of Ein-

[6] This very simple point was apparently unknown to Einstein in 1917, although soon after the publication of his cosmological paper, E. Freundlich and F. Klein pointed out to him that an elliptical topology (arising from the identification of antipodal points) could have been chosen instead of the spherical one considered by Einstein. Although topological questions have been followed with a great interest in recent decades, the chapter by Geroch and Horowitz in "An Einstein Centenary Survey" [31] remains the classic; for more recent texts, see for example [32] and references therein.

stein's vacuum equations with vanishing cosmological constant. This demonstrates the *stability of the Minkowski space* with respect to nonlinear (vacuum) perturbations, and the existence of singularity-free, asymptotically flat radiative vacuum spacetimes. Christodoulou and Klainerman, however, are able to show only a somewhat weaker decay of the field at null infinity than is expected from the usual assumption of a sufficient smoothness at null infinity in the framework of Penrose (see e.g. [40] for a brief account).

Curiously enough, in the case of the vacuum Einstein equations with a *nonvanishing cosmological constant*, a more complete picture has been known for some time. By using his regular conformal field equations, Friedrich [41] demonstrated that initial data sufficiently close to de Sitter data develop into solutions of Einstein's equations with a positive cosmological constant, which are "asymptotically simple" (with a smooth conformal infinity), as required in Penrose's framework. More recently, Friedrich [42] has shown the existence of asymptotically simple solutions to the Einstein vacuum equations with a negative cosmological constant. For the latest review of Friedrich's thorough work on asymptotics, see [43].

Summarizing, thanks to these profound mathematical achievements we know that the *Minkowski, de Sitter*, and *anti de Sitter spacetimes* are the solutions of Einstein's field equations which are *stable with respect to general, nonlinear* (though "weak" in a functional sense) *vacuum perturbations.* A result of this type is not known for any other solution of Einstein's equations.

1.4 On the Interpretation and Characterization of Metrics

Suppose that a metric satisfying Einstein's field equations is known in some region of spacetime and in a given coordinate (reference) system x^μ. A fundamental question, frequently "forgotten" to be addressed in modern theories which extend upon general relativity, is whether the *metric tensor* $g_{\alpha\beta}(x^\mu)$ *is a measurable quantity.* Classical general relativity offers (at least) three ways of giving a positive answer, depending on what objects are considered as "primitive tools" to perform the measurements. The first, elaborated and emphasized primarily by Møller [44], employs standard rigid rods in the measurements. However, a "rigid rod" is not really a simple primitive concept. The second procedure, due to Synge [45], accepts as the basic concepts a "particle" and a "standard clock". If x^μ and $x^\mu + dx^\mu$ are two nearby events contained in the worldline of a clock, then the separation (the spacetime interval) between the events is equal to the interval measured by the clock. The main drawback of this approach appears to lie in the fact that it does not explain why the same functions $g_{\alpha\beta}(x^\mu)$ describe the behavior of the clock as well as paths of free particles, as explained in more detail by Ehlers, Pirani and Schild [46], in the motivation for their own axiomatic but constructive procedure for setting up the spacetime geometry. Their method, inspired by the work of Weyl and others, uses neither rods nor clocks, but instead, light rays and freely falling test particles, which are considered as basic tools for measuring the metric and determining the spacetime geometry. (For a simple description of how this can be performed, see exercise 13.7 in [18]; for some new developments which build upon, among others, the Ehlers-Pirani-Sachs approach, see [47].) After indicating that the metric tensor is a measurable quantity let us briefly turn to the *role of spacetime coordinates.*

In special relativity there are infinitely many global inertial coordinate systems labelling events in the Minkowski manifold $I\!R^4$; they are related by elements of the Poincaré group. The inertial coordinates labels X^0, X^1, X^2, X^3 of a given event do not thus have intrinsic meaning. However, the spacetime interval between two events, determined by the Minkowski metric $\eta_{\mu\nu}$, represents an intrinsic property of spacetime. Since the Minkowski metric is so simple, the differences between inertial coordinates can have a metrical meaning (recall Einstein's reply to Abraham mentioned in Sect. 1.2). In principle, however, both in special and general relativity, it is the metric, the line element, which exhibits intrinsically the geometry, and gives all relevant information. As Misner [48] puts it, if you write down for someone the Schwarzschild metric in the "canonical" form (equation (2) in Sect. 2.2) and receive the reaction "that [it] tells me the $g_{\mu\nu}$ gravitational potentials, now tell me in which (t, r, θ, φ) coordinate system they have these values?", then there are two valid responses: (a) indicate that it is an indelicate and unnecessary question, or (b) ignore it. Clearly, the Schwarzschild metric describes the geometrical properties of the coordinates used in (2). For example, it implies that worldlines with fixed r, θ, φ are timelike at $r > 2M$, orthogonal to the lines with $t =$ constant. It determines local null cones (given by $ds^2 = 0$), i.e. the *causal structure* of the spacetime. In addition, in Schwarzschild coordinates the metric (2) indicates how to *measure* the radial coordinate of a given event, because the proper area of the sphere going through the event is given just by the Euclidean expression $4\pi r^2$ (r is thus often called "the curvature coordinate"). On each sphere the angular coordinates θ, φ have the same meaning as on a sphere in Euclidean space. The Schwarzschild coordinate time t, geometrically preferred by the timelike (for $r > 2M$) Killing vector, which is just equal to $\partial/\partial t$, can be measured by radar signals sent out from spatial infinity ($r \gg 2M$) where t is the proper time (see e.g. [18]). The coordinates used in (2) are in fact "more unique" than the inertial coordinates in Minkowski spacetime, because the only possible continuous transformations preserving the form (2) are rigid rotations of a sphere, and $t \to t +$ constant. Such a simple interpretation of coordinates is exceptional. However, the simple case of the Schwarzschild metric clearly demonstrates that all intrinsic information is contained in the line element.

It is interesting, and for some purposes useful, to consider not just one Schwarzschild metric with a given mass M but the *family* of such metrics for all possible M. In order to cover also the future event horizon let us describe the metrics by using Eddington–Finkelstein ingoing coordinates as in equation (4), Sect. 2.3. This equation can be interpreted as a family of metrics with various values of M given on a *fixed background manifold* $\bar{\mathcal{M}}_1$, with $v \in I\!R, r \in (0, \infty)$, and $\theta \in [0, \pi], \varphi \in [0, 2\pi)$. Alternatively, however, we may use, for example, the Kruskal null coordinates $\tilde{U}, \tilde{V}$ in which the metric is given by equation (6), Sect. 2.3, with $\tilde{U} = V - U, \tilde{V} = V + U$. We may then consider metrics on a background manifold $\bar{\mathcal{M}}_2$ given by $\tilde{U} \in I\!R, \tilde{V} \in (0, \infty)$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$, which corresponds to $\bar{\mathcal{M}}_1$. However, these two background manifolds are *not* the same: the transformation between the Eddington–Finkelstein coordinates and the Kruskal coordinates is not a map from $\bar{\mathcal{M}}_1$ to $\bar{\mathcal{M}}_2$ because it depends on the value of mass M. Therefore, the "background manifold" used frequently in general relativity, for example in problems of conservation of energy, or in quantum gravity, is not defined in a natural, unique manner. The above simple pedagogical observation has recently been made in connection with gauge fixing in quantum gravity by Hájíček [49] in order to explain the old insight by Bergmann and Komar, that the gauge group of general relativity is much larger than the diffeomorphism group of one manifold. To identify points when working with backgrounds, one usually fixes coordinates in all solution manifolds by some gauge condition, and identifies those points of all these manifolds which have the same value of the coordinates.

Returning back to a single solution $(\mathcal{M}, g_{\alpha\beta})$, described by a manifold $\mathcal{M}$ and a metric $g_{\alpha\beta}$ in some coordinates, a notorious (local) *"equivalence problem"* often arises. A given (not necessarily global) solution has the variety of representations which equals the variety of choices of a 4-dimensional coordinate system. Transitions from one choice to another are isomorphic with the group of 4-dimensional diffeomorphisms which expresses the general covariance of the theory.[7] Given another set of functions $g'_{\alpha\beta}(x'^{\gamma})$ which satisfy Einstein's equations, how do we learn that they are not just transformed components of the metric $g_{\alpha\beta}(x^{\gamma})$? In 1869 E. B. Christoffel raised a more general question: under which conditions is it possible to transform a quadratic form $g_{\alpha\beta}(x^{\gamma})dx^{\alpha}dx^{\beta}$ in n-dimensions into another such form $g'_{\alpha\beta}(x'^{\gamma})dx'^{\alpha}dx'^{\beta}$ by means of smooth transformation $x^{\gamma}(x'^{\kappa})$? As Ehlers emphasized in his paper

[7] As pointed out by Kretschmann soon after the birth of general relativity, one can always make a theory generally covariant by taking more variables and inserting them as new dynamical variables into the (enlarged) theory. Thus, standard Yang–Mills theory is covariant with respect to the transformations of Yang–Mills potentials, corresponding to a particular group, say $SU(2)$. However, the theory is usually formulated on a fixed background spacetime with a given metric. The evolution of a dynamical Yang–Mills solution is thus "painted" on a given spacetime. When the metric – the gravitational field – is incorporated as a dynamical variable in the Einstein–Yang–Mills theory, the whole spacetime metric and Yang–Mills field are "built-up" from given data (cf. the article by Friedrich and Rendall in this volume). The resulting theory is covariant with respect to a much larger group. The dual role of the metric, determined only up to 4-dimensional diffeomorphisms, makes the character of the solutions of Einstein's equations unique among solutions of other field theories, which do not consider spacetime as being dynamical.

[50] on the meaning of Christoffel's equivalence problem in modern field theories, Christoffel's results apply to metrics of arbitrary signature, and can be thus used directly in general relativity. Without going into details let us say that today the solution to the equivalence problem as presented by Cartan is most commonly used. For both metrics $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ one has to find a frame (four 1-forms) in which the frame metric is constant, and find the frame components of the Riemann tensor and its covariant derivatives up to – possibly – the 10th order. The two metrics $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are then equivalent if and only if there exist coordinate and Lorentz transformations under which one whole set of frame components goes into the other. In a practical algorithm given by Karlhede [51], recently summarized and used in [52], the number of derivations required is reduced.

A natural first idea of how to solve the equivalence problem is to employ the scalar invariants from the Riemann tensor and its covariant derivatives. This, however, does not work. For example, in all Petrov type *N* and *III* nonexpanding and nontwisting solutions all these invariants vanish as shown recently (see Sect. 8.2), as they do in Minkowski spacetime.

However, even without regarding invariants, at present much can be learnt about an exact solution (at least locally) in geometrical terms, without reference to special coordinates. This is thanks to the progress started in the late 1950s, in which the group of Pascual Jordan in Hamburg has played the leading role, with Jürgen Ehlers as one of its most active members. Ehlers' dissertation[8] [54] from 1957 is devoted to the characterization of exact solutions.

The problem of exact solutions also forms the content of his contribution to the Royaumont GR-conference [55], as well as his plenary talk in the London GR-conference [56]. A detailed description of the results of the Hamburg group on invariant geometrical characterization of exact solutions by using and developing the Petrov classification of Weyl's tensors, groups of isometries, and conformal transformations are contained in the first paper [57] in the (today "golden oldies") series of articles published in the "Abhandlungen der Akademie der Wissenschaften in Mainz". An English version, in a somewhat shorter form, was published by Ehlers and Kundt [53] in the "classic" 1962 book "Gravitation: An Introduction to Current Research" compiled by L. Witten. (We shall meet these references in the following sections.) In the second paper of the "Abhandlungen" [58], among others, algebraically special vacuum solutions are studied, using the formalism of the 2-component spinors, and in particular, geometrical properties of the congruences of null rays are analyzed in terms of their expansion, twist, and shear.

[8] The English translation of the title of the dissertation reads: "The construction and characterization of the solutions of Einstein's gravitational field equations". In [53] the original German title is quoted, as in our citation [54], but "of the solutions" is erroneously omitted. This error then reemerges in the references in [19].

These tools became essential for the discovery by Roy Kerr in 1963 of the solution which, when compared with all other solutions of Einstein's equations found from the beginning of the renaissance of general relativity in the late 1950s until today, has played the most important role. As Chandrasekhar [59] eloquently expresses his wonder about the remarkable fact that all stationary and isolated black holes are *exactly* described by the Kerr solution: "This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time ..." The Kerr solution can also serve as one of finest examples in general relativity of "the incredible fact that a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature..." [60].

The technology developed in the classical works [53,57], and in a number of subsequent contributions, is mostly concerned with the local geometrical characterization of exact spacetime solutions. A well-known feature of the solutions of Einstein's equations, not shared by solutions in other physical theories, is that it is often very complicated to analyze their global properties, such as their extensions, completeness, or topology. If analyzed globally, almost any solution can tell us something about the basic issues in general relativity, like the nature of singularities, or cosmic censorship.

1.5 The Choice of Solutions

Since most solutions, when properly analyzed, can be of potential interest, we are confronted with a richness of material which puts us in danger of mentioning many of them, but remaining on a general level, and just enumerating rather than enlightening. In fact, because of lack of space (and of our understanding) we shall have to adopt this attitude in many places. However, we have selected some solutions, hopefully the fittest ones, and when discussing their role, we have chosen particular topics to be analyzed in some detail, and left other issues to brief remarks and references.

Firstly, however, let us ask what do we understand by the term "exact solution". In the much used "exact-solution-book" [61], the authors "do not intend to provide a definition", or, rather, they have decided that what they "chose to include was, by definition, an exact solution". A mathematical relativist-purist would perhaps consider solutions, the existence of which has been demonstrated in the works of Friedrich or Christodoulou and Klainerman, mentioned at the end of Sect. 1.3, as "good" as the Schwarzschild metric. Most recently, Penrose [62] presented a strong conjecture which may lead to a general vacuum solution described in the complicated (complex) formalism of his twistor theory. Although in this article we do not mean by exact solutions those just mentioned, we also do not consider as exact solutions only those explicit solutions which can be written in terms of elementary functions on half of a page. We prefer, recalling Feynman, simple "special cases", but we also discuss, for example, the late-time behaviour of the Robinson–Trautman solutions for which rigorously convergent series expansions can be obtained, which provide sufficiently rich "special information".

Concerning the selection of the solutions, the builder of general relativity and the gravitational field equations (III) himself indicates which solutions should be preferred [63]: "The theory avoids all internal discrepancies which we have charged against the basis of classical mechanics... But, it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low grade wood (right side of equation). The phenomenological representation of matter is, in fact, only a crude substitute for a representation which would correspond to all known properties of matter. There is no difficulty in connecting Maxwell's theory... so long as one restricts himself to space, free of ponderable matter and free of electric density..."

Of course, Einstein was not aware when he was writing this of Yang–Mills–Higgs fields, or of the dilaton field, etc. However, remaining on the level of field theories with a clear classical meaning, his view has its strength and motivates us to prefer (electro)vacuum solutions. A physical interpretation of the vacuum solutions of Einstein's equations have been reviewed in papers by Bonnor [64], and Bonnor, Griffiths and MacCallum [65] five years ago. Our article, in particular in emphasizing and describing the role of solutions in giving rise to various concepts, conjectures, and methods of solving problems in general relativity, and in the astrophysical impacts of the solutions, is oriented quite differently, and gives more detail. However, up to some exceptions, like, for example, metrics for an infinite line-mass or plane, which are discussed in [64], and new solutions which have been discovered after the reviews [64,65] appeared as, for example, the solution describing a rigidly rotating thin disk of dust, our choice of solutions is similar to that of [64,65].

In selecting particular topics for a more detailed discussion we will be led primarily by following overlapping aspects: (i) the "commonly acknowledged" significance of a solution – we will concentrate in particular on the Schwarzschild, the Kerr, the Taub-NUT, and plane wave solutions, and (ii) the solutions and their properties that I (and my colleagues) have been directly interested in, such as the Reissner–Nordström metric, vacuum solutions outside rotating disks, or radiative solutions such as cylindrical waves, Robinson–Trautman solutions, and the boost-rotation symmetric solutions. Some of these have also been connected with the interests of Jürgen Ehlers, and we shall indicate whenever we are aware of this fact.

Vacuum cosmological solutions are discussed in less detail than they deserve. A possible excuse – from the point of view of being a relativist, a rather unfair one – could be that a special recent issue of Reviews of Modern Physics (Volume 71, 1999), marking the Centennial of the American Physical Society, contains discussion of the Schwarzschild, the Reissner–Nordström and other black hole solutions, and even remarks on the work of Bondi et al. [66] on radiative solutions, but among the cosmological solutions only the standard models are mentioned. A real reason is the author's lack of space, time, and energy. In the concluding remarks we will try to list at least the most important solutions (not only the Friedmann models!) which have not been "selected" and give references to the literature in which more information can be found.

1.6 The Outline

Since the titles of the following sections characterize the contents rather specifically, we restrict ourselves to only a few explanatory remarks. In our discussion of the Schwarschild metric, after mentioning its role in the solar system, we indicate how the Schwarzschild solution gave rise to such concepts as the event horizon, the trapped surface, and the apparent horizon. We pay more attention to the concept of a bifurcate Killing horizon, because this is usually not treated in textbooks, and in addition, Jürgen Ehlers played a role in its first description in the literature. Another point which has not received much attention is Penrose's nice presentation of evidence against Lorentz-covariant field theoretical approaches to gravity, based on analysis of the causal structure of the Schwarzschild spacetime. Among various astrophysical implications of the Schwarzschild solution we especially note recent suggestions which indicate that we may have evidence of the existence of event horizons, and of a black hole in the centre of our Galaxy.

The main focus in our treatment of the Reissner–Nordström metric is directed to the instability of the Cauchy horizon and its relation to the cosmic censorship conjecture. We also briefly discuss extreme black holes and their role in string theory.

About the same amount of space as that given to the Schwarzschild solution is devoted to the Kerr metric. After explaining a few new concepts the metric inspired, such as locally nonrotating frames and ergoregions, we mention a number of physical processes which can take place in the Kerr background, including the Penrose energy extraction process, and the Blandford–Znajek mechanism. In the section on the astrophysical evidence for a Kerr metric, the main attention is paid to the broad iron line, the character of which, as most recent observations indicate, is best explained by assuming that it originates very close to a maximally rotating black hole. The discussion of recent results on black hole uniqueness and on multi-black hole solutions concludes our exposition of spacetimes representing black holes. In the section on axisymmetric fields and relativistic disks a brief survey of various static solutions is first given, then we concentrate on relativistic disks as sources of the Kerr metric and other stationary fields; in particular, we summarize briefly the recent work on uniformly rotating disks.

An intriguing case of Taub-NUT space is introduced by a new constructive derivation of the solution. Various pathological features of this space are then briefly listed.

Going over to radiative spacetimes, we analyze in some detail plane waves – also in the light of the thorough study by Ehlers and Kundt [53]. Some new developments are then noted, in particular, impulsive waves generated by boosting various "particles", their symmetries, and recent use of the Colombeau algebra of generalized functions in the analyses of impulsive waves. A fairly detailed discussion is devoted to various effects connected with colliding plane waves.

In our treatment of cylindrical waves we concentrate in particular on two issues: on the proof that these waves provide explicitly given spacetimes, which admit a smooth global null infinity, even for strong initial data within a $(2+1)$-dimensional framework; and on the role that cylindrical waves have played in the first construction of a midisuperspace model in quantum gravity. Various other developments concerning cylindrical waves are then summarized only telegraphically.

A short section on Robinson–Trautman solutions points out how these solutions with a nonvanishing cosmological constant can be used to give an exact demonstration of the cosmic no-hair conjecture under the presence of gravitational radiation, and also of the existence of an event horizon which is smooth but not analytic.

As the last class of radiative spacetimes we analyze the boost-rotation symmetric solutions representing uniformly accelerated objects. They play a unique role among radiative spacetimes since they are asymptotically flat, in the sense that they admit global smooth sections of null infinity. And as the only known radiative solutions describing finite sources they can provide expressions for the Bondi mass, the news function, or the radiation patterns in explicit forms. They have also been used as test-beds in numerical relativity, and as the model spacetimes describing the production of black hole pairs in strong fields.

Vacuum cosmological solutions such as the vacuum Bianchi models and Gowdy solutions are mentioned, and their significance in the development of general relativity is indicated in the last section. Special attention is paid to their role in understanding the behaviour of a general model near an initial singularity.

In the concluding remarks, several important, in particular *non*-vacuum solutions, which have not been included in the main body of the paper, are at least listed, together with some relevant references. A few remarks on the possible future role of exact solutions ends the article.

Although we give over 360 references in the bibliography, we do not at all pretend to give all relevant citations. When discussing more basic facts and concepts, we quote primarily textbooks and monographs. Only when mentioning more recent developments do we refer to journals. The complete titles of all listed references will hopefully offer the reader a more complete idea of the role the explicit solutions have played on the relativistic stage and in the astrophysical sky.

2 The Schwarzschild Solution

In his thorough "Survey of General Relativity Theory" [67], Jürgen Ehlers begins with an empirical motivation of the theory, goes in depth and detail through his favourite topics such as the axiomatic approach, kinetic theory, geometrical optics, approximation methods, and only in the last section turns to spherically symmetric spacetimes. As T. S. Eliot says, "to make an end is to make a beginning – the end is where we start from", and so here we start with a few remarks on spherical symmetry.

2.1 Spherically Symmetric Spacetimes

In the early days of general relativity spherical symmetry was introduced in an intuitive manner. It is because of the existence of exact solutions which are singular at their centres (such as the Schwarzschild or the Reissner–Nordström solutions), and a realization that spherically symmetric, topologically non-trivial smooth spacetimes without any centre may exist [68], that today the group-theoretical definition of spherical symmetry is preferred (for a detailed analysis, see e.g. [19,26,67]).

Following Ehlers [67], we define a spacetime $(\mathcal{M}, g_{\alpha\beta})$ to be spherically symmetric if the rotation group SO_3 acts on $(\mathcal{M}, g_{\alpha\beta})$ as an isometry group with simply connected, complete, spacelike, 2-dimensional orbits. One can then prove the theorem [67,69] that a spherically symmetric spacetime is the direct product $\mathcal{M} = S^2 \times N$, where S^2 is the 2-sphere manifold with the standard metric g_S on the unit sphere; and N is a 2-dimensional manifold with a Lorentzian (indefinite) metric g_N, and with a scalar r such that the complete spacetime metric $g_{\alpha\beta}$ is "conformally decomposable", i.e. $r^{-2}g_{\alpha\beta}$ is the direct sum of the 2-dimensional parts g_N and g_S. Leaving further technicalities aside (see e.g. [26,67,69]) we write down the final spherically symmetric line element in the form

$$ds^2 = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\; d\varphi^2), \tag{1}$$

where (following [67]) we permit $\phi(r,t)$ and $\lambda(r,t)$ to have an imaginary part $i\pi/2$ so that the signs of dt^2 and dr^2 in (1), and thus the role of r and t as space- and time- coordinates may interchange (a lesson learned from the vacuum Schwarzschild solutions – see below). The "curvature coordinate" r is defined invariantly by the area, $4\pi r^2$, of the 2-spheres r = constant, t = constant. There is no a priori relation between r and the proper distance from the centre (if there is one) to the spherical surface.

2.2 The Schwarzschild Metric and Its Role in the Solar System

Starting from the line element (1) and imposing Einstein's *vacuum* field equations, but allowing spacetime to be in general dynamical, we are led uniquely (cf. Birkhoff's theorem discussed e.g. in [18,26]) to the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right), \quad (2)$$

where $M =$ constant has to be interpreted as a mass, as test particle orbits show. The resulting spacetime is static at $r > 2M$ (no spherically symmetric gravitational waves exist), and asymptotically flat at $r \to \infty$.

Undoubtedly, the Schwarzschild solution, describing the exterior gravitational field of an arbitrary – static, oscillating, collapsing or expanding – spherically symmetric body of (Schwarzschild) mass M, is among the most influential solutions of the gravitational field equations, if not of any type of field equations invented in the 20th century. It is the first exact solution of Einstein's equations obtained – by K. Schwarzschild in December 1915, still before Einstein's theory reached its definitive form and, independently, in May 1916, by J. Droste, a Dutch student of H. A. Lorentz (see [70] for comprehensive survey).

However, in its exact form (involving regions near $r \approx 2M$) the metric (2) has not yet been experimentally tested (a more optimistic recent suggestion will be mentioned in Sect. 2.6). When in 1915 Einstein explained the perihelion advance of Mercury, he found and used only an approximate (to second order in the gravitational potential) spherically symmetric solution. In order to find the value of the deflection of light passing close to the surface of the Sun, in his famous 1911 Prague paper, Einstein used just the equivalence principle within his "Prague gravity theory", based on the variable velocity of light. Then, in 1915, he obtained this value to be twice as big in general relativity, when, in addition to the equivalence principle, the curvature of space (determined from (2) to first order in M/r) was taken into account.

Despite the fact that for the purpose of solar-system observations the Schwarzschild metric in the form (2) is, quoting [18], "too accurate", it has played an important role in experimental relativity. Eddington, Robertson and others introduced the method of expanding the Schwarzschild metric at the order beyond Newtonian theory, and then multiplying each post-Newtonian term by a dimensionless parameter which should be determined by experiment. These methods inspired the much more powerful *PPN ("Parametrized post-Newtonian") formalism* which was developed at the end of the 1960s and the beginning of the 1970s for testing general relativity and alternative theories of gravity. It has been very effectively used to compare general relativity with observations (see e.g. [18,71,72] and references therein). In order to gain at least some concrete idea, let us just write down the simplest generalization of (2), namely the metric

$$ds^2 = -\left[1 - \frac{2M}{r} + 2(\beta - \gamma)\frac{M^2}{r^2}\right]dt^2 + \left(1 + 2\gamma\frac{M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) , \qquad (3)$$

which is obtained by expanding the metric (2) in M/r up to one order beyond the Newtonian approximation, and multiplying each post-Newtonian term by dimensionless parameters which distinguish the post-Newtonian limits of different metric theories of gravity, and should be determined experimentally. (In general, one needs not just two but ten PPN parameters [18,71,72].) In Einstein's theory: $\beta = \gamma = 1$. Calculating from metric (3) the advance of the pericentre of a test particle orbiting a central mass M on an ellipse with semi-major axis a and eccentricity e, one finds $\Delta\phi = \frac{1}{3}(2+2\gamma-\beta)6\pi M/[a(1-e^2)]$, whereas the total deflection angle of electromagnetic waves passing close to the surface of the body is $\Delta\psi = 2(1+\gamma)M/r_0$, where r_0 is the radius of closest approach of photons to the central body.

Measurements of the deflection of radio waves and microwaves by the Sun (recently also of radio waves by Jupiter) at present restrict γ to $\frac{1}{2}(1+\gamma) = 1.0001 \pm 0.001$ [71,72]. Planetary radar rangings, mainly to Mercury, give from the perihelion shift measurements the result $(2\gamma+2-\beta)/3 = 1.00 \pm 0.002$, so that $\beta = 1.000 \pm 0.003$, whereas the measurements of periastron advance for the binary pulsar systems such as PSR 1913+16 implied agreement with Einstein's theory to better than about 1% (see e.g. [71,72] for reviews). There are other solar-system experiments verifying the leading orders of the Schwarzschild solution to a high accuracy, such as gravitational redshift, signal retardation, or lunar geodesic precession. A number of advanced space missions have been proposed which could lead to significant improvements in values of the PPN parameters, and even to the measurements of post-post-Newtonian effects [72].

Hence, though in an approximate form, the Schwarzschild solution has had a great impact on *experimental relativity*. In addition, the observational effects of gravity on light propagation in the solar system, and also today routine observations of gravitational lenses in cosmological contexts [73], have significantly increased our confidence in taking seriously similar predictions of general relativity in more extreme conditions.

2.3 Schwarzschild Metric Outside a Collapsing Star

I recall how Roger Penrose, at the beginning of his lecture at the 1974 Erice Summer School on gravitational collapse, placed two figures side by side. The first illustrated schematically the bending of light rays by the Sun (surprisingly, Penrose did not write "Prague 1911" below the figure). I do not remember exactly his second figure but it was similar to Fig. 1 below: the spacetime

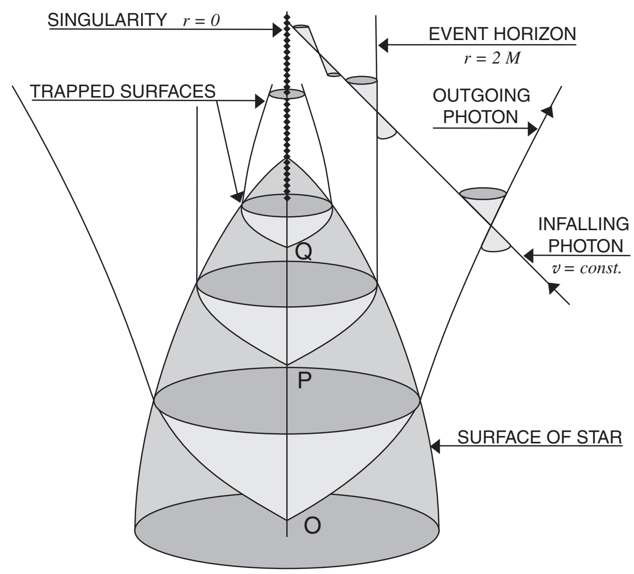

Fig. 1. The gravitational collapse of a spherical star (the interior of the star is shaded). The light cones of the three events, O, P, Q, at the centre of the star, and of the three events outside the star are illustrated. The event horizon, the trapped surfaces, and the singularity formed during the collapse are also shown. Although the singularity appears to lie in a "time direction", from the character of the light cone outside the star but inside the event horizon it is seen that it has a spacelike character.

diagram showing spherical gravitational collapse through the Schwarzschild radius into a spherical black hole.

It is in all modern books on general relativity that the Schwarzschild radius at $R_s = 2M$ is the place where Schwarzschild coordinates t, r are unsuitable, and that metric (2) has a coordinate singularity but not a physical one. One has to introduce other coordinates to extend the Schwarzschild metric through R_s. In order to describe all spacetime outside a collapsing spherical body it is advantageous to use ingoing Eddington–Finkelstein coordinates (v, r, θ, φ) where $v = t + r + 2M \log(r/2M - 1)$. Metric (2) takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right), \tag{4}$$

$(v, \theta, \varphi) =$ constant are ingoing radial null geodesics. Figure 1, plotted in these coordinates, demonstrates well several basic concepts and facts which were introduced and learned after the end of 1950s when a more complete understanding of the Schwarzschild solution was gradually achieved. The metric (4) holds only outside the star, there will be another metric in its interior, for example the Oppenheimer–Snyder collapsing dust solution (i.e. a portion of a collapsing Friedmann universe), but the precise form of the interior solution is not important at the moment. Consider a series of flashes of light emitted from the centre of the star at events $O, P,\ Q$ (see Fig. 1) and assume that the stellar material is transparent. As the Sun has a focusing effect on the light rays, so does matter during collapse. As the matter density becomes higher and higher, the focusing effect increases. At event P a special wavefront will start to propagate, the rays of which will emerge from the surface of the star with zero divergence, i.e. the null vector $k^\alpha = dx^\alpha/dw,\ w$ being an affine parameter, tangent to null geodesics, satisfies $k^\alpha_{;\alpha} = 0$. The wavefront then "stays" at the hypersurface $r = 2M$ in metric (4), and the area of its 2-dimensional cross-section remains constant. The null hypersurface representing the history of this critical wavefront is the (future) *event horizon*. Note that the light cones turn more and more inwards as the event horizon is approached. They become tangential to the horizon in such a way that radial outgoing photons stay at $r = 2M$ whereas ingoing photons fall inwards, and will eventually reach the curvature singularity at $r = 0$. As Fig. 1 indicates, wavefronts emitted still later than the critical one, as for example that emitted from event Q, will be focused so strongly that their rays will start to converge, and will form (closed) *trapped surfaces*. The light cones at trapped surfaces are so turned inwards that both ingoing and outgoing radial rays converge, and their area decreases.

Consider a family of spacelike hypersurfaces $\Sigma(\tau)$ foliating spacetime (τ is a time coordinate, e.g. $v - r$). The boundary of the region of $\Sigma(\tau)$ which contains trapped surfaces lying in $\Sigma(\tau)$ is called the *apparent horizon* in $\Sigma(\tau)$.

In general, the apparent horizon is different from the intersection of the event horizon with $\Sigma(\tau)$, as a nice simple example (based again on an exact solution) due to Hawking [74] shows. Assume that after the spherical collapse of a star a spherical thin shell of mass m surrounding the star collapses and eventually crashes at the singularity at $r = 0$ (Fig. 2). In the vacuum region inside the shell there is the Schwarzschild metric (4) with mass M, and outside the shell with mass $M + m$. Hence the apparent horizon on $\Sigma(\tau_1)$ will be at $r = 2M$ and will remain there until $\Sigma(\tau_2)$ when it discontinuously jumps to $r = 2(M + m)$. One can determine the apparent horizon on a given hypersurface. In order to find the event horizon one has to know the whole spacetime solution. The future event horizon separates events which are visible from future infinity, from those which are not, and thus forms the boundary of a *black hole*.

From the above example of a shell collapsing onto a Schwarzschild black hole we can also learn about the "teleological" nature of the horizon: the motion of the horizon depends on what will happen to the horizon in the future (whether a collapsing shell will cross it or not). This *teleological behaviour of the horizon* has later been discovered in a variety of astrophysically realistic situations such as the behaviour of a horizon perturbed by a mass orbiting a black hole (see [75] for enlightening discussions of such effects).

By studying the Schwarzschild solution and spherical collapse it became evident that one has to turn to *global methods* to gain a full understanding of general relativity. The intuition acquired from analyzing the Schwarzschild metric helped crucially in defining and understanding such concepts as the trapped surface, the event horizon, or the apparent horizon in general situations without symmetry. Nowadays these concepts are explained in several advanced textbooks and monographs (e.g. [18,19,26,32,76]).

Following from the example of spherical collapse one is led to ask whether generic gravitational collapses lead to spacetime singularities and whether these are always surrounded by an event horizon. The Penrose-Hawking singularity theorems [19,26] show that singularities do arise under quite generic circumstances (the occurrence of a closed trapped surface is most significant for the appearance of a singularity). The second question is the essence of the cosmic censorship hypothesis. Various exact solutions have played a role in attempts to "prove" or "disprove" this "one of the most important issues" of classical relativity. We shall meet it in several other places later on, in particular in Sect. 3.1. There a more detailed formulation is given.

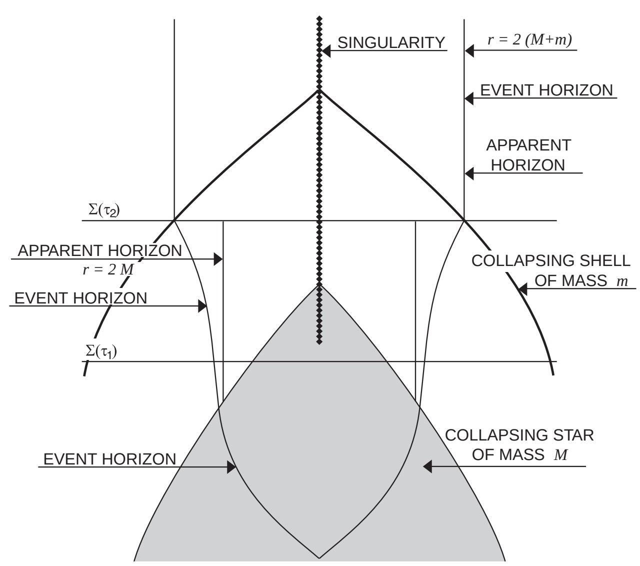

Fig. 2. The "teleological" behaviour of the event horizon during the gravitational collapse of a star, followed by the collapse of a shell. The event horizon moves outwards because it will be crossed by the shell. The apparent horizon moves outwards discontinuously (adapted from [74]).

2.4 The Schwarzschild–Kruskal Spacetime

In the remarks above we considered the Schwarzschild solution outside a static (possibly oscillating, or expanding from $r > 2M$) star, and outside a star collapsing into a black hole. It is not excluded that just these situations will turn out to be physically relevant. Nevertheless, in connection with the Schwarzschild metric it would be heretical not to mention the enormous impact which its maximal vacuum analytic extension into the Schwarzschild–Kruskal spacetime has had. This is today described in detail in many places (see e.g. [18,19,26,76]). We need two sets of the Schwarzschild coordinates to cover the complete spacetime, and we obtain two asymptotically flat spaces, i.e. the spacetime with two ("right" and "left") infinities. The metric in Kruskal coordinates U, V, related to the Schwarzschild r, t (in the regions with $r > 2M$) by

$$\begin{aligned} U &= \pm(r/2M - 1)^{1/2} e^{r/4M} \cosh(t/4M)\,, \\ V &= \pm(r/2M - 1)^{1/2} e^{r/4M} \sinh(t/4M)\,, \end{aligned} \tag{5}$$

takes the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right). \tag{6}$$

The introduction of the Kruskal coordinates which remove the singularity of the Schwarzschild metric (2) at the horizon $r = 2M$ and cover the complete spacetime manifold (every geodesic either hits the singularity or can be continued to the infinite values of its affine parameter), was the most influential example which showed that one has to distinguish carefully between just a coordinate singularity and the real, physical singularity. It also helped us to realize that the definition of a singularity itself is a subtle issue in which the concept of geodesic completeness plays a significant role (see [77] for a recent analysis of spacetime singularities).

The character of the Schwarzschild–Kruskal spacetime is best seen in the Penrose diagram given in Fig. 3, in which the spacetime is compactified by a suitable conformal rescaling of the metric. Both right and left infinities are represented, and the causal structure is well illustrated because worldlines of radial light signals (radial null geodesics) are 45-degree lines in the diagram. In particular the black hole region *II* and a "newly emerged" (as a consequence of the analytical continuation) *white hole* region *IV* (with the white-hole singularity at $r = 0$) are exhibited. For more detailed analyses of the Penrose diagram of the Schwarzschild–Kruskal spacetime the reader is referred to e.g. [18,19,26,76]. Here we wish to turn in some detail to two very important concepts in black hole theory which were first understood by the analytic extension of the Schwarzschild solution, and which are not often treated in standard textbooks. These are the concepts of the *bifurcate horizon* and of the *horizon surface gravity*. Jürgen Ehlers played a somewhat indirect, but important and noble part in their introduction into literature.

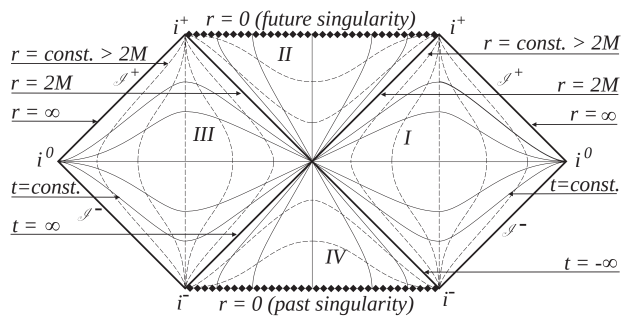

Fig. 3. The Penrose diagram of the compactified Schwarzschild–Kruskal spacetime. Radial null geodesics are 45-degrees lines. Timelike geodesics reach the future (or past) timelike infinities i^+ (or i^-), null geodesics reach the future (or past) null infinities $\mathcal{J}^+$ (or $\mathcal{J}^-$) and spacelike geodesics lead to spatial infinities i^0. (Notice that at i^0 the lines t = constant are tangent to each other – this is often not taken into account in the literature – see e.g. [26,30].)

These concepts were the main subject of the last work of Robert Boyer who became one of the victims of a mass murder on August 1, 1966, in Austin, Texas. Jürgen Ehlers was authorized by Mrs. Boyer to look through the scientific papers of her husband, and together with John Stachel, prepared posthumously the paper [78] from R. Boyer's notes. Ehlers inserted his own discussions, generalized the main theorem on bifurcate horizons, but the paper [78] was published with R. Boyer as the only author.

In the Schwarzschild spacetime there exists the timelike Killing vector, $\partial/\partial t$, which when analytically extended into all Schwarzschild–Kruskal manifold, becomes null at the event horizon $r = 2M$, and is spacelike in the regions II and IV with $r < 2M$. In Kruskal coordinates it is given by

$$k^\alpha = \left(k^V = U/4M,\ \ k^U = V/4M,\ \ k^\theta = 0,\ \ k^\varphi = 0\right). \qquad (7)$$

Hence it vanishes at all points with $U = V = 0$, $\theta \in [0, \pi], \varphi \in [0, 2\pi)$. These points, forming a spacelike 2-sphere which we denote B (in Schwarzschild coordinates given by $r = 2M$, t = constant), are fixed points of the 1-dimensional group G of isometries generated by k^α (see Fig. 3). At the event horizon the corresponding 1-dimensional orbits are null geodesics, with k^α being a tangent vector. However, since k^α vanishes at B, these orbits are incomplete.

This (and similar observations for other black hole solutions) motivated a general analysis of the bifurcate Killing horizons given in [78]. There it is proven for spacetimes admitting a general Killing vector field ξ^α, which generates a 1-dimensional group of isometries, that (i) a 1-dimensional orbit is a complete geodesic if the gradient of the square ξ^2 vanishes on the orbit, (ii) if a geodesic orbit is incomplete, then it is null and $(\xi^2)_{,\alpha} \neq 0$. In addition, if $\xi^\alpha = dx^\alpha/dv$ (v being the group parameter), the affine parameter along the geodesic is $w = e^{\kappa v}$, where $\kappa =$ constant satisfies

$$(-\xi^2)^{,\alpha} = 2\kappa\xi^\alpha. \tag{8}$$

In the Schwarzschild case, with $\xi^\alpha = k^\alpha = (\partial/\partial t)^\alpha$, and considering the part $V = U$ of the horizon, we get $\kappa = 1/4M$. The relation $w = e^{\kappa v}$ is just the familiar equation $\tilde{V} = e^{v/4M}$, where $\tilde{V} = V + U$ is the Kruskal null coordinate and v is the Eddington–Finkelstein ingoing null coordinate used in (4). (Notice that $\tilde{V}$ is indeed the affine parameter along the null geodesics at the horizon $V = U$.) The quantity κ, first introduced in [78], has become fundamental in modern black hole theory, and also in its generalizations in string theory. It is the well-known *surface gravity* of the black hole horizon.

With $\kappa \neq 0$, the limit points corresponding to $v \to -\infty, w = 0$ are fixed points of G. (Unless the spacetime is incomplete, there exists a continuation of each null geodesic beyond these fixed points to $w < 0$.) One can show that the fixed points form a spacelike 2-dimensional manifold B, given by $U = V = 0$ in the Schwarzschild case; this "bifurcation surface" is a totally geodesic submanifold. By the original definition [79], a Killing horizon is a G invariant null hypersurface N on which $\xi^2 = 0$. (A recent definition [80,81] specifies a Killing horizon to be any union of such hypersurfaces.) If $\kappa \neq 0$, at each point of B there is one null direction orthogonal to B which is not tangent to $\bar{N} = N \cup B$. The null geodesics intersecting B in these directions form another null hypersurface, $\tilde{N}$, which is also a Killing horizon. The union $N \cup \tilde{N}$ is called a *bifurcate Killing horizon* (Fig. 4).

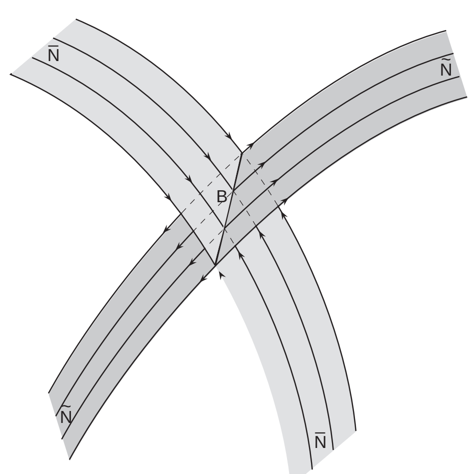

Fig. 4. The bifurcate Killing horizon consisting of two null hypersurfaces $\tilde{N}$ and $\bar{N}$ which intersect in the spacelike 2-dimensional "bifurcation surface" B.

Bifurcate Killing horizons exist also in flat and other curved spacetimes. For example, in the boost-rotation symmetric spacetimes (Sect. 11), null hypersurfaces $z = \pm t$ form the bifurcate Killing horizon corresponding to the boost Killing vector; B, given by $z = t = 0$, is then not compact. (As in the Schwarzschild–Kruskal spacetime, a bifurcate Killing horizon locally divides the spacetime into four wedges.) However, the first motivation for analyzing Killing horizons came from the black hole solutions.

Both Killing horizons and surface gravity play an important role in *black hole thermodynamics* and *quantum field theory on curved backgrounds* [82], in particular in their two principal results: the *Hawking effect* of particle creation by black holes; and the *Unruh effect* showing that a thermal bath of particles will be seen also by a uniformly accelerated observer in flat spacetime when the quantum field is in its vacuum state with respect to inertial observers. Recently, new results were obtained [83] which support the view that a spacetime representing the final state of a black hole formed by collapse has indeed a bifurcate Killing horizon, or the Killing horizon is degenerate ($\kappa = 0$).

2.5 The Schwarzschild Metric as a Case Against Lorentz-Covariant Approaches

There are many other issues on which the Schwarzschild solution has made an impact. Some of astrophysical applications will be very briefly mentioned later on. As the last theoretical point in this section I would like to discuss in some detail the causal structure of the Schwarzschild spacetime including infinity. By analyzing this structure, Penrose [84] presented evidence against various Lorentz (Poincaré)-covariant field theoretical approaches, which regard the physical metric tensor g to be not much different from any other tensor in Minkowski spacetime with flat metric η (see e.g. [85,86]). I thought it appropriate to mention this point here, since Jürgen Ehlers, among others, certainly does not share a field theoretical viewpoint.

The normal procedure of calculating the metric g in these approaches is from a power series expansion of Lorentz-covariant terms (in quantum theory this corresponds to an infinite summation of Feynman diagrams). The derived field propagation has to follow the true null cones of the curved metric g instead of those of η. However, as Penrose shows, in a satisfactory theory the null cones defined by g should not extend outside the null cones defined by η, or "the causality defined by g should not violate the background η-causality". Following [84], let us write this condition as $g < \eta$. Now at first sight we may believe that $g < \eta$ is satisfied in the Schwarzschild field since its effect is to "slow down" the velocity of light (cf. "signal retardation" mentioned in 2.2). However, in the field-theoretical approaches one of the main emphasis is in a consistent formulation of scattering theory. This requires a good behaviour at infinity. But with the Schwarzschild metric, null geodesics with respect to metric g "infinitely deviate" from those with respect to η: for example, the radial outgoing g null geodesics $\theta, \varphi =$ constant, and $u = t - r - 2M \log(r/2M - 1) =$ constant at $r \to \infty$ go "indefinitely far" into the retarded time $t - r$ of η, and hence, do not correspond to outgoing η-null geodesics $t - r =$ constant. One can try to use a different flat metric associated with the Schwarzschild metric g which does not lead to pathological behaviour at infinity, but then it turns out that $g < \eta$ is violated locally. In fact, Penrose [84] proves the theorem, showing that there is an essential incompatibility between the causal structures in the Schwarzschild and Minkowski spacetimes which appears either asymptotically or locally.

This incompatibility is easily understood with the exact Schwarzschild solution, but it is generic, since one is concerned only with the behaviour of the space at large distances from a positive-mass source, i.e. with the causal properties in the neighbourhood of spacelike infinity i^0.

In the present post-Minkowskian approximation methods for the generation of gravitational waves by relativistic sources, a suitable (Bondi-type) coordinate system [66] is constructed at all orders in the far wave zone, which in particular corrects for the logarithmic deviation of the true light cones with respect to the coordinate flat light cones (cf. contribution by L. Blanchet in this volume).

2.6 The Schwarzschild Metric and Astrophysics

In his introductory chapter "General Relativity as a Tool for Astrophysics" for the Seminar in Bad Honnef in 1996 [87], Jürgen Ehlers remarks that "The interest of black holes for astrophysics is obvious... The challenge here is to find observable features that are truly relativistic, related, for example, to horizons, ergoregions... Indications exist, but – as far as I am aware – no firm evidence."

There are many excellent recent reviews on the astrophysical evidence for black holes (see e.g. [88–90]). It is true, that the evidence points towards the presence of dark massive objects – stellar-mass objects in binaries, and supermassive objects in the centres of galaxies – which are associated with deep gravitational potential wells where Newtonian gravity cannot be used, but it does not offer a clear diagnostic of general relativity.

Many investigations of test particle orbits in the strong-gravity regions ($r \leq 10M$) have shown basic differences between the motion in the Schwarzschild metric and the motion in the central field in Newton's theory (e.g. [18,76,91]). For example for $3M < r < 6M$ unstable circular particle orbits exist which are energetically unbound, and thus perturbed particles may escape to infinity; at $r = 3M$ circular photon orbits occur and there are no circular orbits for $r < 3M$. Particles are trapped by a Schwarzschild black hole if they reach the region $r < 3M$.

About ten years ago, the study of the *behaviour of particles and gyroscopes in the Schwarzschild field* revived interest in the "classical" problem of the definition of gravitational, centrifugal, and other inertial "forces" acting on particles and gyros moving on the Schwarzschild or on a more general curved backgrounds, usually axisymmetric and stationary (see e.g. [92,93], and many references therein). One would like to have a split of a covariantly defined quantity (like an acceleration) into non-covariant parts, the physical meaning of which would increase our intuition of relativistic effects in astrophysical problems. If, for example, we adopt the view that the "gravitational force" is velocity-independent, then we find that at the orbits outside the circular photon orbit ($r > 3M$), the centrifugal force is as in classical physics, repulsive, while it becomes attractive inside this orbit, being zero exactly at the orbit.[9]

Relativistic effects will, of course, play a role in many astrophysical situations involving spherical accretion, the structure of accretion disks around compact stars and black holes, their optical appearance etc. They have become an important part of the arsenal of astrophysicists, and they have entered standard literature (see e.g. [95,96]). Though this whole field of science lies beyond the scope of this article, I would like to mention three recent issues which provide us with hope that we may perhaps soon meet the challenge noted in Jürgen Ehlers' remarks made in Bad Honnef in 1996.

The first concerns our Galactic centre. Thanks to new observations of stars in the near infrared band it was possible to detect the transverse motions of stars (for which the radial velocities are also observed) within 0.1 pc in our Galactic centre. The stellar velocities up to 2000 km/sec and their dependence on the radial distance from the centre are consistent with a black hole of mass $2.5 \times 10^6 M_\odot$. In the opinion of some leading astrophysicists, our Galactic centre now provides "the most convincing case for a supermassive hole, with the single exception of NGC 4258" [88]. (In NGC 4258 a disk is observed whose inner edge is orbiting at 1080 km/sec, implying a black hole – "or something more exotic" [88] – with a mass of $3.6 \times 10^7 M_\odot$.) Perhaps we shall be able to observe relativistic effects on the proper motions of stars in our Galactic centre in the not too distant future.

The second issue concerns the fundamental question of whether observations can bring convincing proof of the existence of black hole event horizons. Very recently some astrophysicists [89] claimed that new observations, in particular of X-ray binaries, imply such evidence. The idea is that thin disk accretion cannot explain the spectra of some of X-ray binaries. One has to use a different accretion model, a so called advection-dominated accretion flow model (ADAF) in which most of the gravitational energy released in the infalling gas is carried (advected) with the flow as thermal energy, which falls on the central object. (In thin disks most of this energy is radiated out from the disk.) If the central compact object (for example a neutron star) has a hard surface, the thermal energy stored in the flow is re-radiated after the flow hits the surface. However, some of the X-ray binaries show such low luminosities that a very large fraction of the energy in the flow must be ad-

[9] Curiously enough, Feynman in his 1962-63 lectures on gravitation [94] writes that "inside $r = 2M$ [not $3M$!]... the 'centrifugal force' apparently acts as an attraction rather than a repulsion".

vected through an event horizon into a black hole [89]. Although Rees [88], for example, considers this evidence "gratifyingly consistent with the high-mass objects in binaries being black holes", he believes that it "would still not convince an intelligent sceptic, who could postulate a different theory of strong-field gravity or else that the high-mass compact objects were (for instance) self-gravitating clusters of weakly interacting particles...".

For a sceptical optimistic relativist, the most challenging observational issue related to black holes probably is to find astrophysical evidence for a Kerr metric. We shall come to this point in Sect. 4.3.

The last (but certainly not the least) issue lies more in the future, but eventually should turn out to be most promising. It is connected with both the Numerical Relativity Great Challenge Alliance and the "great challenge" of experimental relativity: to calculate reliable gravitational wave-forms and to detect them. When gravitational waves from stars captured by a supermassive black hole, or from a newly forming supermassive black hole, or, most importantly, from coalescing supermassive holes will be detected and compared with the predictions of the theory, we should learn significant facts about black holes [88,97]. Are these so general remarks entirely inappropriate in the section on the Schwarzschild solution?

One of the most important roles of the Schwarzschild solution in the development of mathematical relativity and especially of relativistic astrophysics stems from its simplicity, in particular from its spherical symmetry. This has enabled us to develop the mathematically beautiful theory of linear perturbations of the Schwarzschild background and employ it in various astrophysically realistic situations (see e.g. [75,76,91], and many references therein). Surprisingly enough, this theory does not only give reliable results in such problems as the calculation of waves emitted by pulsating neutron stars, or waves radiated out from stars falling into a supermassive black hole. Very recently we have learned that one can use perturbation theory of a single Schwarzschild black hole as a "close approximation" to black hole collisions. Towards the end of the collision of two black holes, they will not in fact be two black holes, but will merge into a highly distorted single black hole [98]. When compared with the numerical results on a head-on collision it has been found that this approximation gives predictions for separations Δ as large as $\Delta/M \sim 7$.

3 The Reissner–Nordström Solution

This spherically symmetric solution of the Einstein–Maxwell equations was derived independently[10] by H. Reissner in 1916, H. Weyl in 1917, and G.

[10] In the literature one finds the solution to be repeatedly connected only with the names of Reissner and Nordström, except for the "exact-solutions-book" [61]: there in four places the solution is called as everywhere else, but in one place (p. 257) it is referred to as the "Reissner–Weyl solutions". An enlightening

Nordström in 1918. It represents a spacetime with no matter sources except for a radial electric field, the energy of which has to be included on the right-hand side of the Einstein equations. Since Birkhoff's theorem, mentioned in connection with the Schwarzschild solution in Sect. 2.2, can be generalized to the electrovacuum case, the Reissner–Nordström solution is the unique spherical electrovacuum solution. Similarly to the Schwarzschild solution, it thus describes the exterior gravitational and electromagnetic fields of an arbitrary – static, oscillating, collapsing or expanding – spherically symmetric, charged body of mass M and charge Q. The metric reads

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \quad (9)$$

the electromagnetic field in these spherical coordinates is described by the "classical" expressions for the time component of the electromagnetic potential and the (only non-zero) component of the electromagnetic field tensor:

$$A_t = -\frac{Q}{r}, \quad F_{tr} = -F_{rt} = -\frac{Q}{r^2}. \quad (10)$$

A number of authors have discussed spherically symmetric, static charged dust configurations producing a Reissner–Nordström metric outside, some of them with a hope to construct a "classical model" of a charged elementary particle (see [61] for references). The main influence the metric has exerted on the developments of general relativity, and more recently in supersymmetric and superstring theories (see Sect. 3.2), is however in its analytically extended electrovacuum form when it represents charged, spherical black holes.

3.1 Reissner–Nordström Black Holes and the Question of Cosmic Censorship

The analytic extensions have qualitatively different character in three cases, depending on the relationship between the mass M and the charge Q. In the case $Q^2 > M^2$ (corresponding, for example, to the field outside an electron), the complete electrovacuum spacetime is covered by the coordinates (t, r, θ, φ), $0 < r < \infty$. There is a *naked singularity* (visible from infinity) at $r = 0$ in which the curvature invariants diverge. If $Q^2 < M^2$, the metric (9)

discussion on p. 209 in [61] shows that the solution belongs to a more general "Weyl's electrovacuum class" of electrostatic solutions discovered by Weyl (in 1917) which follow from an Ansatz that there is a functional relationship between the gravitational and electrostatic potentials. As will be noticed also in the case of cylindrical waves in Sect. 9, if "too many" solutions are given in one paper, the name of the author is not likely to survive in the name of an important subclass...

describes a (generic) *Reissner–Nordström black hole*; it becomes singular at two radii:

$$r = r_\pm = M \pm (M^2 - Q^2)^{\frac{1}{2}}. \tag{11}$$

Similarly to the Schwarzschild case, these are only coordinate singularities. Graves and Brill [99] discovered, however, that the analytic extension and the causal structure of the Reissner–Nordström spacetime with $M^2 > Q^2$ is fundamentally different from that of the Schwarzschild spacetime. There are two null hypersurfaces, at $r = r_+$ and $r = r_-$, which are known as the *outer (event) horizon* and the *inner horizon*; the Killing vector $\partial/\partial t$ is null at the horizons, timelike at $r > r_+$ and $r < r_-$, but spacelike at $r_- < r < r_+$. The character of the extended manifold is best seen in the Penrose diagram in Fig. 5, in which the spacetime is compactified by a suitable conformal rescaling of the metric (see, e.g. [18,26,30]). As in the compactified Kruskal-Schwarzschild diagram in Fig. 3, the causal structure is well illustrated because worldlines of radial light signals are 45-degree lines. There are again two infinities illustrated - the right and left - in regions *I* and *III*. However, the maximally extended Reissner–Nordström geometry consists of an *infinite chain of asymptotic regions* connected by "wormholes" between the real singularities (with divergent curvature invariants) at $r = 0$. In Fig. 5, the right and left (past null) infinities in regions I' and III' are still seen - the others are obtained by extending the diagram vertically in both directions.

An important lesson one has learned is that the character of the singularity need not be spacelike as it is in the Schwarzschild case, or with the big bang singularities in standard cosmological models. Indeed, the *singularities* in the Reissner–Nordström geometry are *timelike*: they do not block the way to the future. By solving the geodesic equation one can show that there are test particles which start in "our universe" (region *I*), cross the outer horizon at $r = r_+$ and the inner horizon at $r = r_-$, avoid the singularity and through a "white hole" (the outer horizon between regions IV' and I') emerge into "another universe" I' with its own asymptotically flat region. Such a *gravitational bounce* can occur not only with test particles. The studies of the gravitational collapse of charged spherical shells ([100] and references therein) and of charged dust spheres ([101] and references therein) have shown that a bounce can take place also in fully dynamical cases.[11] The part of Fig. 5 which is "left" from the worldline of the surface of the sphere or the shell is "covered" by the interior of the sphere or flat space inside the shell. As observed in [100], the outcome of the bounce of a shell can be

[11] An intuitive explanation [101] of this bounce is that as the sphere (the shell) contracts, the volume of the exterior region increases, and hence also the total energy in the electric field, which eventually exceeds the energy in the sphere. However, the external plus internal energy does not change during collapse (there are no waves), and so in the neighborhood of a highly contracted charged object, the gravitational field must have a repulsive character corresponding to a negative mass-energy.

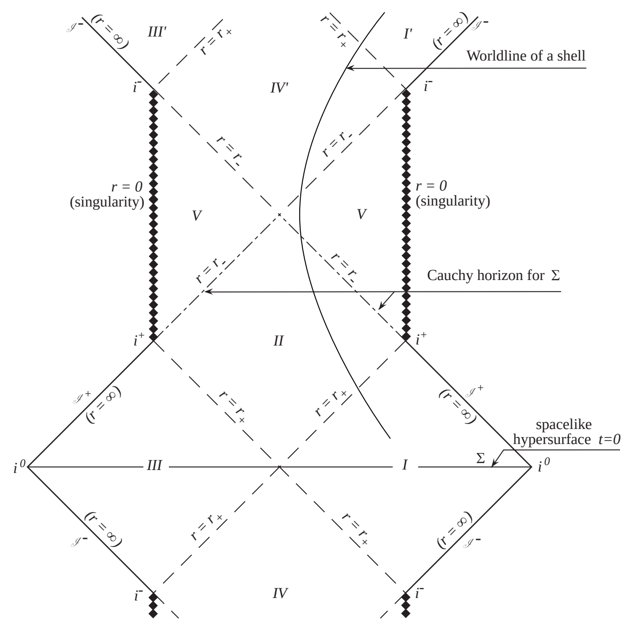

Fig. 5. The compactified Reissner–Nordström spacetime representing a non-extreme black hole consists of an indefinite chain of asymptotic regions ("universes") connected by "wormholes" between timelike singularities. The worldline of a shell collapsing from "universe" I and re-emerging in "universe" I' is indicated. The inner horizon at $r = r_-$ is the Cauchy horizon for spacelike hypersurface Σ. It is unstable and will thus very likely prevent such a process occuring.

different, depending on the value of the shell's total mass, charge and rest mass. The shell may crash into the "right" singularity or it may continue to expand and emerge in region I'. If the rest mass of the shell is negative the collapse may even lead to a naked singularity.

Now even if the shell collapses into a black hole, and after a bounce, emerges in region I', a *locally* naked singularity is present: the timelike singularity at $r = 0$, to the "right" from the wordline of the shell. An observer travelling into the future "between" the shell and the singularity can be surprised by a signal coming from the singularity (see Fig. 5). Penrose's *strong cosmic censorship conjecture* (see e.g. [102]) suggests that this should not happen. In its physical formulation, as given by Wald (see [19] also for the precise formulation), it says that all physically reasonable spacetimes are globally hyperbolic, i.e. apart from a possible initial (big bang-type) singularity, *no singularity is visible to any observer.* It was just the example of the Reissner–Nordström solution (and a similar property of the Kerr solution) which inspired Penrose to formulate the strong cosmic censorship conjecture, in addition to its *weak version* which only requires that from generic nonsingular initial data on a Cauchy hypersurface *no spacetime singularity* develops which is *visible from infinity.* As Penrose [102] puts it, "it seems to be comparatively unimportant whether the observer himself can escape to infinity".

It is evident from Fig. 5 that the Reissner–Nordström spacetime is not globally hyperbolic, i.e. it does not possess a Cauchy hypersurface Σ, the initial data on which (for a test field, say) would determine the development of data in the entire future. If data are given on the spacelike hypersurface Σ "connecting" left and right infinities of regions *III* and *I*, the Cauchy development will predict what happens only in regions *III* and *I* above Σ, and in region *II*, i.e. not beyond the null hypersurfaces (inner horizons) $r = r_-$ between region *II* and regions *V* in the figure. The inner horizons $r = r_-$ represent the *Cauchy horizon* for a typical initial hypersurface like Σ. As noticed above, what is happening at an event in regions *V* is in general influenced not only by data on Σ but also by what is happening at the (locally) naked singularities (which cannot be predicted since the physics at a singularity cannot be controlled).

Penrose was also the first who predicted that the *inner (Cauchy) horizon is unstable* [27]. If this is true, a null singularity, or possibly even a spacelike singularity may arise during a general collapse, so preventing a violation of the strong cosmic censorship conjecture. The instability of the Cauchy horizon can in fact be expected by using first the following simple geometrical-optics argument.

Introduce the ingoing Eddington–Finkelstein null coordinate v by

$$v = t + r^* = t + \int f(r)dr,$$
$$f = 1 - 2M/r + Q^2/r^2, \tag{12}$$

which brings the metric (9) into the form as of equation (4) in the Schwarzschild case. Consider a freely falling observer who is approaching the inner horizon given by $r = r_-, v = \infty$. Denoting the observer's constant specific energy parameter (see e.g. [18]) by $\tilde{E} = -\boldsymbol{U\xi}$, where the Killing vector $\boldsymbol{\xi} = \partial/\partial v$, observer's four-velocity $\boldsymbol{U} = d/d\tau$ (τ - observer's proper time), the geodesic equations imply

$$\dot{r}^2 + f = \tilde{E}^2, \quad \dot{v} = f^{-1}[\tilde{E} - (\tilde{E}^2 - f)^{\frac{1}{2}}], \tag{13}$$

where a dot denotes $d/d\tau$. Between the horizons, $\boldsymbol{\xi}$ is spacelike and $\tilde{E}$ can be negative. Geodesic equations (13) imply $dr/dv \cong \frac{1}{2}f$ for an observer with $\tilde{E} < 0$, approaching $r = r_-$ from region *II*. Expanding f near $r = r_-$, $f \cong f'(r_-)(r - r_-) = -2\kappa(r - r_-)$, where

$$\kappa = (M^2 - Q^2)^{\frac{1}{2}}/r_-^2 \tag{14}$$

is the surface gravity of the inner horizon (as it follows from definition (8)), and integrating, we get f near r_-. From the second equation in (13) we then obtain the "asymptotic formula"

$$\dot{v} \simeq \text{constant} \mid \tilde{E} \mid e^{\kappa\tau} \cong \frac{1}{\kappa\Delta\tau}, \;\; v \to \infty, \Delta\tau \to 0, \tag{15}$$

where $\Delta\tau$ is the amount of proper time the observer needs to reach the inner (Cauchy) horizon.

Imagine now two nearby events A_{out} and B_{out} in the outside world I, for example the emission of two photons from a given fixed $r > r_+$, which are connected by the ingoing null geodesics $v = \text{constant}$ and $v + dv = \text{constant}$ with events A_{in} and B_{in} on the worldline of the observer approaching r_-. The interval of proper time between A_{out} and B_{out} is $d\tau_{out} \sim dv$, whereas (15) implies that the interval of proper time between A_{in} and B_{in}, as measured by the observer approaching r_-, is $d\tau_{in} \sim e^{-\kappa v} dv$. Therefore,

$$\frac{d\tau_{in}}{d\tau_{out}} \sim e^{-\kappa v}, \tag{16}$$

so that as $v \to +\infty$ the events (clocks) in the outside world are measured to proceed increasingly fast by the inside observer approaching the inner horizon. In the limit, when the observer crosses the Cauchy horizon, he sees the whole future history (from some event as A_{out}) of the external universe to "proceed in one flash": $d\tau_{in} \to 0$ with $d\tau_{out} \to \infty$. An intuitive explanation is given by the fact that the observers in region I need infinite proper time to reach $v = \infty$, whereas inside observers only finite proper time. The infalling radiation will thus be unboundedly blue-shifted at the inner horizon which in general will lead to a divergence of the energy density there. This *infinite blueshift at the inner (Cauchy) horizon makes it generally unstable to perturbations ("the blue-sheet instability")*.

There exists an extensive literature analyzing the exciting questions of the *black hole interiors* (see for example [76], the introductory review [103] in the proceedings of the recent workshop devoted entirely to these issues, and other contributions in the proceedings which give also many further references). Some crucial questions are still the subject of much debate. One of the following two approaches to the problem is usually chosen: (i) a linear perturbation analysis of the behaviour of fields at the Cauchy horizon, (ii) the simplified nonlinear, spherically symmetric models of black hole interiors.

In the first approach one considers the evolution of linear perturbations, representing scalar, electromagnetic, or gravitational fields, on the Reissner–Nordström background. Since there is a nonvanishing background electric field, the *electromagnetic and gravitational* perturbations are *coupled*.[12] It is

[12] This leads to various interesting phenomena. For example, the scattering of incident electromagnetic and gravitational waves by the Reissner–Nordström black hole allows for the partial conversion of electromagnetic waves into gravitational waves and vice versa [91]. When studying stationary electromagnetic fields due to sources located outside the Reissner–Nordström black hole, one discovers that

a remarkable fact that "wave equations" for certain gauge-invariant combinations of perturbations can be derived from which all perturbations can eventually be constructed [91,105,106]. In the simplest case of the scalar field Φ on the Reissner–Nordström background, after resolving the field into spherical harmonics and putting $\Phi = r\Psi$, the wave equation has the form [107]

$$\Psi_{,tt} - \Psi_{,r^*r^*} + F_l(r^*)\Psi = 0, \tag{17}$$

where the curvature-induced potential barrier is given by

$$F_l(r^*) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[\frac{2}{r^3}\left(M - \frac{Q^2}{r^2}\right) + \frac{l(l+1)}{r^2}\right], \tag{18}$$

where r is considered to be a function of r^* (cf. (12)). In order to determine the evolution of the field below the outer horizon in a real gravitational collapse, one first concentrates on the evolution of the field outside a collapsing body (star). A nonspherically symmetric scalar test field (generated by a nonspherical distribution of "scalar charge" in the star) serves as a prototype for (small) asymmetries in the external gravitational and electromagnetic fields, which are generated by asymmetries in matter and charge distributions inside the star. Now when a slightly nonspherical star starts to collapse, the perturbations become dynamical and propagate as waves. Their evolution can be determined by solving the wave equation (17). Because of the potential barrier (18) the waves get backscattered and produce slowly decaying radiative tails, as shown in the classical papers by Price [108,109], and generalized to the Reissner–Nordström case in [107,110]. The tails decay in the vicinity of the outer event horizon r_+ (i.e. between regions *I* and *II* in Fig. 5) as $\Psi \sim v^{-2(l+1)}$ for l-pole perturbations.[13] The decaying tails provide the initial data for the "internal problem" – the behaviour of the field near the Cauchy horizon. Calculations show (see [103] and references therein) that near the Cauchy horizon the behaviour of the field remains qualitatively the same: $\Psi(u, v \to \infty) \sim v^{-2(l+1)} + \{\text{slowly varying function of } u\}$, where $u = 2r^* - v$ is constant along outgoing radial null geodesics in region *II*. However, as a consequence of the "exponentially growing blueshift", given by formula (15), the *rate of change of the field diverges as the observer approaches the Cauchy horizon*: $d\Psi/d\tau = (\partial\Psi/\partial x^\alpha)U^\alpha \simeq \Psi_{,v}\,\dot{v} \sim v^{-2l-3}e^{\kappa v}$. Therefore, the measured energy density in the field would also diverge, causing an instability of the Cauchy horizon, which would be expected to create a curvature singularity. More detailed considerations [103] show that the singularity, at least for

closed magnetic field lines not linking any current source may exist, since gravitational perturbations constitute, via the background Maxwell tensor, an effective source [104].

[13] This result is true for a general charged Reissner–Nordström black hole with $Q^2 < M^2$. In the extremal case, $Q^2 = M^2$, the field decays only as $\Psi \sim v^{-(l+2)}$ [107].

large $|u|$, is null and weak (the metric is well-defined, only the Riemann tensor is singular). Any definitive picture of the Cauchy horizon instability can come however only from a fully nonlinear analysis which takes into account the backreaction of spacetime geometry to the growing perturbations.

The second approach to the study of the Cauchy horizon instabilities employs a simplified, *spherically symmetric model which treats the nonlinearities exactly* [111]. The ingoing radiation is modelled by a stream of ingoing charged null dust [112] which is infinitely blueshifted at the inner horizon. There is, however, also an outgoing stream of charged null dust considered to propagate into region *II* towards the inner horizon. The outgoing flux may represent radiation coming from the stellar surface below the outer horizon, as well as a portion of the ingoing radiation which is backscattered in region *II*, and irradiates thus the inner horizon. A detailed analysis based on exact spherically symmetric solutions revealed a remarkable effect: an effective *internal* gravitational-mass parameter of the hole unboundedly increases at the inner (Cauchy) horizon (though the external mass of the hole remains finite). This *"mass inflation phenomenon"* causes the divergence of some curvature scalars at the Cauchy horizon [111]. In reality, the classical laws of general relativity will break down when the curvature reaches Planckian values.

It is outside the scope of this review to discuss further the fascinating issues of black hole interiors. They involve deep questions of classical relativity, of quantum field theory on curved background (as, for example, in discussions of electromagnetic pair production and vacuum polarization effects inside black holes), and they lead us eventually to quantum gravity. We refer again especially to [76] and [103] for more information. Let us only add three further remarks. We mentioned above the work on the inner structure of Reissner–Nordström black holes because this is the most explored (though not closed) area. However, Kerr black holes (Sect. 4) possess also inner horizons and there are many papers concerned with the instabilities of the Kerr Cauchy horizons (see [76,103] for references). Secondly, at the beginning of 1990s, it was shown that the inner horizons of the Reissner–Nordström-de Sitter and Kerr–de Sitter black holes are classically stable in the case when the surface gravity at the inner horizons is smaller than the surface gravity at the cosmological horizon ([103] and references therein, in particular, the review [113]). Penrose [114] even suggested that "it may well be that cosmic censorship requires a zero (or at least a nonpositive) cosmological constant". Very recently, however, three experts in the field [115] have claimed that outgoing modes near to the black hole (outer) event horizon lead to instability for all values of the parameters of Reissner–Nordström-de Sitter black holes. Let me borrow again a statement from Penrose [114]: "My own feelings are left somewhat uncertain by all these considerations".

Finally, a new contribution [116] to the old problem of testing the *weak* cosmic censorship by employing a Reissner–Nordström black hole indicates that one can overcharge a near extreme ($Q^2 \to M^2$) black hole by throwing in a charged particle appropriately. However, the backreaction effects remain to be explored more thoroughly. The question of cosmic censorship thus remains as interesting as ever.

3.2 On Extreme Black Holes, *d*-Dimensional Black Holes, String Theory and "All That"

In the previous section we considered generic Reissner–Nordström black holes with $M^2 > Q^2$. They have outer and inner horizons given by (11), with nonvanishing surface gravities (cf. (14) for the inner horizon). For $M^2 = Q^2$ the two horizons coincide at $r_+ = r_- = M$. Defining the ingoing null coordinate v as in (12), we obtain the ingoing extension of the Reissner–Nordström metric (9) in the form

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dr^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (19)$$

This is the metric of *extreme Reissner–Nordström black holes.* Frequently, these holes are called "degenerate". At the horizon $r = M$, the Killing vector field $\boldsymbol{k} = \partial/\partial v$ obeys the equation $(k^\alpha k_\alpha)_{,\beta} = 0$, so that regarding the general relation (8), the surface gravity $\kappa = 0$, i.e. the *Killing horizon is degenerate.* Using $(k^2)_{,\beta} = 0$ and the Killing equation, we easily deduce that the horizon null generators with tangent $k^\alpha = dx^\alpha/dv$ satisfy the geodesic equation with affine parameter v. The generators have infinite affine length to the past given by $v \to -\infty$ (in contrast to the generators of a bifurcate Killing horizon – cf. Sect. 2.4). This part of the extreme Reissner–Nordström spacetime, given by $r = M, v \to -\infty$, is called an *"internal infinity"*. That there is no "wormhole" joining two asymptotically flat regions and containing a minimal surface 2-sphere like in the non-extreme case can also be seen from the metric in the original Schwarzschild-type coordinates. Considering an embedding diagram $t = \text{constant}, \theta = \pi/2$ in flat Euclidean space one finds that an infinite "tube", or an asymptotically cylindrical region on each $t = \text{constant}$ hypersurface develops. The boundary of the cylindrical region is the internal infinity. It is a compact 2-dimensional spacelike surface. The hypersurfaces $t = \text{constant}$ do not intersect the horizon but only approach such an intersection at the internal infinity. (See [79] for the conformal diagram and a detailed discussion, including analysis of the electrovacuum Robinson–Bertotti universe as the asymptotic limit of the extreme Reissner–Nordström geometry at the internal infinity.)

There has been much interest in the extreme Reissner–Nordström black holes within standard Einstein–Maxwell theory. They admit surprisingly simple solutions of the perturbation equations [117]. Some of them appear to be stable with respect to both classical and quantum processes, and there are attempts to interpret them as solitons [118]. Also, they admit supersymmetry [119].

The quotation marks in the title of this section play a double role: the last two words are just "quoting" from the end of the title of a general review on string theory and supersymmetry prepared for the special March 1999 issue of the Reviews of Modern Physics in honor of the centenary of the American Physical Society by Schwarz and Seiberg [120], but they also should "self-ironically" indicate my ignorance in these issues. In addition, unified theories of the type of string theory appear to be somewhat outside the direct interest of Jürgen Ehlers, who has always emphasized the depth and economy of general relativity because it is a "background-independent" theory: string theories still suffer from the lack of a background-independent formulation. Nevertheless, they are beautiful, consistent, and very challenging constructions, representing one of the most active areas of theoretical physics. Recently, string theory provided an explanation of the Bekenstein-Hawking prediction of the entropy of extreme and nearly extreme black holes. From the point of view of this review we should emphasize that many of the techniques that have been used to obtain exact solutions – mostly exact black hole solutions – in generalized theories like string theory were motivated by classical general relativity. There are also results in classical general relativity which are finding interesting generalizations to string theories, as we shall see with one example below.

Before making a few amateurish comments on new results concerning extreme black holes in string theories, let us point out that in many papers from the last 20 years, *black hole solutions* were studied in spacetimes with the number of *dimensions* either *lower* or *higher* than four. The lower-dimensional cases are usually analyzed as "toy models" for understanding the complicated problems of quantum gravity. The higher-dimensional models are motivated by efforts to find a theory which unifies gravity with the other forces. The most surprising and popular (2+1)-dimensional black hole is the BTZ (Bañados-Teitelboim-Zanelli) black hole in the Einstein theory with a negative cosmological constant. Locally it is isometric to anti de Sitter space but its topology is different. In [121] the properties of (2+1)-dimensional black holes are reviewed. In (1+1)-dimensions one obtains black holes only if one includes at least a simple dilaton scalar field; the motivation for how to do this comes from string theory. In higher dimensions one can find generalizations of all basic black hole solutions in four dimensions [122]. Interesting observations concerning higher-dimensional black holes have been given a few years ago [123]. Perhaps one does not need to quantize gravity in order to remove the singularities of classical relativity. It may well be true that some new classical physics intervenes below Planckian energies. In [123] it is demonstrated that certain singularities of the four-dimensional extreme dilaton black holes can be resolved by passing to a higher-dimensional theory of gravity in which usual spacetime is obtained only below some compactification scale. A useful, brief pedagogical introduction to black holes in unified theories is contained in [76].

One of the most admirable recent results of string theory, which undoubtedly converted some relativists and stimulated many string theorists, has been the derivation of the *exact value of the entropy of extreme and nearly extreme black holes.* I shall just paraphrase a few statements from the March 1999 review for the centenary of the American Physical Society by Horowitz and Teukolsky [124]. There are very special states in string theory called BPS (Bogomol'ny–Prasad–Sommerfield) states which saturate an equality $M \geq c|Q|$, with M being the mass, Q the charge, and c is a fixed constant. The mass of these special states does not get any quantum corrections. The strength of the interactions in string theory is determined by a coupling constant g. One can count BPS states at large Q and small g. By increasing g one increases gravity, and then all of these states become black holes. (The BPS states are supersymmetric and one can thus follow the states from weak to strong coupling.) But they all become identical extreme Reissner–Nordström black holes, because there is only one black hole for given $M = |Q|$. When one counts the number N of BPS states in which an extreme hole can exist, and compares this with the entropy $S_{bh} = \frac{1}{4}A$ of the hole as obtained in black hole thermodynamics [82,125], where A is the area of the event horizon ($A = 4\pi M^2$ for the extreme Reissner–Nordström black hole), one finds exactly the "classical" result: $S_{bh} = \log N!$. The entropy of the classical black hole configuration is given in terms of the number of quantum microstates associated with that configuration, by the basic formula of statistical physics. For more detailed recent reviews, see [126,127], and references therein. Remarkably, the results for the black hole entropy have been obtained also within the canonical quantization of gravity [128]. A comprehensive review [129] of black holes and solitons in string theory appeared very recently.

Allow me to finish this "all that" section with a personal remark. In 1980 L. Dvořák and I found that in the Einstein–Maxwell theory, external magnetic flux lines are expelled from the black hole horizon as the hole becomes an extreme Reissner–Nordström black hole [104]. Hence, extreme black holes exhibit some sort of *"Meissner effect"* known from superconductivity. Last year it was demonstrated by Chamblin, Emparan and Gibbons [130] that this effect occurs also for black hole solutions in string theory and Kaluza–Klein theory. Other extremal solitonic objects in string theory (like p-branes) can also have superconducting properties. Within the Einstein–Maxwell theory this effect was first studied to linear order in magnetic field – we analyzed Reissner–Nordström black holes in the presence of magnetic fields induced by current loops. However, we also used an exact solution due to Ernst [131], describing a charged black hole in a background magnetic field, which asymptotically goes over to a Melvin universe, and found the same effect (see also [132] for the case of the magnetized Kerr–Newman black hole). In [130] the techniques of finding exact solutions of Einstein's field equations are employed within string theory and Kaluza–Klein theory to demonstrate the "Meissner effect" in these theories.

4 The Kerr Metric

The discovery of the Kerr metric in 1963 and the proof of its unique role in the physics of black holes have made an immense impact on the development of general relativity and astrophysics. This can hardly be more eloquently demonstrated than by an emotional text from Chandrasekhar [60]: "In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the Universe..."

In Boyer–Lindquist coordinates the Kerr metric [133] looks as follows (see e.g. [18,30]):

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - 2\frac{2aMr\sin^2\theta}{\Sigma}dt\,d\varphi + \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\mathcal{A}}{\Sigma}\sin^2\theta\,d\varphi^2, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2, \\ \mathcal{A} = \Sigma(r^2+a^2) + 2Mra^2\sin^2\theta, \end{aligned} \tag{21}$$

where M and a are constants.

4.1 Basic Features

The Boyer–Lindquist coordinates follow naturally from the symmetries of the Kerr spacetime. The scalars t and φ can be fixed uniquely (up to additive constants) as parameters varying along the integral curves of (unique) stationary and axial Killing vector fields $\boldsymbol{k}$ and $\boldsymbol{m}$; and the scalars r and θ can be fixed (up to constant factors) as parameters related as closely as possible to the (geometrically preferred) principal null congruences, which in the Kerr spacetime exist (see e.g. [18,30]), and their projections on to the two-dimensional spacelike submanifolds orthogonal to both $\boldsymbol{k}$ and $\boldsymbol{m}$ (see [134] for details). The Boyer–Lindquist coordinates represent the natural generalization of Schwarzschild coordinates. With $a = 0$ the metric (20) reduces to the Schwarzschild metric.

By examining the Kerr metric in the asymptotic region $r \to \infty$, one finds that M represents the mass and $J = Ma$ the angular momentum pointing in the z-direction, so that a is the angular momentum per unit mass. One can arrive at these results by considering, for example, the weak field and slow motion limit, $M/r \ll 1$ and $a/r \ll 1$. The Kerr metric (20) can then be written in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2$$
$$+r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) - \frac{4aM}{r} \sin^2\theta \, d\varphi \, dt, \tag{22}$$

which is the weak field metric generated by a central body with mass M and angular momentum $J = Ma$. A general, rigorous way of interpreting the parameters entering the Kerr metric starts from the *definition of multipole moments* of asymptotically flat, stationary vacuum spacetimes. This is given in physical space by Thorne [135], using his "asymptotically Cartesian and mass centered" coordinate systems, and by Hansen [136], who, generalizing the definition of Geroch for the static case, gives the coordinate independent definition based on the conformal completion of the 3-dimensional manifold of trajectories of a timelike Killing vector $\boldsymbol{k}$. The exact Kerr solution has served as a convenient "test-bed" for such definitions.[14] The mass monopole moment – the mass – is M, the mass dipole moment vanishes in the "mass-centered" coordinates, the quadrupole moment components are $\frac{1}{3}Ma^2$ and $-\frac{2}{3}Ma^2$. The current dipole moment – the angular momentum – is nonvanishing only along the axis of symmetry and is equal to $J = Ma$, while the current quadrupole moment vanishes. All other nonvanishing l-pole moments are proportional to Ma^l [135,136]. Because these specific values of the multipole moments depend on only two parameters, the Kerr solution clearly cannot represent the gravitational field outside a general rotating body. In Sect. 6.2 we indicate how the Kerr metric with general values of M and a can be produced by special disk sources. The fundamental significance of the Kerr spacetime, however, lies in its role as the *only vacuum rotating black hole solution.*

Many texts give excellent and thorough discussions of properties of Kerr black holes from various viewpoints [18,19,26,30,70,75,76,79,91,95,138]. The Kerr metric entered the new edition of "Landau and Lifshitz" [139]. A few years ago, a book devoted entirely to the Kerr geometry appeared [140]. Here we can list only a few basic points.

As with the Reissner–Nordström spacetime, one can make the maximal analytic extension of the Kerr geometry. This, in fact, has much in common with the Reissner–Nordström case. Loosely speaking, the "repulsive" characters of both charge and rotation have somewhat similar manifestations. When $a^2 < M^2$, the metric (20) has coordinate singularities at $\Delta = 0$, i.e. at (cf. (21))

$$r = r_\pm = M \pm \left(M^2 - a^2\right)^{\frac{1}{2}}. \tag{23}$$

[14] For the most complete, rigorous treatment of the asymptotic structure of stationary spacetimes characterized uniquely by multipole moments defined at spatial infinity, see the work by Beig and Simon [137], the article by Beig and Schmidt in this volume, and references therein.

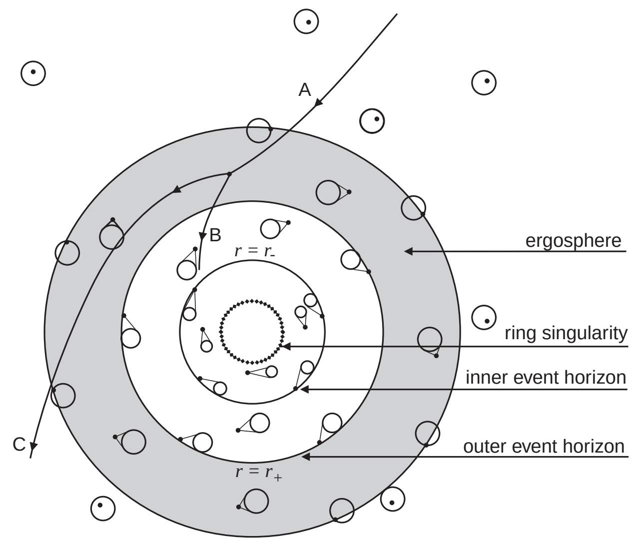

Fig. 6. A schematic picture of a Kerr black hole with two horizons, ergosphere, and the ring singularity; local light wavefronts are also indicated. Particle A, entering the ergosphere from infinity, can split inside the ergosphere into particles B and C in such a manner that C arrives at infinity with a higher energy than particle A came in.

The intrinsic three-dimensional geometry at $r = r_\pm$ reveals that these are null hypersurfaces – the (outer) event horizon and the inner horizon (Fig. 6). As with the Reissner–Nordström metric, the inner horizon – the Cauchy hypersurface – is unstable (see a more detailed discussion in Sect. 3.2 and references there). And as in the Penrose diagram in Fig. 5, one finds infinitely many asymptotically flat regions in the analogous Penrose diagram for the Kerr black hole spacetime.

A crucial difference between the Reissner–Nordström and Kerr geometry is the existence of the *ergosphere* (or, more precisely, ergoregion) in the Kerr case. This is caused by the *dragging of inertial frames* due to a non-vanishing angular momentum. The timelike Killing vector $\boldsymbol{k}$, given in the Boyer–Lindquist coordinates by $\partial/\partial t$, becomes null "sooner", at $r = r_0$, than at the event horizon, $r_0 > r_+$ at $\theta \neq 0, \pi$, as a consequence of this dragging:

$$\begin{aligned} k^\alpha k_\alpha &= -g_{tt} = 1 - 2Mr/\Sigma = 0, \\ r = r_0 &= M + (M^2 - a^2 \cos^2\theta)^{\frac{1}{2}}. \end{aligned} \tag{24}$$

This is the location of the *ergosurface*, the ergoregion being between this surface and the (outer) horizon. In the ergosphere, schematically illustrated in Fig. 6, the "rotating geometry" drags the particles and light (with wavefronts indicated in the figure) so strongly that all physical observers must corotate with the hole, and so rotate with respect to distant observers – "fixed stars" –

at rest in the Boyer–Lindquist coordinates.[15] Static observers, whose worldlines $(r, \theta, \varphi) =$ constant would have $\boldsymbol{k}$ as tangent vectors, cannot exist since $\boldsymbol{k}$ is spacelike in the ergosphere. Indeed, a non-spacelike worldline with r, θ fixed must satisfy the condition

$$g_{tt}dt^2 + g_{\varphi\varphi}d\varphi^2 + 2g_{\varphi t}d\varphi dt \leq 0, \tag{25}$$

in which $g_{tt} = -k^\alpha k_\alpha,\ g_{\varphi\varphi} = m^\alpha m_\alpha,\ g_{\varphi t} = k^\alpha m_\alpha$ are invariants. In the ergosphere, the metric (20), (21) yields $g_{tt} > 0,\ g_{\varphi\varphi} > 0, g_{\varphi t} < 0$, so that $d\varphi/dt > 0$ – an observer moving along a non-spacelike worldline must corotate with the hole. The effect of dragging on the forms of photon escape cones in a general Kerr field (without restriction $a^2 < M^2$) has been numerically studied and carefully illustrated in a number of figures only recently [142].

In order to "compensate" the dragging, the congruence of *"locally nonrotating frames"* (LNRFs), or *"zero-angular momentum observers"* (ZAMOS), has been introduced. These frames have also commonly been used outside relativistic, rapidly rotating stars constructed numerically, but the Kerr metric played an inspiring role (as, after all, in several other issues, such as in understanding the ergoregions, etc.). The four-velocity of these (not freely falling!) observers, given in Boyer–Lindquist coordinates by

$$e^\alpha_{(t)} = \left[(\mathcal{A}/\Sigma\Delta)^{\frac{1}{2}}, 0, 0, 2aMr/(\mathcal{A}\Sigma\Delta)^{\frac{1}{2}}\right], \tag{26}$$

is orthogonal to the hypersurfaces $t =$ constant. The particles falling from rest at infinity with zero total angular momentum fall exactly in the radial direction in the locally nonrotating frames with an orthogonal triad tied to the r, θ, φ coordinate directions (see [143] for the study of the shell of such particles falling on to a Kerr black hole).

Now going down from the ergosphere to the outer horizon, we find that both Killing vectors $\boldsymbol{k}$ and $\boldsymbol{m}$ are tangent to the horizon, and are spacelike there (with $\boldsymbol{k}$ "rotating" with respect to infinity). The null geodesic generators of the horizon are tangent to the null vectors $\boldsymbol{l} = \boldsymbol{k} + \Omega\boldsymbol{m}$, where $\Omega = a/2Mr_+ =$ constant is called the *angular velocity of the hole.* Ω is constant over the horizon so that the *horizon rotates rigidly.* Since $\boldsymbol{l}$ is a Killing vector, the horizon is a Killing horizon (cf. Sect. 2.4).

Another notable difference from the Reissner–Nordström metric is the character of the singularity at $\Sigma = 0$, i.e. at $r = 0,\ \theta = \pi/2$. It is timelike, as in the Reissner–Nordström case, but it is a *ring* singularity (see Fig. 6). In the maximal analytic extension of the Kerr metric one can go through the ring to negative values of the coordinate r and discover *closed timelike*

[15] It is instructive to analyze the somewhat "inverse problem" of gravitational collapse of a slowly rotating dust shell which produces the Kerr metric, linearized in a/M, outside (cf. Eq.(22)), and has flat space inside. Fixed distant stars seen from the centre of such shell appear to rotate due to the dragging of inertial frames, as was discussed in detail recently [141].

lines since $g_{\varphi\varphi} < 0$ there. If the Kerr parameters are such that $a^2 > M^2$, the Kerr metric does not represent a black hole. It describes the gravitational field with a *naked ring singularity*. The Kerr ring singularity has a repulsive character near the rotation axis. It gives particles outward accelerations and collimates them along the rotation axis [144], which might be relevant in the context of the formation and precollimation of cosmic jets. However, the cosmic censorship conjecture is a very plausible, though difficult "to prove" hypothesis, and Kerr naked singularities are unlikely to form in nature. However, the Kerr geometry with $a^2 > M^2$, with a region containing the ring singularity "cut out", can be produced by thin disks; though if they should be composed of physical matter, they cannot be very relativistic (see Sect. 6.2 and references therein).

If $a^2 = M^2$, the Kerr solution represents an *extreme Kerr black hole*, as is the analogous Reissner–Nordström case with $Q^2 = M^2$. The inner and outer horizons then coincide at $r = M$. The horizon is degenerate with infinite affine length. Almost extreme Kerr black holes probably play the most important role in astrophysics (see below). In realistic astrophysical situations accreting matter will very likely have a sufficient amount of angular momentum to turn a Kerr hole to an almost extreme state.

There exists a charged, electrovacuum generalization of the Kerr family found by Newman et al. [145]. The *Kerr–Newman metric* in the Boyer–Lindquist coordinates can be obtained from the Kerr metric (20) if all of the terms $2Mr$ explicitly appearing in (20), (21) are replaced by $2Mr - Q^2$, with Q being the charge. The metric describes *charged, rotating black holes* if $M^2 > a^2 + Q^2$, with two horizons located at $r_\pm = M \pm (M^2 - a^2 - Q^2)^{\frac{1}{2}}$. These become extreme when $M^2 = a^2 + Q^2$, and with $M^2 < a^2 + Q^2$ one obtains naked (ring) singularities. The analytic extension, the presence of ergoregions and the structure of the singularity is similar to the Kerr case.

In addition to the gravitational field, there exists a stationary electromagnetic field which is completely determined by the charge Q and rotation parameter a. The vector potential of this field is given by the 1-form

$$A_\alpha dx^\alpha = -(Qr/\Sigma)(dt - a \sin^2\theta \, d\varphi), \tag{27}$$

so that if $a \neq 0$ the electric field is supplemented by a magnetic field. At large distances ($r \to \infty$) the field corresponds to a *monopole electric field* with charge Q and a *dipole magnetic field* with magnetic moment $\mu = Qa$. Since the gyromagnetic ratio of a charged system with angular momentum J is defined by $\gamma = \mu/J$, one finds the charged-rotating-black hole *gyromagnetic ratio* to satisfy the relation $\gamma = Q/M$, i.e. it is twice as large as that of classical matter, and the same as that of an electron. By examining a black hole with a loop of rotating charged matter around it, the radius of the loop changing from large values to the size of the horizon, it is possible to gain some understanding of this result [146].

4.2 The Physics and Astrophysics Around Rotating Black Holes

In the introduction to their new 770 page monograph on black hole physics, Frolov and Novikov [76] write: "... there are a lot of questions connected with black hole physics and its applications. It is now virtually impossible to write a book where all these problems and questions are discussed in detail. Every month new issues of Physical Review D, Astrophysical Journal, and other physical and astrophysical journals add scores of new publications on the subject of black holes..." Although Frolov and Novikov have also black hole-like solutions in superstring and other theories on their minds, we would not probably be much in error, in particular in the context of astrophysics, if we would claim the same just about Kerr black holes. Hence, first of all, we must refer to the same literature as in the previous Sect. 4.1. A few more references will be given below.

A remarkable fact which stands at the roots of these developments is that the wave equation is separable, and the geodesic equations are integrable in the Kerr geometry. Carter [79], who explicitly demonstrated the separability of the Hamilton-Jacobi equation governing the geodesic motion, has emphasized that one can in fact *derive* the Kerr metric as the simplest nonstatic generalization of the Schwarzschild solution, by requiring the separability of the covariant Klein-Gordon wave equation.[16]

A thorough and comprehensive analysis of the behaviour of freely falling particles in the Kerr field would produce material for a book. We refer to e.g. [18,76,91,134,138,144,147] for fairly detailed accounts and a number of further references. From the point of view of astrophysical applications the following items appear to be most essential: in contrast to the Schwarzschild case, where the stable circular orbits exist only up to $r = 3r_+ = 6M$, in the field of rotating black holes, the stable direct (i.e. with a positive angular momentum) circular orbits in the equatorial plane can reach regions of "deeper potential well". With an extreme Kerr black hole the last stable direct circular orbit occurs at $r = r_+ = M$. (See [147] for a clear discussion of the positions of the innermost stable, innermost bound, and photon orbits as the hole becomes extreme and a long cylindrical throat at the horizon develops.) A *"spin-orbit-coupling"* effect *increases the binding energy of the direct orbits* and decreases the binding energy of the retrograde (with a negative angular momentum) orbits relative to the Schwarzschild values. The binding energy of the last stable direct circular orbit is $\Delta E = 0.0572\mu$ (μ is the particle's proper mass) in the Schwarzschild case, whereas $\Delta E = 0.4235\mu$ for an extreme Kerr hole. A particle slowly spiralling inward due the emission of gravitational waves

[16] Although this is still not a "constructive, analytic derivation of the Kerr metric which would fit its physical meaning", as required by Landau and Lifshitz [139], it is certainly more intuitive than the original derivation by Kerr. On the other hand, despite various hints like the existence of the Killing tensor field (in addition to the Killing vectors) in the Kerr geometry (see e.g. [134]), it does not seem to be clear *why* the Kerr geometry makes it possible to separate these equations.

would radiate the total energy equal to this binding energy; hence much more – 42% of its rest energy – in the Kerr case. The second significant effect is the dragging of the particles moving on orbits outside of the equatorial plane. The *dragging*[17] will make the orbit of a star around a supermassive black hole to precess with angular velocity $\sim 2J/r^3$. The star may go through a disk around the hole, subsequently crossing it at different places [150]. One can also show that as a result of the joint action of the gravomagnetic effect and the viscous forces in an accretion disk, the disk tends to be oriented in the equatorial plane of the central rotating black hole (the "Bardeen–Petterson effect").

The above examples demonstrate specific effects in the Kerr background which very likely play a significant role in astrophysics (see also Sect. 4.3 below). The best known process in the field of a rotating black hole is probably astrophysically unimportant, but is of principal significance in the black hole physics. This is the *Penrose process for extracting energy from rotating black holes.* It is illustrated schematically in Fig. 6: particle A comes from infinity into the ergosphere, splits into two particles, B and C. Whereas C is ejected back to infinity, B falls inside the black hole. The process can be arranged in such a way that particle C comes back to infinity with higher energy than with which particle A was coming in. The gain in the energy is caused by the decrease of rotational energy of the hole. Such process is possible because the Killing vector $\boldsymbol{k}$ becomes spacelike in the ergosphere, so that the (conserved) energy of particle B "as measured at infinity" (see e.g. [18]), $E_B = -k_\alpha p^\alpha_B$, can be negative. Unfortunately, the "explosion" of particle B requires such a big internal energy that the process is not realistic astrophysically.

More general considerations of the interaction of black holes with matter outside have led to the formulation of the four laws of *black hole thermodynamics* [19,76,82,125]. These issues, in particular after the discovery of the *Hawking effect* that black holes emit particles thermally with temperature $T = \kappa\hbar/2\pi kc$ (κ-surface gravity, k-Boltzmann's constant), have been an inspiration in various areas of theoretical physics, going from general relativity and statistical physics, to quantum gravity and string theory (see [125,126] and some remarks and references in Sect. 3.2). The Kerr solution played indeed the most crucial role in these developments. I recall how during my visits to Moscow in the middle of the 1970s Zel'dovich and his colleagues were somewhat regretfully admitting that they were on the edge of discovering the Hawking effect. They realized that an analogue of the Penrose process occurs with the waves (in so called *superradiant scattering*) which get amplified if their energy per unit angular momentum is smaller than the angular velocity

[17] Relativists often consider the effects produced by moving mass currents as "the dragging of inertial frames", but the concept of the gravomagnetic field, or *gravomagnetism* has some advantages, as has been stressed recently [148]. The gravomagnetic viewpoint, however, has also been used in many works in the past – see, e.g. [71,75], and in particular [149] and references therein.

Ω of the horizon. Zel'dovich then suggested that there should be spontaneous emission of particles in the corresponding modes but did not study quantum fields on a nonrotating background (cf. an account of these developments, including the visit of Hawking to Moscow in 1973, by Israel [70]).

Returning back to Earth or, rather, up to heavens, it is not so well known that an astrophysically more realistic example of the Penrose process exists: this is the *Blandford–Znajek mechanism* – see [75,88,151] – in which a magnetic field threading a rotating hole (the field being maintained by external currents in an accretion disk, for example) can extract the hole's rotational energy and convert it into a Poynting flux and electron-positron pairs. A Kerr black hole with angular momentum parallel to an external magnetic field acts (by "unipolar induction") like a rotating conductor in an external field. There will be an induced electric field and a potential difference between the pole and the equator. If these are connected, an electric current will flow and power will be dissipated. In fact, this appears to be until now the most plausible process to explain gigantic relativistic jets emanating from the centres of some of the most active galaxies. The BZ-mechanism has its problems: extremely rotating black holes expel magnetic flux [75,152] – there probably exists a value of the angular momentum $J_0 < J_{max}$ for which the power extracted will be greatest. It is not clear whether the process can be efficiently maintained [153]; and perhaps more importantly, new astrophysical estimates of seed magnetic fields seem to be too low to make the mechanism efficient [154]. The BZ-mechanism will probably attract more attention in the coming years, in particular in view of the recent discovery of two "microquasars" in our own Galaxy, which generate double radio structures and apparent superluminal jets similar to extragalactic strong radio sources [155].

A remarkable achievement of pure mathematical physics, with a great impact on astrophysics, has not only been the discovery of the Kerr solution itself but also the development of *the theory of Kerr metric perturbations* [75,76,91,156]. By employing the Newman–Penrose null tetrad formalism, invented and extensively used in mathematical relativity, in particular in gravitational radiation theory, it has been possible to separate completely all perturbation equations for non-zero spin fields. In particular, a *single* "master equation" – called the *Teukolsky equation* – governs scalar, electromagnetic and gravitational perturbations of a Kerr black hole.[18] If no sources are present on the right-hand side, the equation looks as follows:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$

[18] In the case of a Kerr–Newman black hole, the electromagnetic and gravitational perturbations necessarily couple. Until now, in contrast to the spherical Reissner–Nordström case, a way of how to decouple them has not been discovered.

$$-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi}{\partial t}+(s^2\cot^2\theta-s)\psi=0. \qquad (28)$$

The coordinates are the Boyer–Lindquist coordinates used in (20), Δ is defined in (21), and s is the spin weight of the perturbing field; $s=0,\pm1,\pm2$. The variables in the Teukolsky equation can be separated by decomposing ψ according to

$${}_s\psi_{lm}=(1/\sqrt{2\pi})\,{}_sR_{lm}(r,\omega)\,{}_sS_{lm}(\theta)e^{im\varphi}e^{-i\omega t}, \qquad (29)$$

where ${}_sS_{lm}$ are so called spin weighted spheroidal harmonics. By solving the radial Teukolsky equation for ${}_sR_{lm}$ with appropriate boundary conditions one can find answers to a number of (astro)physical problems of interest like the structure of stationary electromagnetic or gravitational fields due to external sources around a Kerr black hole (e.g. [75,157]), the emission of gravitational waves from particles plunging into the hole (e.g. [76,97,158]), or the scattering of the waves from a rotating black hole (e.g. [76,156] and references therein). At present, the Teukolsky equation is being used to study the formation of a rotating black hole from a head-on collision of two holes of equal mass and spin, initially with small separation, to find the wave forms of gravitational radiation produced in this process [159]. The first studies of second-order perturbations of a Kerr black hole are also appearing [160].

To find all gravitational (metric) perturbations by solving the complete system of equations in the Newman–Penrose formalism is in general a formidable task. As Chandrasekhar's "last observation" at the end of his chapter on gravitational perturbations of the Kerr black hole reads [91]: "The treatment of the perturbations of the Kerr spacetime in this chapter has been prolixious in its complexity. Perhaps, at a later time, the complexity will be unravelled by deeper insights. But mean time, the analysis has led us into a realm of the rococo: splendorous, joyful, and immensely ornate."

4.3 Astrophysical Evidence for a Kerr Metric

Very recently new observations seem to have opened up real possibilities of testing gravity in the strong-field regime. In particular, it appears feasible to distinguish the Kerr metric from the Schwarzschild, i.e. to measure a/M. Our following remarks on these developments are based on the review by M. Rees [88], and in particular, on the very recent survey by A. Fabian [161], an authority on diagnosing relativistic rotation from the character of the emission lines of accretion disks around black holes.

The interest here is not in optical lines since the optical band comes from a volume much larger than the hole. However, the X-rays are produced in the innermost parts of an accretion flow, and should thus display substantial gravitational redshifts as well as Doppler shifts. This only became possible to observe quite recently, when the ASCA X-ray satellite started to operate, and the energy resolution and sensitivity became sufficient to analyze line shapes.

Typically the profile of a line emitted by a disk from gas orbiting around a compact object has a double-horned shape. The disk can be imagined to be composed of thin annuli of orbiting matter – the total line is then the sum of contributions from each annulus. If the disk is not perpendicular to our line of sight its approaching sides will – due to classical Doppler shifts – produce blue peaks, receding sides red peaks. The broadest parts of the total line come from the innermost annuli because the motion there is fastest. In addition, there are relativistic effects: they imply that the emission is beamed in the direction of motion, transverse Doppler shifted, and gravitationally redshifted. As a result, the total line is broad and skewed in a characteristic manner. Such lines are seen in the X-ray spectra of most Seyfert 1 galaxies. In the Seyfert galaxy MCG-6-30-15 the fluorescent iron line was observed to be (red)shifted further to lower energies.[19]

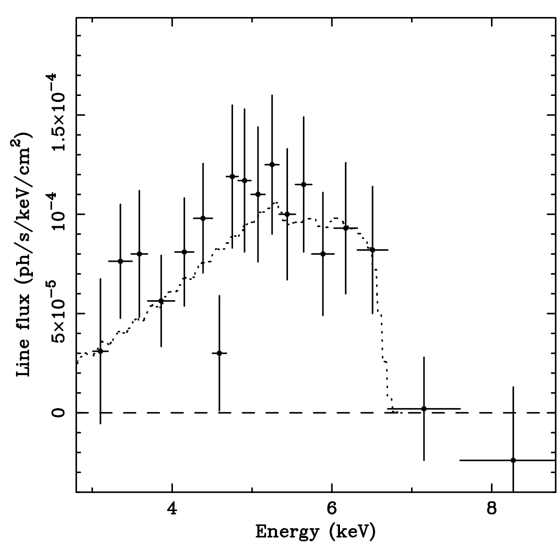

Fig. 7. The broad iron line from MCG-6-30-15. The best-fitting, maximally spinning Kerr black hole model is shown (from Iwasawa, K. et al. (1996), Mon. Not. Roy. Astron. Soc. **282**, 1038).

This suggests that the emission took place below $3R_s = 6M$, i.e. below the innermost stable orbit for a Schwarzschild black hole. In 1996, the line shape was well fitted by the assumption that the line is produced in a close orbit

[19] In Seyfert 1 galaxies hard flares occur which irradiate the accretion disk, and produce a reflection component of continuum peaking at ~ 30 keV and the fluorescent iron line at about 6.5 keV.

around maximally rotating (extreme) Kerr black hole. In 1997, the parameter a/M was quantified as exceeding 0.95. Hence, it has been "tentatively concluded that the line was the first spectroscopic evidence for a Kerr hole" [161].

There are alternative models for a broad skew iron line, including Comptonization by cold electrons, or the emission from irradiated matter falling from the inner edge of the disk around a nonrotating Schwarzschild black hole. It appears, however, that the data speak against these possibilities [161]. In any case, with future X-ray detectors, which will yield count rates orders of magnitude higher than ASCA, the line shapes should reveal in a much greater detail specific features of the Kerr metric.

In addition, other possibilities to determine a/M exist. These include:
(i) Observations of stars in relativistic orbits going through a disk around a supermassive rotating black hole [88,150].
(ii) Characteristic frequencies of the vibrational modes in disks or tori around rotating black holes [88,162].
(iii) The precession of a disk which is tilted with respect to the hole's spin axis. This precession arises because of frame dragging and produces a periodic modulation of the X-ray luminosity.
(iv) Astrophysically most important would be a discovery showing that the properties of cosmic jets depend on the value of a/M. This could indicate that the Blandford–Znajek mechanism (see Sect. 4.2) is really going on. Its likelihood would increase if jets were found with Lorentz factors γ significantly exceeding 10 (see [88] for more details).
(v) Last but not least, future observations of gravitational waves from black hole collisions [97] offer great hopes of a clean observation of a black hole geometry, without astrophysical complications.

It is hard to point out any other exact solution of Einstein's field equations (or of any kind of field equations?) discovered in the second half of the 20th century which has had so many impacts on so many diverse areas of physics, astrophysics, astronomy, and even space science as has had the Kerr metric.

5 Black Hole Uniqueness and Multi-black Hole Solutions

Since black holes can be formed from the collapse of various matter configurations, it is natural to expect that there will be many solutions of Einstein's equations describing black holes. It is expected that the asymptotic final state of a collapse can be represented by a *stationary* spacetime, i.e. one which admits a 1-dimensional group of isometries whose orbits are timelike near infinity. Strong arguments show [26] that the event horizon of a stationary black hole must be a Killing horizon. One of the most remarkable and surprising results of black hole theory are the sequence of theorems showing rigorously that the only stationary solution of the Einstein electrovacuum equations that is asymptotically flat and has a regular event horizon is the Kerr–Newman solution. There is a number of papers on this issue – recent detailed reviews are given in [80,81]. The intuition gained from exact black hole solutions in proving the theorems has been essential.

Roughly speaking, the uniqueness proof consists of the following three parts. First, one demonstrates the "rigidity theorem", which claims that nondegenerate ($\kappa \neq 0$) stationary electrovacuum analytic black holes are either static or axially symmetric. One then establishes that the Reissner–Nordström nondegenerate electrovacuum black holes are all static (nonrotating) nondegenerate black holes in electrovacuum. Finally, one separately proves that the nondegenerate Kerr–Newman black holes represent all nondegenerate axially symmetric stationary electrovacuum black holes.

Although such results were proved more than 10 years ago, recently there has been new progress in the understanding of the global structure of stationary black holes. Again, exact solutions have been inspiring: by gluing together two copies of the Kerr spacetime in a certain way, Chruściel [80] constructed a black hole spacetime which is stationary but not axisymmetric, demonstrating thus that the standard formulation and proof of the rigidity theorem [26] is not correct. (The reason being essentially that when one extends the isometries from a neighbourhood of the horizon by analytic continuation one has no guarantee that the maximal analytic extension is unique.) Chruściel proved "a corrected version of the black hole rigidity theorem"; in the connected case one can prove a uniqueness theorem for static electrovacuum black holes with *degenerate* horizons. The uniqueness theorem for static degenerate black holes which demonstrates that the extreme Reissner–Nordström black hole is the only case, is of importance also in string theory. The most unsatisfactory feature of the rigidity theorem is the assumption of analyticity of the metric in a neighborhood of the event horizon. In this context, Chruściel [80] mentions the case of Robinson–Trautman exact analytic metrics, which can be smoothly but not analytically extended through an event horizon [163]. We shall discuss this issue in somewhat greater detail in Sect. 10.

The black hole uniqueness theorems indicated above are concerned with only single black holes. (Corresponding spacetimes contain an asymptotically flat hypersurface Σ with compact interior and compact connected boundary $\partial\Sigma$ which is located on the event horizon.) Consequently a question naturally arises as to whether one can generalize the theorems to some multi-black hole solutions. In classical physics a solution exists in which a system of n arbitrarily located charged mass points with charges q_i and masses m_i, such that $|q_i| = \sqrt{G}m_i$, is in static equilibrium. In relativity the metric

$$ds^2 = -V^{-2}dt^2 + V^2\left(dx^2 + dy^2 + dz^2\right), \tag{30}$$

with time-independent V satisfying Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,, \tag{31}$$

is a solution of the Einstein–Maxwell equations with the electric field

$$E = \nabla V^{-1}, \tag{32}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. (In standard units $E = \sqrt{G}\nabla V^{-1}$.) The simplest solution of this form is the Majumdar–Papapetrou metric, corresponding to a linear combination of n "monopole sources" with masses $m_i > 0$ and charges $q_i = m_i$, located at arbitrary points $\boldsymbol{x}_i$:

$$V = 1 + \sum_{i=1}^{n} \frac{m_i}{|\,\boldsymbol{x} - \boldsymbol{x}_i\,|}. \tag{33}$$

Hartle and Hawking [164] have shown that every such spacetime can be analytically extended to a spacetime representing n degenerate charged black holes in static equilibrium. The points $\boldsymbol{x} = \boldsymbol{x}_i$ are actually event horizons of area $4\pi m_i^2$. For the case of one black hole, the metric (30) is just the extreme Reissner–Nordström black hole in isotropic coordinates.

A uniqueness theorem for the Majumdar–Papapetrou metrics is not available, although some partial answers are known (see [165,166] for more details). It is believed that these are the only asymptotically flat, regular multi-black hole solutions of the Einstein–Maxwell equations. In fact, such a result would exclude an interesting possibility that *a repulsive gravitational spin-spin interaction* between two (or more) rotating, possibly charged, black holes can overcome their gravitational attraction and thus that there exists in Einstein's theory of gravitation – in contrast to Newton's theory – a stationary solution of the two-body problem.

Among new solutions discovered by modern generating techniques, there are the solutions of Kramer and Neugebauer which represent a nonlinear superposition of Kerr black holes (see [167] for a review). These solutions have been the subject of a number of investigations which have shown that spin-spin repulsion is not strong enough to overcome attraction. In particular, two symmetrically arranged equal black holes cannot be in stationary equilibrium. The situation might change if one considers two Kerr–Newman black holes [168]. Here one has four forces to reckon with: gravitational and electromagnetic Coulomb-type interactions, and gravitational and electromagnetic spin-spin interactions. One can then satisfy the conditions which render the system of two Kerr–Newman black hole free of singularities on the axis, and make the total mass of the system positive. However, there persists a singularity in the plane of symmetry away from the axis [168]. In view of this result we have conjectured that even with electromagnetic forces included one cannot achieve balance for two black holes, except for the exceptional case of two nonrotating extreme (degenerate) Reissner–Nordström black holes. Recently, some new rigorous results concerning the (non)existence of multi-black hole stationary, axisymmetric electrovacuum spacetimes have been obtained [169] (see also [80]), but the "decisive theorem" is still missing.

In connection with the problem of the balance of gravity by a gravitational spin-spin interaction, we should mention that there exists the solution of Dietz and Hoenselaers [170] in which balance of the two rotating "particles" is achieved. However, the "sources" are complicated naked singularities which become Curzon–Chazy "particles" (see Sect. 6.1) if the rotation goes to zero, and it is far from clear whether appropriate physical interior solutions can be constructed.

In 1993, Kastor and Traschen [171] found an interesting family of solutions to the Einstein–Maxwell equations with a non-zero cosmological constant Λ. They describe an arbitrary number of charged black holes in a "background" de Sitter universe. In the limit of $\Lambda = 0$ these solutions become Majumdar–Papapetrou static metrics. In contrast to these metrics, the *cosmological multi-black hole solutions with $\Lambda > 0$ are dynamical.* Remarkably, one can construct solutions which describe coalescing black holes. In some cases cosmic censorship is violated – a naked singularity is formed as a result of the collision [172]. Although these solutions do not have smooth horizons, the singularities are mild, and geodesics can be extended through them. The metric is always at least C^2. Since the solutions are dynamical, one may interpret the non-smoothness of the horizons as a consequence of gravitational and electromagnetic radiation. In this sense, the situation is analogous to the case of the Robinson–Trautman spacetimes discussed in Sect. 10. In five or more dimensions, however, one can construct *static* multi-black hole solutions with $\Lambda = 0$, which do not have smooth horizons [173]. The solutions of Kastor and Traschen also inspired a new and careful analysis [174] of the global structure of the Reissner–Nordström–de Sitter spacetimes characterized by mass, charge, and cosmological constant. The structure is considerably richer than that with $\Lambda = 0$. Most recently, the hoop conjecture (giving the criterion as to whether a black hole forms from a collapsing system) was discussed [175] by analyzing the solution of Kastor and Traschen.

6 On Stationary Axisymmetric Fields and Relativistic Disks

6.1 Static Weyl Metrics

The static axisymmetric vacuum metrics in Weyl's canonical coordinates $\rho \in [0, \infty), z, t \in \mathbb{R}, \varphi \in [0, 2\pi)$ have the form

$$ds^2 = e^{-2U}\left[e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right] - e^{2U} dt^2. \tag{34}$$

The function $U(\rho, z)$ satisfies flat-space Laplace's equation

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{35}$$

The function $k(\rho, z)$ is determined from U by quadrature up to an additive constant. The axis $\rho = 0$ is free of conical singularities at places where $\lim_{\rho \to 0} k = 0$.

The mathematically simplest example is the Curzon–Chazy solution in which $U = -m/\sqrt{\rho^2 + z^2}$ is the Newtonian potential of a spherical point particle. The spacetime, however, is not spherically symmetric. In fact, one of the lessons which one has learned from this solution is the *directional character of the singularity* at $\rho^2 + z^2 = 0$. For example, the limit of the invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ depends on the direction of approach to the singularity. The singularity has a character of a ring through which some timelike geodesics may pass to a Minkowski region [176].

Various studies of the Weyl metrics indicated explicitly how important it is always to check whether a result is not just a consequence of the choice of coordinates. There is the subclass of Weyl metrics generated by the Newtonian potential of a constant density line mass ("rod") with total mass M and (coordinate) length l, which is located along the z-axis with the middle point at the origin. These are Darmois-Zipoy-Vorhees metrics, called also the γ-metrics [64]. The Schwarzschild solution (a spherically symmetric metric!) is a special case in this subclass: it is given by the potential of the rod with $l = 2M$. Clearly, in general there is no correspondence between the geometry of the physical source and the geometry of the Newtonian "source" from the potential of which a Weyl metric is generated.

A survey of the best known Weyl metrics, including some specific solutions describing fields due to circular disks is contained in [64]. More recently, Bičák, Lynden-Bell and Katz [177] have shown that most vacuum static Weyl solutions, including the Curzon and the Darmois-Vorhees-Zipoy solutions, can arise as the metrics of counterrotating relativistic disks (see [177] also for other references on relativistic disks). The simple idea which inspired their work is commonly used in Newtonian *galactic dynamics* [6]: imagine a point mass placed at a distance b below the centre $\rho = 0$ of a plane $z = 0$. This gives a solution of Laplace's equation above the plane. Then consider the potential obtained by reflecting this $z \geq 0$ potential in $z = 0$ so that a symmetrical solution both above and below the plane is obtained. It is continuous but has a discontinuous normal derivative on $z = 0$, the jump in which gives a positive surface density on the plane. In galactic dynamics one considers general line distributions of mass along the negative z-axis and, employing the device described above, one finds the potential-density pairs for general axially symmetric disks. In [178], an infinite number of new static solutions of Einstein's equations were found starting from realistic potentials used to describe flat galaxies, as given recently by Evans and de Zeeuw [179].

Although these disks are Newtonian at large distances, in their central regions interesting relativistic features arise, such as velocities close to the velocity of light, and large redshifts. In a more mathematical context, some particular cases are so far the only explicit examples of spacetimes with a "polyhomogeneous" null infinity (cf. [180] and Sect. 9), and spacetimes with a meaningful, but infinite ADM mass [178]. New Weyl vacuum solutions generated by Newtonian potentials of flat galaxies correspond to both finite and semi-infinite rods, with the line mass densities decreasing according to general power laws. It is an open question what kinds of singularities rods with different density profiles represent.

Very recently, new interesting examples of the static solutions describing self-gravitating disks or rings, and disks or rings around static black holes have been constructed [181–183] and the effects of the fields on freely moving test particles studied [181]. Exact disks with electric currents and magnetic fields have also been considered [184].

Employing the Weyl formalism, one can describe nonrotating *black holes strongly distorted* by the surrounding matter. The influence of the matter can be so strong that it may even cause the horizon topology to be changed from spherical to toroidal (see [75] and references therein).

Finally, we have to mention two solutions in the Weyl class, which were found soon after the birth of general relativity, and have not lost their influence even today. The first, discovered by Bach and Weyl, is assigned by Bonnor [64] as "probably the most perspicacious of all exact solutions in GR". It refers to two Curzon–Chazy "monopoles" on the axis of symmetry. One finds that the metric function k has the property that $\lim_{\rho\to 0} k \neq 0$, so that there is a stress described by a conical singularity between the particles, which holds particles apart. A similar solution can be constructed for the Schwarzschild "particles" (black holes) held apart by a stress. These cases can serve as one of the simplest demonstrations of the difference between the Einstein theory and field theories like the Maxwell theory: it is only in general relativity in which field equations involve also *equations of motion*.

The second "old" solution which has played a very significant role is the metric discovered by Levi–Civita. It belongs to the class of degenerate (type D) static vacuum solutions which form a subclass of the Weyl solutions. In the invariant classification of the degenerate solutions by Ehlers and Kundt [53], this solution is contained in the last, third subclass. That is why Ehlers and Kundt called it the C-metric, and it is so well known today. We shall discuss the C-metric later (Sect. 11) in greater detail since, as it has been learned in the 1970s, it is actually a radiative solution representing uniformly accelerated black holes. What Levi–Civita found and Ehlers and Kundt analyzed is only a portion of spacetime in which the boost Killing vector is timelike, and the coordinates can thus be found there (analogous to the coordinates in a uniformly accelerated frame in special relativity) in which the metric is time-independent.

6.2 Relativistic Disks as Sources of the Kerr Metric and Other Stationary Spacetimes

Thanks to the black hole uniqueness theorems (Sect. 5), the Kerr metric represents the unique solution describing all rotating vacuum black holes. Nevertheless, although the cosmic censorship conjecture, on which the physical relevance of the Kerr metric rests, is a very plausible hypothesis, it remains, as was noted in several places above, one of the central unresolved issues in relativity. It would thus support the significance of the Kerr metric if a physical source were found which produces the Kerr field. The situation would then resemble the case of the spherically symmetric Schwarzschild metric which can represent both a black hole and the external field due to matter.

This has been realized by many workers. The review on the "Sources for the Kerr Metric" [185] written in 1978, contains 71 references, and concludes with: "Destructive statements denying the existence of a material source for the Kerr metric should be rejected until (if ever) they are reasonably justified." The work from 1991 gives "a toroidal source", consisting of "a toroidal shell ..., a disk ... and an annulus of matter interior to the torus" [186]. The masses of the disk and annulus are negative. To summarize in Hermann Bondi's way, the sources suggested for the Kerr metric have not been the easiest materials to buy in the shops ...

The situation is somewhat different in the special case of the *extreme* Kerr metric, where there is a definite relationship between mass and angular momentum. The numerical study [187] of uniformly rotating disks indicated how the extreme Kerr geometry forms around disks in the "ultrarelativistic" limit. These numerical results have been supported by important analytical work (see Sect. 6.3). However, in the case of a general Kerr metric physical sources had not been found before 1993.

A method similar to that of constructing disk sources of static Weyl spacetimes (described in Sect. 5.1) has been shown to work also for axisymmetric, reflection symmetric, and *stationary* spacetimes [188,189]. It is important to realize that although now no metric function solves Laplace's equation as in the static case, we may view the procedure described in Sect. 5.1 as the *identification* of the surface $z = b$ with the surface $z = -b$. The field then remains continuous, but the jump of its normal derivatives induces a matter distribution in the disk which arises due to the identification of the surfaces. What remains to be seen, is whether the material can be "bought in the shops". This idea can be employed for all known asymptotically flat stationary vacuum spacetimes, for example for the Tomimatsu-Sato solutions, for the "rotating" Curzon solution, or for other metrics (cf. [61] for references).

Any stationary axisymmetric vacuum metric can be written in canonical coordinates (t, φ, ρ, z) in the form [61]

$$ds^2 = e^{-2U}\left[e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right] - e^{2U}(dt + Ad\varphi)^2 \ , \qquad (36)$$

where U, k, and A are functions of ρ, z. For the Kerr solution (mass M, specific angular momentum $a \geq 0$), the functions U, k, A are ratios of polynomials when expressed in spheroidal coordinates [61].

Now, identify the "planes" $z = b = \text{constant} > 0$ and $z = -b$ (this identification leads to disks with zero radial pressure). With the Kerr geometry the matching is more complicated than in the static cases, and therefore, one has to turn to Israel's covariant formalism (see [190] for its recent exposition). Using this formalism one is able to link the surface stress-energy tensor of the disk arising from this identification, to the jump of normal extrinsic curvature across the timelike hypersurface given by $z = b$ (with the jump being determined by the discontinuities in the normal derivatives in functions U, k, A).

The procedure leads to physically plausible disks made of two streams of collisionless particles, that circulate in opposite directions with differential velocities [188,189]. Although extending to infinity, the disks have finite mass and exhibit interesting relativistic properties such as high velocities, large redshifts, and dragging effects, including ergoregions. Physical disk sources of Kerr spacetimes with $a^2 > M^2$ can be constructed (though these are "less relativistic"). And the procedure works also for electrovacuum stationary spacetimes. The disks with electric current producing Kerr–Newman spacetimes are described in [191], where the conditions for the existence of (electro)geodesic streams are also discussed.

The power and beauty of the Einstein field equations is again illustrated: the character of exact vacuum fields determines fully the physical characteristics of their sources. In a more sophisticated way, this is seen in the problem of relativistic rigidly (uniformly) rotating disks of dust.

6.3 Uniformly Rotating Disks

The structure of an infinitesimally thin, finite relativistic disk of dust particles which rotate uniformly around a common centre was first explored by J. Bardeen and R. Wagoner 25 years ago [187]. By developing an efficient expansion technique in the quantity $\delta = z_c/(1 + z_c)$, z_c denoting the central redshift, they obtained numerically a fairly complete picture of the behaviour of the disk, even in the ultrarelativistic regime ($\delta \to 1$). In their first letter from 1969 they noted that "there may be some hope of finding an analytic solution". Today such a hope has been substantiated, thanks to the work of G. Neugebauer and R. Meinel (see [192,193] and references therein). The solution had, in fact, to wait until the "soliton-type-solution generating techniques" for nonlinear partial differential equations had been brought over from applied mathematics and other branches of physics to general relativity, starting from the end of the 1970s.

These techniques have been mainly applied only in the vacuum cases so far, but this is precisely what is in this case needed: the structure of the thin disk enters the field equations only through the boundary conditions at $z = 0$, $0 \leq \rho \leq a$ (a is the radius of the disk). The specific procedures which enabled Neugebauer and Meinel to tackle the problem are sophisticated and lengthy. Nevertheless, we wish to mention them telegraphically at least, since they represent the first example of solving *the boundary value problem* for a *rotating* object in Einstein's theory by analytic methods.

In the stationary axisymmetric case, Einstein's vacuum field equations for the metric (18) imply the well-known Ernst equation (see e.g. [61]) – a nonlinear partial differential equation for a complex function f of ρ and z:

$$(\mathrm{Re} f)\left[f_{,\rho\rho} + f_{,zz} + \frac{1}{\rho} f_{,\rho}\right] = f_{,\rho}^2 + f_{,z}^2, \tag{37}$$

where the Ernst potential

$$f(\rho, z) = e^{2U} + ib, \tag{38}$$

with $U(\rho, z)$ being the function entering the metric (36), function $b(\rho, z)$ is a "potential" for $A(\rho, z)$ in (37),

$$A_{,\rho} = \rho\, e^{-4U} b_{,z}, \qquad A_{,z} = -\rho\, e^{-4U} b_{,\rho}, \tag{39}$$

and the last function $k(\rho, z)$ in (37) can be determined from U and b by quadratures.

The Ernst equation can be regarded *as the integrability condition* of a system of *linear* equations for a complex matrix Φ, which is a function of $\rho + iz, \rho - iz$, and of a (new) complex parameter λ. Knowing Φ, one can determine f from Φ at $\lambda = 1$. Now the problem of solving the linear system can be reformulated as the so called *Riemann–Hilbert problem* in complex function theory. (This, very roughly, means the following: let K be a closed curve in the complex plane and $F(K)$ a matrix function given on K; find a matrix function Φ_{in} which is analytic inside L, and Φ_{out} analytic outside K such that $\Phi_{in}\Phi_{out} = F$ on K.) The Riemann–Hilbert problem can be formulated as an integral equation. The hardest problem with which Neugebauer and Meinel were faced was in connecting the specific physical boundary values of f on the disk with the functions entering the Riemann–Hilbert problem (with contour K being determined by the position of the disk in the ρ, z plane), and with the corresponding integral equation. The fact that they succeeded and found the solution of their integral equation is a remarkable achievement in mathematical physics. The gravitational field and various physical characteristics of the disk (e.g. the surface density) are given up to quadratures in terms of ultraelliptic functions [192], which can be numerically evaluated without difficulties. This result, however, may appear as a "lucky case": it does not imply that one will be able to tackle similarly more complicated situations as, for example, thin disks with pressure, with non-uniform rotation, or 3-dimensional rotating bodies such as neutron stars.

Many physical characteristics of uniformly rotating relativistic disks such as their surprisingly high binding energies, the high redshifts of photons emitted from the disks, or the dragging of inertial frames in the vicinity of the disks, were already obtained with remarkable accuracy in [187], as the exact solution now verifies. Here we only wish to demonstrate the fundamental difference between the Newtonian and relativistic case, as it is illustrated

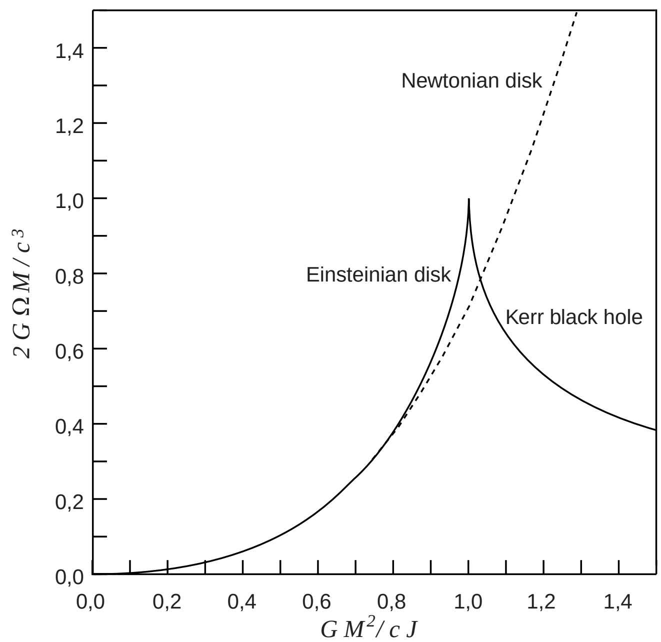

Fig. 8. The general relativistic ("Einsteinian") thin disk of rigidly rotating dust constructed by Neugebauer and Meinel, compared with the analogous disk in Newtonian theory. If the angular momentum is too low, the disk forms a rotating (Kerr) black hole. (From [193].)

in Fig. 8. The rigidly rotating disk of dust of Neugebauer and Meinel represents the relativistic analogue of a classical Maclaurin disk. For the Maclaurin disk, it is easy to show that the (dimensionless) quantities $y = 2G\Omega M/c^3$ and $x = GM^2/cJ$ (M and J are the total mass and angular momentum respectively, and Ω is the angular velocity) are related by $y = (9\pi^2/125)x^3$. For a fixed M the angular velocity $\Omega \sim y$ can be increased arbitrarily, with J being correspondingly decreased. For relativistic disks, however, there is an upper bound on Ω given by $\Omega_{max} = c^3/2GM$, whereas J is restricted by the lower bound $J_{min} = GM^2/c$. With an angular momentum too low, a *rigidly* rotating disk cannot exist. If we "prepare" such a disk, it immediately begins to collapse and forms – assuming the cosmic censorship – a rotating Kerr black hole with $x = GM^2/cJ > 1$. (Notice that the assumption of rigid rotation is here crucial: the differentially rotating disks considered in the preceding section can have an arbitrary value of x.) Since one can define the angular velocity $\Omega(M, J)$ of the horizon of a Kerr hole, one may consider $y(x)$ for black hole states with $x > 1$ (cf. Fig 8). The rigidly rotating disk states and the black hole states just "meet" at $y = x = 1$. In the ultrarelativistic limit the gravitational field outside the disk starts to be unaffected by the detailed structure of the disk – it approaches the field of an extremely rotating Kerr black hole with $x = 1$. Such a result had already been obtained by Bardeen and Wagoner. However, it is only now, with the exact solution available, that it can be investigated with full rigor. It gives indirect evidence that Kerr black holes are really formed in the gravitational collapse of rotating bodies.

As noticed also by Bardeen and Wagoner, in the ultrarelativistic limit the disk itself "becomes buried in the horizon of the extreme Kerr metric, surrounded by its own infinite, non asymptotically flat universe" (see [194] for a recent detailed analysis of the ultrarelativistic limit). Similar phenomena arise also in the case of some spherical solutions of Einstein–Yang–Mills–Higgs equations (cf. [195] and Sect. 13).

7 Taub-NUT Space

The name of this solution of vacuum Einstein's equations fits both to the names of its discoverers (Taub–Newman–Unti–Tamburino) and to its curious properties. Owing to these properties (which induced Misner [196] to consider the solution "as a counterexample to almost anything"), this spacetime has played a significant role in exhibiting the type of effects that can arise in strong gravitational fields.

Taub [197] discovered an empty universe with four global Killing vectors almost half a century ago, during his pioneering study of metrics with several symmetries. By continuing the Taub universe through its horizon one arrives in NUT space. NUT space, however, was only discovered in 1963 by a different method [198]. In fact, it could have been obtained earlier by applying the transformation given in Jürgen Ehlers' dissertation [54] and his talk at the GR2 conference in Royaumont in 1959 [55]. This transformation gives the recipe for obtaining stationary solutions from static ones. How the NUT space can be obtained by applying this transformation to the Schwarzschild metric was demonstrated explicitly by Ehlers at GR4 in London in 1965 [56].

7.1 A New Way to the NUT Metric

Here we shall briefly mention a simple, physically appealing new derivation of the NUT metric given recently by Lynden-Bell and Nouri-Zonoz (LBNZ) [199]. Their work also shows how even uncomplicated solutions may still be of interest in unexpected contexts. LBNZ's inspiration to study the NUT space has in fact come from Newton's Principia! In one of his scholia Newton discusses motion under the standard central force plus a force which is normal to the surface swept out by the radius vector to the body which is describing the non-coplanar path. A simple interesting case is the motion of mass m_0 satisfying the equation

$$m_0 \, \frac{d^2\boldsymbol{r}}{dt^2} = -V^{'}(r)\ \hat{\mathbf{r}} + \frac{m_0}{c}\ \boldsymbol{v} \times \boldsymbol{B}_g, \tag{40}$$

where $\hat{\mathbf{r}} = \boldsymbol{r}/r$,

$$\boldsymbol{B}_g = -Q\ \hat{\mathbf{r}}/r^2,\ Q = \tilde{Q}\ c/m_0, \tag{41}$$

Q and $\tilde{Q}$ are constants, and c is the velocity of light. Here we write c explicitly though $c = 1$, to make the analogy with magnetism. Indeed, $\boldsymbol{B}_g$ is the field of a "gravomagnetic" monopole of strength Q. The classical orbits of particles lie on cones which, if the monopole is absent, flatten into a plane [199].

It was known that NUT space corresponds to the mass with a gravomagnetic monopole, but this was never used in such a physical way as by LBNZ for its derivation. The main point is to start from the well-known split of the stationary metrics as described in Landau and Lifshitz [139] (see [200] for a covariant approach, and the contribution of Beig and Schmidt in the present volume)

$$ds^2 = -e^{-2\nu}(dt - A_i dx^i)^2 + \gamma_{ij} dx^i dx^j, \tag{42}$$

where ν, A_i, γ_{ij} are independent of t. This form is unique up to the choice of time zero: $t' = t + \chi(x^i)$ implies again the metric in the form (42) in (t', x^i), with the "vector potential" undergoing a gauge transformation $A'_i = A_i + \nabla_i \chi$. Writing down the equation of motion of a test particle in metric (42), in analogy with the equation of motion of a charged particle in an electromagnetic field, one is naturally led to define the "gravoelectric" and "gravomagnetic" fields by

$$\boldsymbol{E}_g = \nabla \nu,\ \ \boldsymbol{B}_g = \nabla \times \boldsymbol{A}, \tag{43}$$

where "$\nabla\times$" is with respect to γ_{ij}. Following the problem of §95 in [139] one then rewrites all Einstein's equations in terms of the fields (43), the metric γ_{ij}, and their derivatives. To find the vacuum *spherically symmetric* spatial γ-metric one takes $\gamma_{ij} dx^i dx^j = e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\varphi^2)$, and one assumes $\nu = \nu(r)$, and $B_g^r = -Qe^{-\lambda}/r^2$. The Einstein equations then imply the spacetime metric, which is *not* spherically symmetric, in the form

$$\begin{aligned} ds^2 = -e^{-2\nu}\Big(dt - 2q\,(1+\cos\theta)\,d\varphi\Big)^2 &+ \left(1 - q^2/r^2\right)^{-1} e^{2\nu} dr^2 \\ &+\ r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right), \end{aligned} \tag{44}$$

where $q = Q/2 = \text{constant}$,

$$e^{-2\nu} = 1 - 2r^{-2}\left(q^2 + m\sqrt{r^2 - q^2}\right), \tag{45}$$

and the vector potential $A_\varphi = 2q(1 + \cos\theta)$ satisfies (43). (The factor $(1 - q^2/r^2)$ should be raised to the power -1 in equation (3.22) in [199], as it is clear from (3.20).) Equation (44) is the NUT metric, with r being the curvature coordinate of spheres $r = \text{constant}$. With $q = 0$ the metric (44) becomes the Schwarzschild metric in the standard Schwarzschild coordinates.

More commonly the metric (44) is written in the form

$$ds^2 = -V\left(d\tilde{t} + 4q\sin^2\frac{\theta}{2}\,d\varphi\right)^2 + V^{-1}d\tilde{r}^2 + \left(\tilde{r}^2 + q^2\right)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{46}$$

$$V = 1 - 2\,\frac{m\tilde{r} + q^2}{\tilde{r}^2 + q^2}, \tag{47}$$

which can be obtained from (44) by putting

$$\tilde{r} = \sqrt{r^2 - q^2}, \qquad \tilde{t} = t - 4q\varphi. \tag{48}$$

Recently, Ehlers [201] considered a Newtonian limit of NUT space within his frame theory which encompasses general relativity and Newton–Cartan theory (a slight generalization of Newton's theory). The main purpose of Ehlers' frame theory is to define rigorously what is meant by the statement that a one parameter family of relativistic spacetime models converges to a Newton–Cartan model or, in particular, to a strictly Newtonian model.

The strictly Newtonian limit occurs when the Coriolis angular velocity field $\boldsymbol{\omega}$, related to the connection coefficients Γ^i_{tj} in the Newton–Cartan theory, depends on time only. NUT spacetimes approach a truly Newton–Cartan limit with spatially *non*-constant radial Coriolis field $\omega^{\tilde{r}} = -q/\tilde{r}^2$, which in this limit coincides with the Newtonian gravomagnetic field. As in the analogous classical problem with the equation of motion (40), the geodesics in NUT space lie on cones. This result has been used to study gravitational lensing by gravomagnetic monopoles [199]: they twist the rays that pass them in a characteristic manner, different from that due to rotating objects.

The metrics (44) and (46) appear to have a preferred axis of fixed points of symmetry. This is a false impression since we can switch the axis into any direction by a gauge transformation. For example, the metric (44) has a conical singularity at $\theta = 0$ but is regular at $\theta = \pi$, whereas the metric (46) has a conical singularity at $\theta = \pi$ but is regular at $\theta = 0$. The metrics are connected by the simple gauge transformation, i.e. $t \to \tilde{t} = t - 4q\varphi$. A mass endowed with a gravomagnetic monopole appears as a spherically symmteric object but the spacetime is *not* spherically symmetric according to the definition given in Sect. 2.1. Nevertheless, there exist equivalent coordinate systems in which the axis can be made to point in any direction – just as the axis of the vector potential of a magnetic monopole can be chosen arbitrarily. For further references on interpreting the NUT metric as a gravomagnetic monopole, see [199] and the review by Bonnor [64].

7.2 Taub-NUT Pathologies and Applications

By introducing two coordinate patches, namely the coordinates of metric (44) to cover the south pole ($\theta = \pi$) and those of (46) the north pole ($\theta = 0$), the rotation axis can be made regular. However since φ is identified with period

2π, equation (48) implies that t and $\tilde{t}$ have to be identified with the period $8\pi q$. Then observers with $(\tilde{r}, \theta, \varphi) =$ constant follow *closed timelike lines* if V in (47) is positive, i.e. if $\tilde{r} > \tilde{r}_0 = m + (m^2 + q^2)^{\frac{1}{2}}$. The hypersurface $\tilde{r} = \tilde{r}_0$ is the null hypersurface – horizon, below which lines $(t, \theta, \varphi) =$ constant become spacelike. Because of the periodic identification of t and $\tilde{t}$, the hypersurfaces of constant $\tilde{r}$ change the topology from $S^2 \times R^1$ to S^3, on which $t/2q,\ \theta,\ \varphi$ become Euler angle coordinates.

The region with $V < 0$ is the Taub universe: it has homogeneous but non-isotropic space sections $\tilde{r} =$ constant. The coordinate $\tilde{r}$, allowed to run from $-\infty$ to $+\infty$, is a timelike coordinate, and is naturally denoted by t in the Taub region.

In addition to the closed timelike lines in the NUT region there are further intriguing pathologies exhibited by the Taub-NUT solutions. Here we just list some of them and refer to the relevant literature [26,27,196]. The Taub region is globally hyperbolic: its entire future and past history can be determined from conditions given on a spacelike Cauchy hypersurface. However, this is not the case with the whole Taub-NUT spacetime. As in the Reissner–Nordström spacetimes (Sect. 3), there are Cauchy horizons $H^{\pm}(\Sigma)$ of a particular spacelike section Σ of maximal proper volume lying between the globally hyperbolic Taub regions and the causality violating NUT regions. $H^{\pm}(\Sigma)$ are smooth, compact null hypersurfaces diffeomorphic to S^3 – the generators of such null surfaces are *closed null geodesics*. The Taub region is limited between $t_- \leq t \leq t_+$, where $t_\pm$ are roots of V in equation (47) (with the interchange $t \leftrightarrow \tilde{r}$). This region is compact but there are timelike and null geodesics which remain within it and are not complete. (See [27] for a nice picture of these geodesics spiralling around and approaching $H_+(\Sigma)$ asymptotically.) This pathological behaviour of "the incomplete geodesics imprisoned in a compact neighbourhood of the horizon" was inspirational in the definition of singularities [77] – one meets here the example in which the geodesic incompleteness is not necessarily connected with strong gravitational fields. It can be shown, however, that after an addition of even the slightest amount of matter this pathological behaviour will not take place – true singularities arise.

This enables one to consider the time between t_- and t_+ as the lifetime of the Taub universe. Wheeler [202] constructed a specific case of the Taub universe which will live as long as a typical Friedmann closed dust model ($\sim 10^{10}$ years) but will have a volume at maximum expansion smaller by a factor of 5×10^{10}. This example thus appears to be a difficulty for the anthropic principle.

Taub space seems also to be the only known example giving the possibility of making inequivalent NUT-like extensions which lead to a non-Hausdorff spacetime manifold [26,200,203].

The Taub-NUT solution plays an important role in cosmology and quantum gravity. Here we wish to note yet two other recent applications of this space. About ten years ago, interest was revived in closed timelike lines, time machines, and wormholes. One of the leaders in this activity, Kip Thorne, explains in [204] in a pedagogical way the main recent results on closed timelike curves and wormholes by using "Misner space" – Minkowski spacetime with identification under a boost, which Misner introduced as a simplified version of Taub-NUT space.

The second application of the Taub-NUT space is still more remarkable – it plays an important role outside general relativity. The asymptotic motion of monopoles in (super-)Yang–Mills theories corresponds to the geodesic motion in *Euclidean* Taub-NUT space [205]. Euclidean Taub-NUT spaces have been discussed in many further works on monopoles in gauge theories. One of the latest of these works [206], on the exact T-duality (which relates string theories compactified on large and small tori) between Taub-NUT spaces and so called "calorons" (instantons at finite temperature defined on $R^3 \times S^1$), gives also references to previous contributions.

8 Plane Waves and Their Collisions

8.1 Plane-Fronted Waves

The history of gravitational plane waves had began already by 1923 with the paper on spaces conformal to flat space by Brinkmann. Interest in these waves was revived in 1937 by Rosen, and in the late 1950s by Bondi, Pirani and Robinson, Holy, and Peres (see [53,61] for references). A comprehensive geometrical approach to these spacetimes soon followed in the classical treatise by Jordan, Ehlers and Kundt [57], and in the subsequent well-known chapter by Ehlers and Kundt [53]. As an application of various newly developed methods to analyze gravitational radiation, and as a simple background to test various physical theories, plane waves have proved to be a useful and stimulating arena which offers interesting contests even today, as we shall indicate by a few examples in Sect. 8.2.

Consider a congruence of null geodesics (rays) $x^\alpha(v)$ such that $dx^\alpha/dv = k^\alpha, k_\alpha k^\alpha = 0, k_{\alpha;\beta}k^\beta = 0$, v being an affine parameter. In general a geodesic congruence is characterized by its expansion θ, shear $|\sigma|$ and twist ω given by (see e.g. [19])

$$\theta = \frac{1}{2}k^\alpha_{;\alpha}, \qquad |\sigma| = \sqrt{\frac{1}{2}k_{(\alpha;\beta)}k^{\alpha;\beta} - \theta^2}, \tag{49}$$

$$\omega = \sqrt{\frac{1}{2}k_{[\alpha;\beta]}k^{\alpha;\beta}}. \tag{50}$$

According to the definition given by Ehlers and Kundt [53] a vacuum spacetime is a *"plane-fronted gravitational wave"* if it contains a shearfree $|\sigma| = 0$ geodesic null congruence, and if it admits "plane wave surfaces" (spacelike 2-surfaces orthogonal to k^α). This definition is inspired by plane electromagnetic waves in Maxwell's theory. Electromagnetic plane waves are *null fields* ("pure radiation fields"): there exists a null vector k^α, tangent to the rays, which is transverse to the electromagnetic field $F_{\alpha\beta}$, i.e. $F_{\alpha\beta}k^\beta = 0$, $F^*_{\alpha\beta}k^\beta = 0$, and the quadratic invariants of which vanish, $F_{\alpha\beta}F^{\alpha\beta} = 0 = F_{\alpha\beta}F^{*\alpha\beta}$, where $F^*_{\alpha\beta}$ is dual to $F_{\alpha\beta}$. Analogously, Petrov type N gravitational fields (see [61]) are null fields with rays tangent to k^α (the "quadruple Debever-Penrose null vector"), and with the Riemann tensor satisfying $R_{\alpha\beta\gamma\delta}k^\delta = 0$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 0$, and $R^{\alpha\beta\gamma\delta}R^*_{\alpha\beta\gamma\delta} = 0$.[20] Then the Bianchi identities and the Kundt-Thompson theorem for type N solutions in vacuum spacetimes (also more generally, under the presence of a nonvanishing cosmological constant) imply that the shear of k^α must necessarily vanish (see [61,207]). Because of the existence of plane wave surfaces, the expansion (49) and twist (50) must vanish as well, $\theta = \omega = 0$. In this way we arrive at the Kundt class of nonexpanding, shearfree and twistfree gravitational waves [61]. The best known subclass of these waves are *"plane-fronted gravitational waves with parallel rays" (pp-waves)* which are defined by the condition that the null vector k^α is covariantly constant, $k_{\alpha;\beta} = 0$. Thus, automatically k^α is the Killing vector, and $\theta = |\sigma| = \omega = 0$.

Ehlers and Kundt [53] give several equivalent characterizations of the pp-waves and show, following their previous work [57], that in suitable null coordinates with a null coordinate u such that $k_\alpha = u_{,\alpha}$ and $k^\alpha = (\partial/\partial v)^\alpha$, the metric has the form

$$ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - 2H(u,\zeta,\bar{\zeta})du^2, \tag{51}$$

where H is a real function dependent on u, and on the complex coordinate ζ which spans the wave 2-surfaces $u =$ constant, $v =$ constant. These 2-surfaces with Euclidean geometry are thus contained in the wave hypersurfaces $u =$ constant and cut the rays given by $(u, \zeta) =$ constant, v changing. The vacuum field equations imply 2-dimensional Laplace's equation

$$H_{,\zeta\bar{\zeta}} = 0, \tag{52}$$

so that we can write

$$2H = f(u,\zeta) + \bar{f}(u,\bar{\zeta}), \tag{53}$$

where $f(u, \zeta)$ is an arbitrary function of u, analytic in ζ. To characterize the curvature in the waves and their effect on test particles it is convenient to introduce the null complex tetrad, such that at each spacetime point, together with the preferred null vector k^α, we have a null vector $l^\alpha, l^\alpha k_\alpha = -1$, and complex spacelike vector m^α satisfying $m_\alpha \bar{m}^\alpha = 1, m_\alpha k^\alpha = m_\alpha l^\alpha = 0$. For

[20] This algebraic (local) analogy between null fields exists also between electromagnetic and gravitational shocks (possible discontinuities across null hypersurfaces), and in the asymptotic behaviour of fields at large distances from sources (the "peeling property" – see e.g. [19,27]).

the metric (51) the only nonvanishing projection of the Weyl (in the vacuum case, the Riemann) tensor onto this tetrad is the (Newman–Penrose) scalar

$$\Psi_4 = C_{\alpha\beta\gamma\delta} l^\alpha \bar{m}^\beta l^\gamma m^\delta = H_{,\bar{\zeta}\bar{\zeta}} \ , \tag{54}$$

which denotes a *transverse* component of the wave propagating in the k^α direction. As shown by Ehlers and Kundt [53] (see also e.g. [61]), though in a somewhat different notation, we can use again an analogy with the electromagnetic field – described for an analogous plane wave by the transverse component $\phi_2 = F_{\alpha\beta}\bar{m}^\alpha l^\beta$ – and write $\Psi_4 = A\, e^{i\Theta}$, where real $A > 0$ is considered as the *amplitude* of the wave, and at each spacetime point associate Θ with the plane of polarization. Vacuum pp-waves with $\Theta =$ constant are called *linearly polarized*.

Consider a free test particle (observer) with 4-velocity $\mathbf{u}$ and a neighbouring free test particle displaced by a "connecting" vector $Z^\alpha(\tau)$. Introducing then the physical frame $\mathbf{e}_{(i)}$ which is connected with the observer such that $\mathbf{e}_{(0)} = \mathbf{u}$ and $\mathbf{e}_{(i)}$ are connected with the null tetrad vectors by

$$\begin{aligned} \mathbf{m} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(1)} + i\mathbf{e}_{(2)}\right), \qquad & \bar{\mathbf{m}} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(1)} - i\mathbf{e}_{(2)}\right), \\ \mathbf{l} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{u} - \mathbf{e}_{(3)}\right), \qquad & \mathbf{k} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{u} + \mathbf{e}_{(3)}\right), \end{aligned} \tag{55}$$

we find that the equation of geodesic deviation in spacetime with only $\Psi_4 \neq 0$ implies (see [207])

$$\ddot{Z}^{(1)} = -A_+ Z^{(1)} + A_\times Z^{(2)},\ \ddot{Z}^{(2)} = A_+ Z^{(2)} + A_\times Z^{(1)},\ \ddot{Z}^{(3)} = 0, \tag{56}$$

where $A_+ = \frac{1}{2}\,\mathrm{Re}\,\Psi_4, A_\times = \frac{1}{2}\,\mathrm{Im}\,\Psi_4$ are amplitudes of "+" and "×" polarization modes, and $Z^{(i)}$ are the frame components of the connecting vector $\mathbf{Z}$. Since the frame vector $\mathbf{e}_{(3)}$ is chosen in the longitudinal direction (the direction of the rays), equation (56) clearly exhibits the transverse character of the wave. If particles, initially at rest, lie in the $(\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ plane, there is no motion in the longitudinal direction of $\mathbf{e}_{(3)}$. The ring of particles is deformed into an ellipse, the axes of different polarizations are shifted one with respect to the other by $\frac{\pi}{4}$ (such behaviour is typical for linearized gravitational waves – cf. e.g. [18]). Making a rotation in the transverse plane by an angle ϑ,

$$\mathbf{e}'_{(1)} = \cos\vartheta\,\mathbf{e}_{(1)} + \sin\vartheta\,\mathbf{e}_{(2)}\ , \quad \mathbf{e}'_{(2)} = -\sin\vartheta\,\mathbf{e}_{(1)} + \cos\vartheta\,\mathbf{e}_{(2)}\ , \tag{57}$$

and taking $\vartheta = \vartheta_+(\tau) = -\frac{1}{2}\ \mathrm{Arg}\ \Psi_4 = -\frac{1}{2}\Theta$, then $A'_+ = \frac{1}{2}|\Psi|$, $A'_\times = 0$ – the wave is purely "+" polarized. If $\Theta =$ constant, the rotation angle is independent of time – the wave is rightly considered as linearly polarized.

Hence, with the discovery of pp-waves, the understanding of the properties of gravitational radiation has become deeper and closer to physics. In addition, the pp-waves can easily be "linearized" by taking the function H in the metric (51) to be so small that the spacetime can be considered as a perturbation of Minkowski space within the linearized theory. Such an "easy way" from the linear to fully nonlinear spacetimes is of course paid by their simplicity.

In general, in fact, the pp-waves have only the single isometry generated by the Killing vector $k^\alpha = (\partial/\partial v)^\alpha$. However, a much larger *group of symmetries* may exist for various particular choices of the function $H(u, \zeta, \bar{\zeta})$. Jordan, Ehlers and Kundt [57] (see also [53,61]) gave a complete classification of the pp-waves in terms of their symmetries and corresponding special forms of H. For example, in the best known case of plane waves to which we shall turn in greater detail below, Ψ_4 is independent of ζ, so that after removing linear terms in ζ by a coordinate transformation, we have

$$H(u, \zeta, \bar{\zeta}) = A(u)\zeta^2 + \bar{A}(u)\bar{\zeta}^2, \tag{58}$$

with $A(u)$ being an arbitrary function of u. This spacetime admits five Killing vectors.

Recently, Aichelburg and Balasin [208,209] generalized the classification given in [57] by admitting distribution-valued profile functions and allowing for non-vacuum spacetimes with metric (51), but with H which in general does not satisfy (52). They have shown that with H in the form of delta-like pulses,

$$H(u, \zeta, \bar{\zeta}) = f(\zeta, \bar{\zeta})\delta(u), \tag{59}$$

new symmetry classes arise even in the vacuum case.

The main motivation to consider impulsive pp-waves stems from the metrics describing a black hole or a "particle" boosted to the speed of light. The simplest metric of this type, given by Aichelburg and Sexl [210], is a Schwarzschild black hole with mass m boosted in such a way that $\mu = m/\sqrt{1-w^2}$ is held constant as $w \to 1$. It reads

$$ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - 4\mu \log(\zeta\bar{\zeta})\delta(u)du^2, \tag{60}$$

with H clearly in the form (59). This is not a vacuum metric: the energy-momentum tensor $T_{\alpha\beta} = \mu\delta(u)\delta(\zeta)k_\alpha k_\beta$ indicates that there is a "point-like particle" moving with the speed of light along $u = 0$. The Aichelburg-Sexl metric and its more recent generalizations have found interesting applications even outside of general relativity. Some of them will be briefly mentioned in Sect. 8.2.

Let us now turn to the simplest class of pp-waves, which comprises of the best known and illuminating examples of exact gravitational waves. These are the *plane waves*. They are defined as homogeneous pp-waves in the sense that the curvature component Ψ_4 (see (54)) is constant along the wave surfaces so that function H is in the form (58). One can write H as in (53) where

$$f(u, \zeta) = \frac{1}{2}\mathcal{A}(u)e^{i\Theta(u)}\zeta^2, \tag{61}$$

with linear terms being removed by a coordinate transformation. Just as a plane electromagnetic wave, a plane gravitational wave is thus completely represented by its amplitude $\mathcal{A}(u)$ and polarization angle $\Theta(u)$ as functions of the phase u.

The plane waves, including their generalization into the Einstein–Maxwell theory (an additional term $B(u)\zeta\bar{\zeta}$ then appears in H, both Ψ_4 and the electromagnetic quantity Φ_2 being independent of ζ), were already studied in 1926 (see [61]). A real understanding however came only in the late 1950s. Ehlers and Kundt [53] give various characterizations of this class. For example, they prove that a non-flat vacuum field is a pp-wave if and only if the curvature tensor is complex recurrent, i.e. if $P_{\alpha\beta\gamma\delta,\mu} = P_{\alpha\beta\gamma\delta}q_\mu$, where $P_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + i\,{}^{*}\!R_{\alpha\beta\gamma\delta}$; and it is a plane wave if and only if the recurrence vector q_μ is collinear with a real null vector. They also state a nice theorem showing that the plane wave spacetimes defined by the metric (51), H and f given by (53), (69), $\zeta = x + iy$, and with coordinate ranges $-\infty < x, y, u, v < \infty$, are geodesically complete if functions $\mathcal{A}(u)$ and $\Theta(u)$ are C^1-functions. Quoting directly from [53], *"there exist ... complete solutions free of sources (singularities), proving to think of a graviton field independent of any matter by which it be generated. This corresponds to the existence of source-free photon fields in electrodynamics"*. Ehlers and Kundt [53] also state an open problem which, as far as I am aware, has not yet been solved: to prove that plane waves are the only geodesically complete pp-waves.

The most telling examples of plane waves are *sandwich waves*. The amplitude $\mathcal{A}(u)$ in (61) need not be smooth: either it can only be continuous and nonvanishing on a finite interval of u (sandwich), or a step function (shock), or a delta function (impulse). A physical interpretation of such waves is better achieved in other coordinate systems, in which the metric "before" and "after" the wave is not Minkowskian but has a higher degree of smoothness. For linearly polarized waves (Θ equal to zero), a convenient coordinate system can be introduced by setting (see e.g. [211]) $\zeta = (1/\sqrt{2})(px + iqy)$, $v = (1/2)(t + z + pp'x^2 + qq'y^2)$, $u = t - z$, where $' = d/du$, and functions $p = p(u)$ and $q = q(u)$ solve equations $p'' + \mathcal{A}(u)p = 0$ and $q'' - \mathcal{A}(u)q = 0$. In these coordinates the metric turns out to be

$$ds^2 = -dt^2 + p^2dx^2 + q^2dy^2 + dz^2. \tag{62}$$

In double-null coordinates $\tilde{u}$, $\tilde{v}$, with $\tilde{u} = u = t - z$, $\tilde{v} = t + z$, and with a general polarization, the metric can be cast into the form (see e.g. [65,212])

$$ds^2 = -d\tilde{u}d\tilde{v} + e^{-U}(e^V \cosh W dx^2 + e^{-V} \cosh W dy^2 - 2\sinh W dxdy), \tag{63}$$

where U, V, W depend on $\tilde{u}$ only. This so called Rosen form was used in the classical paper on exact plane waves by Bondi, Pirani and Robinson [213].

A simple, textbook example [214] of a sandwich wave is the wave with a "square profile": $\mathcal{A}(u) = 0$ for $u < 0$ and $u > a^2, \mathcal{A}(u) = a^{-2} =$ constant for $0 \le u \le a^2$. The functions p and q which enter (62) are then $p = q = 1$ at $u \le 0, p = \cos(u/a), q = \cosh(u/a)$ at $0 \le u \le a^2$, and $p = -(u/a)\sin a +$

constant, $q = (u/a)\sinh a + \text{constant}$ at $a^2 \le u$. This example can be used to demonstrate explicitly various typical features of plane sandwich gravitational waves within the exact theory: (i) the wave fronts travel with the speed of light; (ii) the discontinuities of the second derivatives of the metric tensor are permitted along a null hypersurface, but must have a special structure; (iii) the waves have a transverse character and produce relative accelerations in test particles; (iv) the waves focus astigmatically initially parallel null congruences (rays) that are pointing in other directions than the waves themselves; (v) as a consequence of the focusing, Rosen-type line elements contain coordinate singularities on a hypersurface behind the waves, and in general caustics will develop there [214].

The focusing effects imply a remarkable property of plane wave spacetimes: no spacelike global hypersurface exists on which initial data can be specified, i.e. *plane wave spacetimes contain no global Cauchy hypersurface.* This can be understood from Fig. 9. Considering a point Q in flat space in front of the wave, Penrose [215] has shown that its future null cone is distorted as it passes through the wave in such a manner that it is refocused to either a point R or a line passing through R parallel to the wave front. Any possible Cauchy hypersurface going through Q must lie below the future null cone through Q, i.e. below the past null cone of R. Hence, it cannot extend as a spacelike hypersurface to spatial infinity.

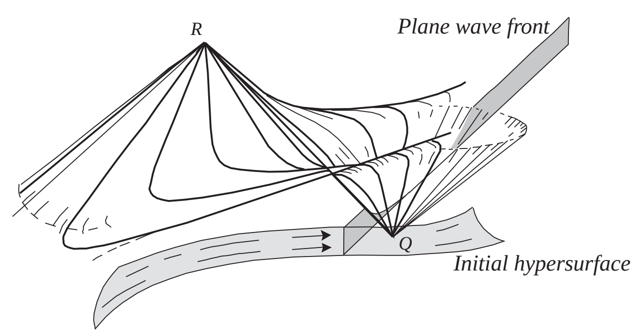

Fig. 9. The future null cone of the event Q is distorted as it passes through the plane wave, and refocused at the event R in such a manner that no Cauchy initial hypersurface going through Q exists. (From [215].)

8.2 Plane-Fronted Waves: New Developments and Applications

The interest in impulsive waves generated by boosting a "particle" at rest to the velocity of light by means of an appropriate limiting procedure persists up to the present. The ultrarelativistic limits of Kerr and Kerr–Newman black holes were obtained in [216–218], and recently, boosted static multipole

(Weyl) particles were studied [219]. Impulsive gravitational waves were also generated by boosting the Schwarzschild–de Sitter and Schwarzschild–anti de Sitter metrics to the ultrarelativistic limit [220,221].

These types of spacetimes, especially the simple Aichelburg-Sexl metrics, have been employed in current problems of the generation of gravitational radiation from axisymmetric black hole collisions and black hole encounters. The recent monograph by d'Eath [222] gives a comprehensive survey, including the author's new results. There is good reason to believe that spacetime metrics produced in high speed collisions will be simpler than those corresponding to (more realistic) situations in which black holes start to collide with low relative velocities. The spacetimes corresponding to the collisions at exactly the speed of light is an interesting limit which can be treated most easily. Aichelburg-Sexl metrics are used to describe limiting "incoming states" of two black holes, moving one against the other with the speed of light. An approximation method has been developed in which a large Lorentz boost is applied so that one has a weak shock propagating on a strong shock. One finds an estimate of 16.8 % for the efficiency of gravitational wave generation in a head-on speed-of-light collision [222].

Great interest has been stimulated by 't Hooft's [223] work on the quantum scattering of two pointlike particles at centre-of-mass energies higher or equal to the Planck energy. This quantum process has been shown to have close connection with classical black hole collisions at the speed of light (see [222,224] and references therein).

Recently, the Colombeau algebra of generalized functions, which enables one to deal with singular products of distributions, has been brought to general relativity and used in the description of impulsive pp-waves in various coordinate systems [225], and also for a rigorous solution of the geodesic and geodesic deviation equations for impulsive waves [226]. The investigation of the equations of geodesics in non-homogeneous pp-waves (with $f \sim \zeta^3$) has shown that the motion of test particles in these spacetimes is described by the Hénon-Heiles Hamiltonian which implies that the motion is chaotic [227].

Plane-fronted waves have been used as simple metrics in various other contexts, for example, in quantum field theory on a given background (see [228] for recent work), and in string theory [229]. As emphasized very recently by Gibbons [230], since for pp-waves and type N Kundt's class (see the beginning of Sect. 8.1) all possible invariants formed from the Weyl tensor and its covariant derivatives vanish [231], these metrics suffer no quantum corrections to all loop orders. Thus they may offer insights into the behaviour of a full quantum theory. The invariants vanish also in type *III* spacetimes with nonexpanding and nontwisting rays [232].

8.3 Colliding Plane Waves

As with a number of other issues in gravitational (radiation) theory, the pioneering ideas on colliding plane gravitational waves are connected with

Roger Penrose. It does not seem to be generally recognized that the basic idea appeared six years before the well-known paper by Khan and Penrose [233] in which the metric describing the general spacetime representing a collision of two parallel-polarized impulsive gravitational waves was obtained. Having demonstrated the surprising fact that general relativistic plane wave spacetimes admit no Cauchy hypersurface due to the focusing effect the waves exert on null cones, Penrose [215] (in footnote 12) remarks: "This fact has relevance to the question of two colliding weak plane sandwich waves. Each wave warps the other until singularities in the wave fronts ultimately appear. This, in fact, causes the spacetime to acquire genuine physical singularities in this case. The warping also produces a scattering of each wave after collision so that they cease to be sandwich waves when they separate (and they are no longer plane – although they have a two-parameter symmetry group)."

The first detailed study of colliding plane waves, independently of Khan and Penrose, was also undertaken by Szekeres (see [234,235]). He formulated the problem as a characteristic initial value problem for a system of hyperbolic equations in two variables (null coordinates) u, v with data specified on the pair of null hypersurfaces $u = 0, v = 0$ intersecting in a spacelike 2-surface (Fig. 10). In the particular case of spacetimes representing plane waves propagating before the collision in a flat background, Szekeres has shown that coordinates (of the "Rosen type", as known from the case of one wave – see Eq. (63)) exist in which the metric reads

$$\begin{aligned} ds^2 = &- e^{-M} du\, dv + \\ &+ e^{-U} \left[e^{V} \cosh W dx^2 + e^{-V} \cosh W dy^2 - 2 \sinh W dx\, dy \right] , \end{aligned} \tag{64}$$

where M, U, V and W are functions of u and v. Coordinates x and y are aligned along the two commuting Killing vectors $\partial/\partial x$ and $\partial/\partial y$, which are assumed to exist in the whole spacetime representing the colliding waves (cf. the note by Penrose above). In almost all recent work on colliding waves, region *IV* in Fig. 10, where $u < 0$, $v < 0$, is assumed to be flat. The null lines $u = 0, v < 0$ and $v = 0, u < 0$ are wavefronts, and in regions *II* $(u < 0, v > 0)$ and *III* $(u > 0, v < 0)$ one has the standard plane wave metric corresponding to two approaching plane waves from opposite directions. In region *II*, functions M, U, V, W depend on v only, and in region *III* only on u. The waves collide at the 2-surface $u = v = 0$, in region *I* they interact. The spacetime here can be determined by the initial data posed on the $v \geq 0$ portion of the hypersurface $u = 0$ (which in Fig. 10 are "supplied" by the wave propagating to the right) and by the data on the $u \geq 0$ portion of the hypersurface $v = 0$ (given by the wave propagating to the left). Unfortunately, the integration of such an initial value problem does not seem to be possible for general incoming wave forms and polarizations. If, however, the approaching waves have constant and aligned (parallel) polarizations, one may set the function $W = 0$ globally. The solution of the initial value problem then reduces to a one dimensional integral for the function V, and two quadratures for the function

M. (The function $\exp(-U)$ must have the form $f(u)+g(v)$ as a consequence of the field equations everywhere, and it can be determined easily from the initial data.) Despite these simplifications it is very difficult to obtain exact solutions in closed analytic form. Szekeres [235] found a solution (as he puts it "more or less by trial and error") which, as special cases, includes the solution given by himself earlier [234] and the solution obtained independently and simultaneously by Khan and Penrose [233]. Although Szekeres' formulation of a general solution for the problem of colliding parallel-polarized waves is difficult to use for constructing other specific explicit examples, it has been employed in a general analysis of the structure of the singularities produced by the collision [236], which will be discussed in the following.

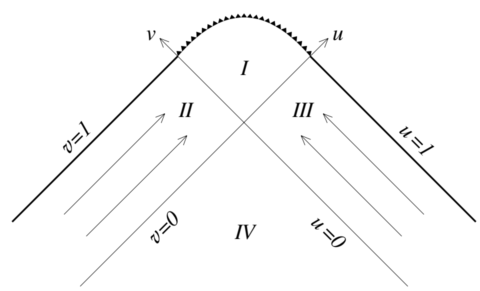

Fig. 10. The spacetime diagram indicating the collision of two plane-fronted gravitational waves which come from regions *II* and *III*, collide in region *I*, and produce a spacelike singularity. Region *IV* is flat.

It has also inspired an important, difficult piece of mathematical physics which was developed at the beginning of the 1990s in the series of papers by Hauser and Ernst [237]. Their new method of analyzing the initial value problem can be used also for the case when the polarization of the approaching waves is not aligned. They formulated the initial value problem in terms of the equivalent matrix Riemann–Hilbert problem in the complex plane. Their techniques are related to those used by Neugebauer and Meinel to analyze and construct the rotating disk solution as a boundary value problem (Sect. 6.3). No analogous solution for colliding waves in the noncollinear case is available at present, but investigations in this direction are still in progress. Most recently, Hauser and Ernst prepared an extensive treatise [238] in which they give a general description and detailed mathematical proofs of their study of the solutions of the hyperbolic Ernst equation.

The approach of Khan and Penrose for obtaining exact solutions describing colliding plane waves starts in the region I where the waves interact: (i) find a solution with two commuting spacelike Killing vectors $\partial/\partial x$ and $\partial/\partial y$, transform to null coordinates, and look back in time whether this solution can be extended across the null hypersurface $u = 0, v = 0$ so that it describes a plane wave propagating in the u-direction in region II and another plane wave propagating in the v-direction in region *III* ; (ii) satisfy boundary conditions not only across boundaries between regions *I* and *II*, and regions *I* and *III*, but also across the boundaries between *II* and *IV*, and *III* and *IV* in such a manner that *IV* is flat. The original prescription of Khan and Penrose for extending the solution from region *I* to regions *II* and *III* consists in the substitutions $uH(u)$ and $vH(v)$ in place of u and v everywhere in the metric coefficients; here $H(u) = 1$ for $u \geq 0$, $H = 0$ for $u < 0$ is the usual Heaviside function. We then get the metric as a function of v (respectively u) in region *II* (respectively *III*) corresponding to the wave propagating to the right (respectively to the left) in Fig. 10. Finally, it remains to investigate carefully the structure of discontinuities and possible singularities on the null boundaries between these regions. In the original Khan and Penrose solutions the Riemann tensor has a δ-function character on the boundaries between *II* and *IV*, and *III* and *IV*; but inside regions *II* and *III* themselves the spacetime is flat (the collision of impulsive plane waves). In the solution obtained by Szekeres [235], regions *II* and *III* are not flat, and the Riemann tensor at the boundaries between *II* (respectively *III*) and *IV* is just discontinuous (the collision of shock waves).

Nutku and Halil [239] constructed an exact solution describing the collision of two impulsive plane waves with non-aligned polarizations. In the limit of collinear polarizations their solution reduces to the solution of Khan and Penrose. All of these solutions reveal that the spacelike singularity always develops in region *I* (given by $u^2 + v^2 = 1$ in Fig. 10) – in agreement with the original suggestion of Penrose. Moreover, the singularity "propagates backward" and so called *fold singularities*, analyzed in detail in 1984 by Matzner and Tipler [240], appear also at $v = 1$ and $u = 1$ in regions *II* and *III*. This new type of singularity provides evidence of how even relatively recent studies of explicit exact solutions may reveal unexpected global features of relativistic spacetimes.

The remarkable growth of interest in colliding plane waves owes much to the systematic (and symptomatic) effort of S. Chandrasekhar who, since 1984, together with V. Ferrari, and with B. Xanthopoulos, published a number of papers on colliding plane vacuum gravitational waves [241,242], and on gravitational waves coupled with electromagnetic waves, with null dust, and with perfect fluid (see [212] for references). The basic strategy of their approach follows that of Khan and Penrose: first a careful analysis of the possible solution is done in the interaction region *I*, and then one works backward in time, extending the solutions to regions *II*, *III* and *IV*.

The main new input consists in carrying over the techniques known from stationary, axisymmetric spacetimes with one timelike and one spacelike Killing vector to the case of two spacelike Killing vectors, $\partial/\partial x, \partial/\partial y$, and exploring new features.

Taking a simple linear solution of the Ernst equation, $E = P\eta + iQ\mu$, where P and Q are real constants which satisfy $P^2 + Q^2 = 1$, and η, μ are suitable time and space coordinates, Chandrasekhar and Ferrari [241] show that one arrives at the Nutku-Halil solution. In particular, if $Q = 0$, the Khan-Penrose solution emerges. Since by starting from the same simplest form of the Ernst function in the axisymmetric stationary case one arrives at the Kerr solution (or at the Schwarzschild solution for the real Ernst function), we may conclude that in region I the solutions of Khan and Penrose and of Nutku and Halil are, for spacetimes with two spacelike Killing vectors, the analogues of the Schwarzschild and Kerr solutions. This mathematical analogy can be generalized to colliding electromagnetic and gravitational waves within the Einstein–Maxwell theory – Chandrasekhar and Xanthopoulos [243] found the analogue of the charged Kerr–Newman solution. Such a generalization is also of interest from a conceptual viewpoint: the δ-function singularity in the Weyl tensor of an impulsive gravitational wave might imply a similar singularity in the Maxwell stress tensor, which would seem to suggest that the field itself would contain "square roots of the δ-function".

In the most important paper [242] of the series, Chandrasekhar and Xanthopoulos, starting from the simplest linear solution for the Ernst conjugate function $E^+ = P\eta + iQ\mu, P^2 + Q^2 = 1$, obtained a new exact solution for colliding plane impulsive gravitational waves accompanied by shock waves. This solution results in the development of a nonsingular Killing–Cauchy horizon instead of a spacelike curvature singularity. The metric can be analytically extended across this horizon to produce a maximal spacetime which contains timelike singularities. (The spacelike singularity in region I in Fig. 10 is changed into the horizon, to the future of which timelike singularities occur.) In the region of interaction of the colliding waves, the spacetime is isometric to the spacetime in a region interior to the ergosphere.

Many new interesting solutions were discovered by using the Khan and Penrose approach. In addition, inverse scattering (soliton) methods and other tools from the solution generation techniques were applied. They are reviewed in detail in [65,212,244].

Although very attractive mathematical methods are contained in these works, one feels that physical interpretation has receded into the background – as seemed to be the case when the new solution generating techniques were exploited in all possible directions for stationary axisymmetric spacetimes. It is therefore encouraging that a more physical and original approach to the problem has been initiated by Yurtsever. In a couple of papers he discusses Killing–Cauchy horizons [245] and the structure of the singularities produced by colliding plane waves [236]. Similar to the Cauchy horizons in black hole physics, one finds that the Killing–Cauchy horizons are unstable. We thus expect that the horizon will be converted to a spacelike singularity. By using the approach of Szekeres described at the beginning of this section, it is possible to relate the asymptotic form of the metric near the singularity – which approaches an inhomogeneous Kasner solution (see Sect. 12.1) – to the initial data given along the wavefronts of the incoming waves. For specific choices of initial data the singularity degenerates into a coordinate singularity and a Killing–Cauchy horizon arises. However, Yurtsever's analysis [236] shows that such horizons are unstable (within *full nonlinear* theory) against small but generic perturbations of the initial data. These results are stronger than those on the instability of the inner horizons of the Reissner–Nordström or Kerr black holes. In particular, Yurtsever constructs an interesting (though unstable) solution which, when analytically extended across its Killing–Cauchy horizon, represents a Schwarzschild black hole created out of the collision between two plane sandwich waves propagating in a cylindrical universe [236].

Yurtsever also introduced "almost plane wave spacetimes" and analyzed collisions of almost plane waves [246]. These waves have a finite but very large transverse sizes. Some general results can be proved (for example, that almost plane waves cannot have a sandwich character, but always leave tails behind), and an order-of-magnitude analysis can be used in the discussion of the outcome of the collision of two almost plane waves; i.e. whether they will focus to a finite minimum size and then disperse, or whether a black hole will be created. Although in the case of almost plane waves one can hardly hope to find an exact spacetime in an explicit form, this is a field which was inspired by exact explicit solutions, and may play a significant role in other parts of general relativity.

9 Cylindrical Waves

In 1913, before the final formulation of general relativity, Einstein remarked in a discussion with Max Born that, in the weak-field limit, gravitational waves exist and propagate with the velocity of light (Poincaré pioneered the idea of gravitational waves propagating with the velocity of light in 1905 – see [15]). Yet, in 1936 Einstein wrote to Born [247]: "... gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows the nonlinear general relativistic field equations can tell us more, or, rather, limit us more than we have believed up to now. If only it were not so damnably difficult to find rigorous solutions". However, after finding a mistake in his argumentation (with the help of H. Robertson) and discovering with Nathan Rosen cylindrical gravitational waves [248] as the first exact radiative solutions to his vacuum field equations, Einstein changed his mind. In fact, cylindrical waves were found more than 10 years before Einstein and Rosen by Guido Beck in Vienna [249]. Beck was mainly interested in time-independent axisymmetric Weyl fields, but he realized that through a complex transformation of coordinates ($z \to it, t \to iz$) one obtains cylindrically symmetric time-dependent fields which represent cylindrical gravitational waves, and wrote down equations (71) and (72) below. The work of Einstein and Rosen is devoted explicitly to gravitational waves. It investigates conditions for the existence of standing and progressive waves, and even notices that the waves carry away energy from the mass located at the axis of symmetry. We shall thus not modify the tradition and will call this type of waves Einstein–Rosen waves (which some readers may wish to shorten to EROS-waves).

This type of waves, symmetric with respect to the transformation $z \to -z$ (z – the axis of symmetry), contains one degree of freedom of the radiation field and corresponds to a fixed state of polarization. The metric can be written in the form

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + e^{2\psi}dz^2 + \rho^2 e^{-2\psi}d\varphi^2, \qquad (65)$$

where ρ and t are invariants ("Weyl-type canonical coordinates"), and $\psi = \psi(t,\rho)$, $\gamma = \gamma(t,\rho)$. The Killing vectors $\partial/\partial\varphi$ and $\partial/\partial z$ are both spacelike and hypersurface orthogonal.

The metric containing a second degree of freedom was discovered by Jürgen Ehlers (working in the group of Pascual Jordan), who used a trick similar to Beck's on the generalized (stationary) Weyl metrics, and independently by Kompaneets (see the discussion in [250]). In the literature (e.g. [251,252]) one refers to the Jordan-Ehlers-Kompaneets form of the metric:

$$ds^2 = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + e^{2\psi}\left(dz + \omega d\varphi\right)^2 + \rho^2 e^{-2\psi}d\varphi^2. \qquad (66)$$

Here, the additional function $\omega(t,\rho)$ represents the second polarization.

Despite the fact that cylindrically symmetric waves cannot describe exactly the radiation from bounded sources, both the Einstein–Rosen waves and their generalization (66) have played an important role in clarifying a number of complicated issues, such as the energy loss due to gravitational waves [253], the interaction of waves with cosmic strings [254,255], the asymptotic structure of radiative spacetimes [250], the dispersion of waves [256], testing the quasilocal mass-energy [257], testing codes in numerical relativity [251], investigation of the cosmic censorship [258], and quantum gravity in a simplified but field theoretically interesting context of midisuperspaces [259–261].

In the following we shall discuss in some detail the asymptotic structure and midisuperspace quantization since in these two issues cylindrical waves have played the pioneering role. Some other applications of cylindrical waves will be briefly mentioned at the end of the section.

9.1 Cylindrical Waves and the Asymptotic Structure of 3-Dimensional General Relativity

In recent work with Ashtekar and Schmidt [262,263], which started thanks to the hospitality of Jürgen Ehlers' group, we considered gravitational waves with a space-translation Killing field ("generalized Einstein–Rosen waves"). In (2+1)-dimensional framework the Einstein–Rosen subclass forms a simple instructive example of *explicitly given spacetimes which admit a smooth global null (and timelike) infinity even for strong initial data*. Because of the symmetry, the 4-dimensional Einstein vacuum equations are equivalent to the 3-dimensional Einstein equations with certain matter sources. This result has roots in the classical paper by Jordan, Ehlers and Kundt [57] which includes "reduction formulas" for the calculation of the Riemann tensor of spaces which admit an Abelian isometry group.

Vacuum spacetimes which admit a spacelike, hypersurface orthogonal Killing vector $\partial/\partial z$ can be described conveniently in coordinates adapted to the symmetry:

$$ds^2 = V^2(x)dz^2 + \bar{g}_{ab}(x)dx^a dx^b, \;\; a, b, \ldots = 0, 1, 2, \tag{67}$$

where $x \equiv x^a$ and $\bar{g}_{ab}$ is a metric with Lorentz signature. The field equations can be simplified if one uses a metric in the 3-space which is rescaled by the norm of the Killing vector, and writes the norm of the Killing vector as an exponential. Then (67) becomes

$$ds^2 = e^{2\psi(x)}dz^2 + e^{-2\psi(x)}g_{ab}(x)dx^a dx^b, \tag{68}$$

and the field equations,

$$R_{ab} - 2\nabla_a\psi\nabla_b\psi = 0, \qquad g^{ab}\nabla_a\nabla_b\psi = 0, \tag{69}$$

where ∇ denotes the derivative with respect to the metric g_{ab}, can be reinterpreted as Einstein's equations in 3 dimensions with a scalar field $\Phi = \sqrt{2}\psi$ as source. Thus, 4-dimensional vacuum gravity is equivalent to 3-dimensional gravity coupled to a scalar field. In 3 dimensions, there is no gravitational radiation. Hence, the local degrees of freedom are all contained in the scalar field. One therefore expects that Cauchy data for the scalar field will suffice to determine the solution. For data which fall off appropriately, we thus expect the 3-dimensional Lorentzian geometry to be asymptotically flat in the sense of Penrose [27,264], i.e. that there should exist a 2-dimensional boundary representing null infinity.

In general cases, this is analyzed in [262]. Here we shall restrict ourselves to the Einstein–Rosen waves by assuming that there is a further spacelike, hypersurface orthogonal Killing vector $\partial/\partial\varphi$ which commutes with $\partial/\partial z$. Then, as is well known, the equations simplify drastically. The 3-metric is given by

$$d\sigma^2 = g_{ab}dx^a dx^b = e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\varphi^2, \tag{70}$$

the field equations (69) become

$$\gamma' = \rho(\dot{\psi}^2 + \psi'^2), \qquad \dot{\gamma} = 2\rho\dot{\psi}\psi', \tag{71}$$

and

$$-\ddot{\psi} + \psi'' + \rho^{-1}\psi' = 0, \tag{72}$$

where the dot and the prime denote derivatives with respect to t and ρ respectively. The last equation is the wave equation for the non-flat 3-metric (70) as well as for the flat metric obtained by setting $\gamma = 0$.

Thus, we can first solve the axisymmetric wave equation (72) for ψ on Minkowski space and then solve (71) for γ – the only unknown metric coefficient – by quadratures. The "method of descent" from the Kirchhoff formula in 4 dimensions gives the representation of the solution of the wave equation in 3 dimensions in terms of Cauchy data $\Psi_0 = \psi(t=0,x,y), \Psi_1 = \psi_{,t}(t=0,x,y)$ (see [262]). We assume that the Cauchy data are axially symmetric and of compact support.

Let us look at the behaviour of the solution at future null infinity $\mathcal{J}^+$. Let ρ, φ be polar coordinates in the plane, and introduce the retarded time coordinate $u = t-\rho$ to explore the fall-off along constant u null hypersurfaces. For large ρ, the function ψ at $u =$ constant admits a power series expansion in ρ^{-1}:

$$\psi(u,\rho) = \frac{f_0(u)}{\sqrt{\rho}} + \frac{1}{\sqrt{\rho}} \sum_{k=1}^{\infty} \frac{f_k(u)}{\rho^k}. \tag{73}$$

The coefficients in this expansion are determined by integrals over the Cauchy data. At $u \gg \rho_0$, ρ_0 being the radius of the disk in the initial Cauchy surface in which the data are non-zero, we obtain

$$f_0(u) = \frac{k_0}{u^{\frac{3}{2}}} + \frac{k_1}{u^{\frac{1}{2}}} + \dots, \tag{74}$$

where k_0, k_1 are constants which are determined by the data. If the solution happens to be time-symmetric, so that Ψ_1 vanishes, we find $f_0 \sim u^{-\frac{3}{2}}$ for large u. Similarly, we can also study the behaviour of the solution near the timelike infinity i^+ of 3-dimensional Minkowski space by setting $t = U + \kappa\rho$, $\kappa > 1$, and investigating ψ for $\rho \to \infty$ with U and κ fixed. We refer to [262] for details.

In Bondi-type coordinates $(u = t-\rho, \rho, \varphi)$, equation (70) yields

$$d\sigma^2 = e^{2\gamma}(-du^2 - 2du d\rho) + \rho^2 d\varphi^2. \tag{75}$$

The Einstein equations take the form

$$\gamma_{,u} = 2\rho\, \psi_{,u}(\psi_{,\rho} - \psi_{,u}), \qquad \gamma_{,\rho} = \rho\, \psi^2_{,\rho}, \tag{76}$$

and the wave equation on ψ becomes

$$-2\psi_{,u\rho} + \psi_{,\rho\rho} + \rho^{-1}(\psi_{,\rho} - \psi_{,u}) = 0. \tag{77}$$

The asymptotic form of $\psi(t,\rho)$ is given by the expansion (73). Since we can differentiate (73) term by term, the field equations (76) and (77) imply

$$\gamma_{,u} = -2[\dot{f}_0(u)]^2 + \sum_{k=1}^{\infty} \frac{g_k(u)}{\rho^k}, \tag{78}$$

$$\gamma_{,\rho} = \sum_{k=0}^{\infty} \frac{h_k(u)}{\rho^{k+2}}, \tag{79}$$

where the functions f_k, h_k are products of the functions $f_0, f_k, \dot{f}_0, \dot{f}_k$. Integrating (79) and fixing the arbitrary function of u in the result using (78), we obtain

$$\gamma = \gamma_0 - 2\int_{-\infty}^{u} \left[\dot{f}_0(u)\right]^2 du - \sum_{k=1}^{\infty} \frac{h_k(u)}{(k+1)\rho^{k+1}}. \tag{80}$$

Thus, γ also admits an expansion in ρ^{-1}, where the coefficients depend smoothly on u. It is now straightforward to show that the spacetime admits a smooth future null infinity, $\mathcal{J}^+$. Setting $\tilde{\rho} = \rho^{-1}, \tilde{u} = u, \tilde{\varphi} = \varphi$ and rescaling g_{ab} by a conformal factor $\Omega = \tilde{\rho}$, we obtain

$$d\tilde{\sigma}^2 = \Omega^2 d\sigma^2 = e^{2\tilde{\gamma}}(-\tilde{\rho}^2 d\tilde{u}^2 + 2d\tilde{u}d\tilde{\rho}) + d\tilde{\varphi}^2, \tag{81}$$

where $\tilde{\gamma}(\tilde{u}, \tilde{\rho}) = \gamma(u, \tilde{\rho}^{-1})$. Because of (80), $\tilde{\gamma}$ has a smooth extension through $\tilde{\rho} = 0$. Therefore, $\tilde{g}_{ab}$ is smooth across the surface $\tilde{\rho} = 0$. This surface is the future null infinity, $\mathcal{J}^+$. Hence, the (2+1)-dimensional curved spacetime has a smooth (2-dimensional) null infinity. Penrose's picture works for arbitrarily strong initial data Ψ_0, Ψ_1.

Using (81), we find that at $\mathcal{J}^+$ we have:

$$\gamma(u, \infty) = \gamma_0 - 2\int_{-\infty}^{u} \dot{f}_0^2 du. \tag{82}$$

Since one can make sure that $\gamma = 0$ at i^+ [263], one finds the simple result that

$$\gamma_0 = 2\int_{-\infty}^{+\infty} \dot{f}_0^2 du. \tag{83}$$

At spatial infinity (t = constant, $\rho \to \infty$), the metric is given by

$$d\sigma^2 = e^{2\gamma_0}(-dt^2 + d\rho^2) + \rho^2 d\varphi^2. \tag{84}$$

For a non-zero data, constant γ_0 is positive, whence the metric has a "conical singularity" at spatial infinity. This conical singularity, present at spatial infinity, is "radiated out" according to equation (82). The future timelike infinity, i^+, is smooth. In (2+1)-dimensions, modulo some subtleties [262], equation (82) plays the role of the *Bondi mass loss formula* in (3+1)-dimensions, relating the decrease of the total (Bondi) mass-energy at null infinity to the flux of gravitational radiation. We can thus conclude that *cylindrical waves in (2+1)-dimensions give an explicit model of the Bondi–Penrose radiation theory which admits smooth null and timelike infinity for arbitrarily strong initial data*. There is no other such model available. The general results on the existence of $\mathcal{J}$ in 4 dimensions, mentioned at the end of Sect. 1.3, assume weak data.

It is of interest to investigate cylindrical waves also in a (3+1)-dimensional context. The asymptotic behaviour of these waves was discussed by Stachel

[250] many years ago. However, his work deals solely with asymptotic directions, which are perpendicular to the axis of symmetry, i.e. to the $\partial/\partial z$ – Killing vector. Detailed calculations show that, in contrast to the perpendicular directions, where null infinity in the (3+1)-dimensional framework does not exist, it *does* exist in other directions for data of compact support. If the data are not time-symmetric, the fall-off is so slow that (local) null infinity has a *polyhomogeneous* (logarithmic) character [180] – see [263] for details.

We have concentrated on the simplest case of Einstein–Rosen waves. They served as a prototype for developing a general framework to analyze the asymptotic structure of spacetime at null infinity in three spacetime dimensions. This structure has a number of quite surprising features which do not arise in the Bondi–Penrose description in four dimensions [262]. One of the motivations for developing such a framework is to provide a natural point of departure for constructing the stage for asymptotic quantization and the S-matrix theory of an interesting midisuperspace in quantum gravity.

9.2 Cylindrical Waves and Quantum Gravity

As the editors of the Proceedings of the 117th WE Heraeus Seminar on canonical gravity in 1993 [265], Jürgen Ehlers and Helmut Friedrich start their Introduction realistically: "When asking a worker in the field about the progress in quantum general relativity in the last decade, one shouldn't be surprised to hear: 'We understand the problems better'. If it referred to a lesser task, such an answer would sound ironic. But the search for quantum gravity... has been going on now for more than half a century and in spite of a number of ingenious proposals, a satisfactory theory is still lacking..." Although I am following the subject from afar, I believe that one would not be too wrong if one repeated the same words in 1999. However, apart from general theoretical developments, many interesting quantum gravity models have been studied, and exact solutions have played a basic role in them. In particular, in the investigations of (spherical) gravitational collapse and in quantum cosmology based typically on homogeneous cosmological models (cf. Sect. 12.1), one starts from simple classical solutions – see e.g. [266–268] for reviews and [269] for a bibliography up to 1990. A common feature of such models is the reduction of infinitely many degrees of freedom of the gravitational field to a *finite* number. In quantum field theory (such as quantum electrodynamics) a typical object to be quantized is a wave with an infinite number of degrees of freedom. The first radiative solutions of the gravitational field equations which were subject to quantization were the Einstein–Rosen waves. Kuchař [259] applied the methods of canonical quantization of gravity to these waves, using the methods employed earlier in the minisuperspace models, i.e. restricting himself only to geometries (fields) preserving the symmetries.

The Einstein–Rosen cylindrical waves have an *infinite* number ∞^1 of degrees of freedom contained in one polarization, one degree of freedom for each cylindrical surface drawn around the axis of symmetry. Moreover, the slicing of spacetime by spacelike (cylindrically symmetric) hypersurfaces is not fixed completely by the symmetry – an arbitrary cylindrically symmetric deformation of a given slice leads again to an allowed slice. Such a deformation represents an ∞^1 "fingered time". Hence, the resulting space of 3-geometries on cylindrically symmetric slices is infinitely richer than the minisuperspaces of quantum cosmology. The exact Einstein–Rosen waves thus inspired the first construction of what Kuchař [259] called the "*midisuperspace*".

Let us briefly look at the main steps in Kuchař's procedure.[21] The symmetry of the problem implies that the spatial metric has the form

$$g_{11} = e^{\gamma - \Phi}, \qquad g_{22} = R^2 e^{-\Phi}, \qquad g_{33} = e^{\Phi}, \tag{85}$$

where γ, Φ, and R are functions of a single cylindrical coordinate $x^1 = r$ ($x^2 = \varphi$, $x^3 = z$). Similarly the lapse function $N = N(r)$ depends only on r, and the shift vector has the only nonvanishing radial component $N^1 = N^1(r), N^2 = N^3 = 0$. We have adopted here Kuchař's notation. When we put $R = r = \rho$, $\Phi = 2\psi$, $\gamma \to 2\gamma$, $N = e^{\gamma-\Phi}$, and $N^1 = 0$, we recover the standard Einstein–Rosen line element (65); however, in general the radial and time coordinates t and r differ from the canonical Einstein–Rosen radial and time coordinates in which the metric has the standard form (65). The symmetry implies that the canonical momentum π^{ik} is diagonal and expressible by three functions $\pi_\gamma, \pi_R, \pi_\Phi$ of r; for example, $\pi^{11} = \pi_\gamma e^{\Phi-\gamma}$, and similarly for the other components. After the reduction to cylindrical symmetry, the action functional assumes the canonical form

$$S = 2\pi \int_{-\infty}^{\infty} dt \int_0^{\infty} dr (\pi_\gamma \dot{\gamma} + \pi_R \dot{R} + \pi_\Phi \dot{\Phi} - N\mathcal{H} - N^1 \mathcal{H}_1) \ , \tag{86}$$

in which γ, R, Φ are the canonical coordinates and $\pi_\gamma, \pi_R, \pi_\Phi$ the conjugate momenta (the integration over z has been limited by $z = z_0$ and $z = z_0 + 1$). The superhamiltonian $\mathcal{H}$ and supermomentum $\mathcal{H}_1$ are rather complicated functions of the canonical variables:

$$\mathcal{H} \ = e^{\frac{1}{2}(\Phi-\gamma)} \left(-\pi_\gamma \pi_R + \tfrac{1}{2} R^{-1} \pi_\Phi^2 + 2R'' - \gamma' R' + \tfrac{1}{2} R \Phi'^2 \right) , \tag{87}$$

$$\mathcal{H}_1 = -2\pi_\gamma' + \gamma' \pi_\gamma + R' \pi_R + \Phi' \pi_\Phi . \tag{88}$$

The most important step now is the replacement of the old canonical variables $\gamma, \pi_\gamma, R, \pi_R$ by a new canonical set T, Π_T, R, Π_R through a suitable canonical transformation. We shall write here only one of its components (see [259,270] for the complete transformation):

$$T(r) = T(\infty) + \int_{\infty}^{r} [-\pi_\gamma(r)] \, dr . \tag{89}$$

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[21] For the basic concepts and ideas of canonical gravity, we refer to e.g. [18,19] and especially to Kuchař's review [270], where the canonical quantization of cylindrical waves is also analyzed.

By integrating the Hamilton equations following from the action (86), rewritten in the new canonical coordinates, one finds that T and R are the Einstein–Rosen privileged time and radial coordinates, i.e. those appearing in the canonical form (65) of the Einstein–Rosen metric (with $T = t, R = \rho$). According to (89), the Einstein–Rosen time can be reconstructed, in a non-local way, from the momentum π_γ, which characterizes the extrinsic curvature of a given hypersurface. In this way, the concept of the *"extrinsic time representation"* entered canonical gravity with cylindrical gravitational waves.

In terms of the new canonical variables, the superhamiltonian and supermomentum become

$$\mathcal{H} = e^{\frac{1}{2}(\Phi-\gamma)} \left(R' \Pi_T + T' \Pi_R + \tfrac{1}{2} \left(R^{-1} \pi_\Phi^2 + R \Phi'^2 \right) \right), \tag{90}$$

$$\mathcal{H}_1 = T' \Pi_T + R' \Pi_R + \Phi' \pi_\Phi. \tag{91}$$

Since $\mathcal{H}$ and $\mathcal{H}_1$ are linear in Π_T and Π_R, the classical constraints $\mathcal{H} = 0, \mathcal{H}_1 = 0$ can immediately be resolved with respect to these momenta, conjugate to the "embedding" canonical variables $T(r)$ and $R(r)$:

$$-\Pi_T = \left(R'^2 - T'^2 \right)^{-1} \left[\tfrac{1}{2} (R^{-1} \pi_\Phi^2 + R \Phi'^2) R' - \Phi' \pi_\Phi T' \right] = 0, \tag{92}$$

and similarly for Π_R. It is easy to see [259,270] that the constraints have the same form as the constraints for a massless scalar field Φ propagating on a flat background foliated by arbitrary spacelike hypersurfaces $T = T(r), R = R(r)$. The canonical variables Φ, π_Φ represent the true degrees of freedom, and the remaining canonical variables play the role of spacelike embeddings of a Cauchy hypersurface into spacetime.

After turning the canonical momenta Π_T, Π_R, π_Φ, into variational derivatives, e.g. $\Pi_T = -i\delta/\delta T(r)$, one can impose the classical constraints $\mathcal{H} = 0, \mathcal{H}_1 = 0$ as restrictions on the state functional $\Psi(T, R, \Phi)$: $\mathcal{H}\Psi = 0$, $\mathcal{H}_1 \Psi = 0$. In particular, the Wheeler-DeWitt equation $\mathcal{H}\Psi = 0$ in the extrinsic time representation assumes the form of a many-fingered time counterpart of an ordinary Schrödinger equation. This reduces to the ordinary Schrödinger equation for a single massless scalar field in Minkowski space if we adopt the standard foliation $T =$ constant (see [259,270] for details).

The described procedure, first realized in the case of the Einstein–Rosen waves, has opened a new route in canonical and quantum gravity. In contrast to the Arnowitt-Deser-Misner approach, in which the gravitational dynamics is described relative to a fixed foliation of spacetime, in this new approach (called "bubble time" dynamics of the gravitational field or the "internal time formalism" [271]) one tries to extract the many-fingered time (i.e. embeddings of Cauchy hypersurfaces) from the gravitational phase space, but does not fix the foliation in the "target manifold" by coordinate conditions. However, the definition of the target manifold by a gauge (coordinate) condition is needed.

This new approach has been so far successfully applied to a few other models (based on exact solutions) with infinite degrees of freedom, for example, plane gravitational waves, bosonic string, and as late as 1994, to spherically symmetric vacuum gravitational fields [272]. The internal time formalism for spacetimes with two Killing vectors was developed in [273] (therein references to previous works can also be found). Recently, canonical transformation techniques have been applied to Hamiltonian spacetime dynamics with a thin spherical null-dust shell [274]. One would like to construct a midisuperspace model of spherical gravitational collapse, or more specifically, a model for Hawking radiation with backreaction. The extensive past work on Hamiltonian approaches to spherically symmetric geometries (see [274] for more than 40 references in this context) have not yet led to convincing insights. The very basic question of existence of the "internal time" formalism in a general situation has been most recently addressed by Hájíček [49]; the existence has been proven, and shown to be related to the choice of gauge.

9.3 Cylindrical Waves: a Miscellany

Chandrasekhar [247] constructed a formalism for cylindrical waves with two polarizations (cf. the metric (66)), similar to that used for the discussion of the collision of plane-fronted waves (Sect. 8.3). He obtained the "cylindrical" Ernst equation and corroborated (following the suggestion of O. Reula) the physical meaning of Thorne's *C-energy* [253] – the expression for energy suggested for cylindrical fields – by defining a Hamiltonian density corresponding to the Lagrangian density from which the Ernst equation can be derived. A brief summary of older work on the mass loss of a cylindrical source radiating out cylindrical waves and its relation to the C-energy is given in [65]. It should be pointed out, however, that although C-energy is a useful quantity, it was constructed by exploiting the local field equations, without direct reference to asymptotics. The physical energy (per unit z length) at both spatial and null infinity, which is the generator of the time translation, is in fact a non-polynomial function of the C-energy. In the weak field limit the two agree, but in strong fields they are quite different [262].

In [256], an exact solution was constructed with which one can study the *dispersion* of waves: a cylindrical wave packet, which though initially impulsive, after reflection at the axis disperses, and develops shock wave fronts when the original wave meets the waves that are still ingoing. Cylindrical waves have been also analyzed in the context of *phase shifts* occurring in gravitational soliton interactions (see [275] and references therein).

An exact explicit solution for cylindrical waves with two degrees of polarization has been obtained [252] from the Kerr solution after transforming the metric into "cylindrical" coordinates and using the substitution $t \to i\tilde{z}, z \to i\tilde{t}, a \to i\tilde{a}$. Both this solution and the well-known Weber-Wheeler-Bonnor pulse [65] have been employed as *test beds in numerical relativity* [276], in particular in the approach which combines a Cauchy code for determining the dynamics of the central source with a characteristic code for determining the behaviour of radiation [251].

In a number of works cylindrical waves have been considered in interaction with *cosmic strings* [254,255]. The strings are usually modelled as infinitely thin conical singularities. Recently Colombeau's theory of generalized functions was used to calculate the distributional curvature at the axis for a time-dependent cosmic string [277].

A somewhat surprising result concerning cosmic strings and radiation theory should also be noted: although an infinite, static cylindrically symmetric string does not, of course, radiate, it generates a nonvanishing (though "non-radiative") contribution to the Bondi news function [278,279]. Recently, the asymptotics at null infinity of cylindrical waves with both polarizations (and, in general, an infinite cosmic string along the axis) has been analyzed in the context of axisymmetric electrovacuum spacetimes with a translational Killing vector at null infinity [280].

Finally, the cylindrically symmetric electrovacuum spacetimes with both polarizations, satisfying certain completeness and asymptotic flatness conditions in spacelike directions have been shown rigorously to imply that strong cosmic censorship holds [258]. This means that for generic (smooth) initial data the maximal globally hyperbolic development of the data is inextendible (no Cauchy horizons as for example, those discussed in Sect. 3.1 for the Reissner–Nordström spacetime arise). This global existence result is nontrivial since with two polarizations and electromagnetic field present, all field equations are nonlinear.

10 On the Robinson–Trautman Solutions

Robinson–Trautman metrics are the general radiative vacuum solutions which admit a geodesic, shearfree and twistfree null congruence of diverging rays. In the standard coordinates the metric has the form [281]

$$ds^2 = 2r^2 P^{-2} d\zeta d\bar{\zeta} - 2du\, dr - [\Delta \ln P - 2r(\ln P)_{,u} - 2mr^{-1}]\, du^2, \qquad (93)$$

where ζ is a complex spatial (stereographic) coordinate (essentially θ and φ), r is the affine parameter along the rays, u is a retarded time, m is a function of u (which can be in some cases interpreted as the mass of the system), $\Delta = 2P^2(\partial^2/\partial\zeta\partial\bar{\zeta})$, and $P = P(u, \zeta, \bar{\zeta})$ satisfies the fourth-order Robinson–Trautman equation

$$\Delta\Delta(\ln P) + 12\, m\, (\ln P)_{,u} - 4m_{,u} = 0. \qquad (94)$$

The best candidates for describing radiation from isolated sources are the Robinson–Trautman metrics of type *II* with the 2-surfaces S^2 given by $u, r =$ constant and having spherical topology. The Gaussian curvature of S^2 can be expressed as $K = \Delta \ln P$. If $K =$ constant, we obtain the Schwarzschild solution with mass equal to $K^{-\frac{3}{2}}$.

These spacetimes have attracted increased attention in the last decade – most recently in the work by Chruściel, and Chruściel and Singleton [282]. In these studies the Robinson–Trautman spacetimes have been shown to exist globally for all positive "times", and to converge asymptotically to a Schwarzschild metric. Interestingly, the extension of these spacetimes across the "Schwarzschild-like" event horizon can only be made with a finite degree of smoothness. All these rigorous studies are based on the derivation and analysis of an asymptotic expansion describing the long-time behaviour of the solutions of the nonlinear parabolic equation (94).

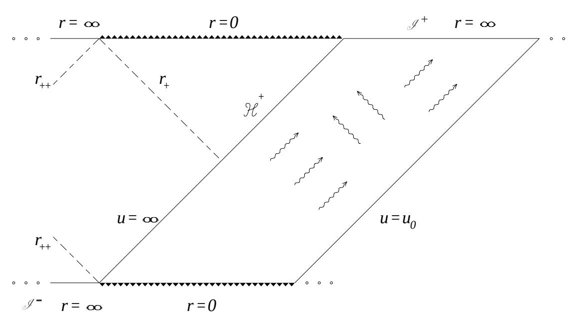

Fig. 11. The evolution of the cosmological Robinson–Trautman solutions with a positive cosmological constant. A black hole with the horizon $\mathcal{H}^+$ is formed; at future infinity $\mathcal{J}^+$ the spacetime approaches a de Sitter spacetime exponentially fast, in accordance with the cosmic no-hair conjecture.

In our recent work [163,283] we studied Robinson–Trautman radiative spacetimes with a positive cosmological constant Λ. The results proving the global existence and convergence of the solutions of the Robinson–Trautman equation (94) can be taken over from the previous studies since Λ does not explicitly enter this equation. We have shown that, starting with arbitrary, smooth initial data at $u = u_0$ (see Fig. 11), these cosmological Robinson–Trautman solutions converge exponentially fast to a Schwarzschild–de Sitter solution at large retarded times ($u \to \infty$). The interior of a Schwarzschild–de Sitter black hole can be joined to an "external" cosmological Robinson–Trautman spacetime across the horizon $\mathcal{H}^+$ with a higher degree of smoothness than in the corresponding case with $\Lambda = 0$. In particular, in the extreme case with $9\Lambda m^2 = 1$, in which the black hole and cosmological horizons coincide, the Robinson–Trautman spacetimes can be extended *smoothly* through $\mathcal{H}^+$ to the extreme Schwarzschild–de Sitter spacetime with the same values of Λ and m. However, such an extension is *not analytic* (and not unique).

We have also demonstrated that the cosmological Robinson–Trautman solutions represent explicit models exhibiting the cosmic no-hair conjecture: a geodesic observer outside of the black hole horizon will see, that inside his past light cone, these spacetimes approach the de Sitter spacetime exponentially fast as he approaches the future (spacelike) infinity $\mathcal{J}^+$. For a freely falling observer the observable universe thus becomes quite bald. This is what the cosmic no-hair conjecture claims. As far as we are aware, these models represent the only exact analytic demonstration of the cosmic no-hair conjecture under the presence of gravitational waves. They also appear to be the only exact examples of black hole formation in nonspherical spacetimes which are not asymptotically flat. Hopefully, these models may serve as tests of various approximation methods, and as test beds in numerical studies of more realistic situations in cosmology.

11 The Boost–Rotation Symmetric Radiative Spacetimes

In this section we would like to describe briefly the only explicit solutions available today which are radiative and represent the fields of finite sources. Needless to say, we cannot hope to find explicit analytic solutions of the Einstein equations without imposing a symmetry. A natural first assumption is axial symmetry, i.e. the existence of a spacelike rotational Killing vector $\partial/\partial\varphi$. However, it appears hopeless to search for a radiative solution with only one symmetry. We are now not interested in colliding plane waves since these do not represent finite sources; we wish our spacetime to be as "asymptotically flat as possible". The unique role of the boost-rotation symmetric spacetimes is exhibited by a theorem, formulated precisely and proved for the vacuum case with hypersurface orthogonal Killing vectors in [284], and generalized to electrovacuum spacetimes with Killing vectors which need not be hypersurface orthogonal in [279] (see also references therein). This theorem roughly states that in *axially* symmetric, locally asymptotically flat spacetimes (in the sense that a null infinity satisfying Penrose's requirements exists, but it need not necessarily exist globally), the only *additional* symmetry that does not exclude radiation is the *boost* symmetry.

In Minkowski spacetime the boost Killing vector has the form

$$\zeta_{boost} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \tag{95}$$

so that orbits of symmetry to which the Killing vector is tangent are hyperbolas $z^2 - t^2 = B =$ constant, x, $y =$ constant. Orbits with $B > 0$ are timelike; they can represent worldlines of uniformly accelerated particles in special relativity. Imagine, for example, a charged particle, axially symmetric about the z-axis, moving with a uniform acceleration along this axis. The electromagnetic field produced by such a source will have boost-rotation symmetry.

Figure 12 shows two particles uniformly accelerated in opposite directions along the z-axis. In the space diagram (left), the "string" connecting the particles is also indicated. In the spacetime diagram, the particles' worldlines are shown in bold. Thinner hyperbolas represent the orbits of the boost Killing vector (95) in the regions $t^2 > z^2$ where it is spacelike. In Fig. 13 the corresponding compactified diagram indicates that null infinity cannot be smooth everywhere since it contains four singular points in which particles' worldlines "start" and "end". Notice that in electromagnetism the presence of *two* particles, one moving along $z > 0$, the other along $z < 0$, makes the field symmetric also with respect to inversion $z \to -z$. The electromagnetic field can be shown to be analytic everywhere, except for the places where the particles occur. These two particles move independently of each other, since their worldlines are divided by two null hypersurfaces $z = t, z = -t$. This is analogous to the boost-rotation symmetric spacetimes in general relativity that we are now going to discuss.

Specific examples of solutions representing "uniformly accelerated particles" have been analyzed for 35 years, starting with the first solutions of this type obtained by Bonnor and Swaminarayan [285], and Israel and Khan [286]. In a curved spacetime the "uniform acceleration" is understood with respect to a fictitious Minkowski background, and the "particles" mean singularities or black holes. For a more extensive description of the history of these specific solutions discovered before 1985, see [287]. From a unified point of view, boost-rotation symmetric spacetimes (with hypersurface orthogonal Killing vectors) were defined and treated geometrically in [288]. We refer to this detailed work for rigorous definitions and theorems. Here we shall only sketch some of the general properties and some applications of these spacetimes.

The metric of a general boost-rotation symmetric spacetime in "Cartesian-type" coordinates $\{t, x, y, z\}$ reads:

$$\begin{aligned} ds^2 = \; & \frac{1}{x^2+y^2}\left[(e^{\lambda}x^2 + e^{-\mu}y^2)dx^2 + 2xy(e^{\lambda} - e^{-\mu})dxdy\right] + \\ & + \frac{1}{x^2+y^2}(e^{\lambda}y^2 + e^{-\mu}x^2)dy^2 + \frac{1}{z^2-t^2}(e^{\lambda}z^2 - e^{\mu}t^2)dz^2 - \\ & - \frac{1}{z^2-t^2}\left[2zt(e^{\lambda} - e^{\mu})dzdt + (e^{\mu}z^2 - e^{\lambda}t^2)dt^2\right] \;, \qquad (96) \end{aligned}$$

where μ and λ are functions of $\rho^2 = x^2 + y^2$ and $z^2 - t^2$. As a consequence of the vacuum Einstein equations, the function μ must satisfy an equation of the form which is identical to the flat-space wave equation; and function λ is determined in terms of μ by quadrature. Now it can easily be seen that the metric (96) admits axial and boost Killing vectors which have exactly the same form as in Minkowski space, i.e. the axial Killing vector $\partial/\partial\varphi$ and the boost Killing vector (95). In fact, the whole structure of group orbits in boost-rotation symmetric *curved* spacetimes outside the sources (or singularities) is the same as the structure of the orbits generated by the axial and boost Killing vectors in Minkowski space. In particular, the boost Killing vector (95) is timelike in the region $z^2 > t^2$. The invariance of a metric (or of any other field) in a time-direction (determined in a coordinate-free manner by a timelike Killing vector) means stationarity, and of course, we could hardly expect to find radiative properties there. Intuitively, the existence of a timelike Killing vector in the region $z^2 > t^2$ is understandable because there (generalized) uniformly accelerated reference frames can be introduced in which sources are at rest, and the fields are time independent.

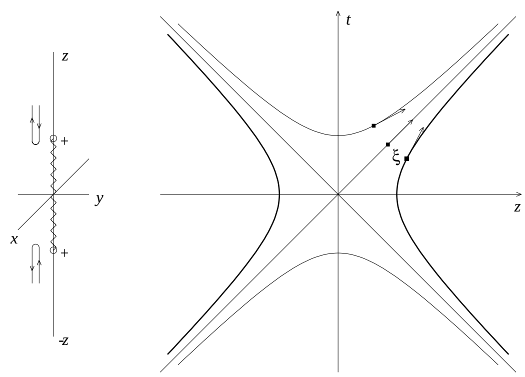

Fig. 12. Two particles uniformly accelerated in opposite directions. The orbits of the boost Killing vector (thinner hyperbolas) are spacelike in the region $t^2 > z^2$.

However, in the other "half" of the spacetime, $t^2 > z^2$, the boost Killing vector (95) is spacelike (see the lines representing orbits of the boost Killing vector in Fig. 12). Hence in this region the metric (96) is nonstationary. Here we expect to discover radiative properties. Indeed, it can be shown that for $t^2 > z^2 + \rho^2$ the metric (96) can *locally* be transformed into the metric of Einstein–Rosen cylindrical waves. Although locally in the whole region $t^2 > z^2$ the metric (96) can be transformed into a radiative metric, the global properties of the boost-rotation symmetric solutions are quite different from those of cylindrical waves. Again, we have to refer to the work [288] for a detailed analysis. Let us only say that the boost-rotation symmetric solutions, if properly defined – with appropriate boundary conditions on functions λ and μ – always admit asymptotically flat null infinity $\mathcal{J}$ at least locally. Starting with arbitrary solutions λ and μ, and adding suitable constants to both λ and μ (Einstein's equations are then still satisfied), we can always guarantee that even *global* $\mathcal{J}$ exists in the sense that it admits smooth spherical sections. For the special type of solutions for λ and μ, *complete* $\mathcal{J}$ satisfies Penrose's requirements, except for four points in which the sources "start" and "end" (cf. Fig. 13). In all cases one finds that the gravitational field in smooth regions of the null infinity is radiative [279,289]. In particular, the leading term of the Riemann curvature tensor, proportional to r^{-1} (where $r^2 = \rho^2 + z^2$), is nonvanishing and has the same algebraic structure as the Riemann tensor of plane waves. This is fully analogous to the asymptotic properties of radiative electromagnetic fields outside finite sources. Recently, general forms of the news functions have been obtained for electrovacuum spacetimes with boost-rotation symmetry and with Killing vectors which need not be hypersurface orthogonal [279].

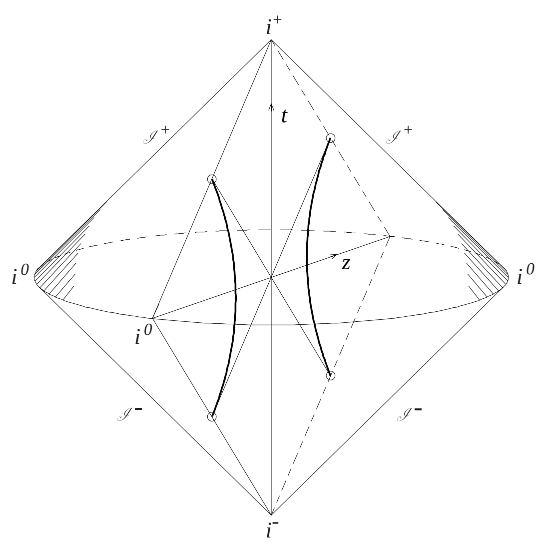

Fig. 13. The Penrose compactified diagram of a boost-rotation symmetric spacetime. Null infinity can admit smooth sections.

It is well known that in general relativity the "causes" of motion are always incorporated in the theory – in contrast to electrodynamics where they need not even be describable by Maxwell's theory. In a general case of the boost-rotation symmetric solutions there exist nodal (conical) singularities of the metric distributed along the z-axis which can be considered as "strings", and cause particles to accelerate. They reveal themselves also at $\mathcal{J}$. However, the distribution of nodes can always be arranged in such a manner that $\mathcal{J}$ admits smooth regular sections as mentioned above.

In exceptional cases, when $\mathcal{J}$ is regular except for four points, either the particles are "self-accelerating" due to their "inner" multipole structure, which has to include a negative mass; or there are more particles distributed along $z > 0$ (and symmetrically along $z < 0$) with the signs and the magnitudes of their masses and accelerations chosen appropriately. (For the concept of a negative mass in general relativity, and the first discussion of a "chasing" pair of a positive and a negative mass particle, see classical papers by Bondi, and Bonnor and Swaminarayan [285].) An infinite number of different analytic solutions representing self-accelerating particles was constructed explicitly [290]. Although a negative mass cannot be bought easily in the shop (as Bondi liked to say), these solutions are the only exact solutions of Einstein's equations available today for which one can find such quantities of physical interest as radiation patterns (angular distribution of gravitational radiation), or total radiation powers [287]. From a mathematical point of view, these solutions represent the only known spacetimes in which *arbitrarily strong* (boost-rotation symmetric) *initial data* can be chosen on a hyperboloidal hypersurface in the region $t^2 > z^2$, which will lead to a complete smooth null infinity and a regular timelike future infinity. With these specific examples one thus does not have to require weak-field initial data as one has to in the work of Friedrich, and Christodoulou and Klainerman, mentioned at the end of Sect. 1.3.

The boost-rotation symmetric radiative spacetimes can be used as test beds for approximation methods or numerical relativity. Bičák, Reilly and Winicour [291] found the explicit boost-rotation symmetric "initial null cone solution", which solves initial hypersurface and evolution equations in "radiative" coordinates employed in the null cone version of numerical relativity. This solution has been used for checking and improving numerical codes for computing gravitational radiation from more realistic sources; a new solution of this type has also been found [292]. Recently, the specific boost-rotation symmetric spacetimes constructed in [290] were used as test beds in the standard version of numerical relativity based on spacelike hypersurfaces [293].

There exist "generalized" boost-rotation symmetric spacetimes which are *not* asymptotically flat, but are of considerable physical interest. They describe accelerated particles in asymptotically "uniform" external fields. One can construct such solutions from asymptotically flat boost-rotation symmetric solutions for the pairs of accelerated particles by a limiting procedure, in which one member of the pair is "removed" to infinity, and its mass parameter is simultaneously increased [294]. Since the resulting spacetimes are not asymptotically flat, their radiative properties are not easy to analyze. Only if the external field is weak will there exist regions in which the spacetimes are approximately flat; and here their radiative properties might be investigated. So far no systematic analysis of these spacetime has been carried out. Nevertheless, they appear to offer the best rigorous examples of the motion of relativistic objects. No nodal singularities or negative masses are necessary to cause an acceleration.

As an eloquent example of such a spacetime consider a charged (Reissner–Nordström) black hole with mass M and charge Q, immersed in an electric field "uniform at infinity", characterized by the field-strength parameter E.

An exact solution of the Einstein–Maxwell equations exists which describes this situation [131]. It goes over into an approximate solution obtained by perturbing the charged black hole spacetime by a weak external electric field which is uniform at infinity [295]. One of the results of the analysis of this solution is very simple: a charged black hole in an electric field starts to accelerate according to Newton's second law, $Ma = QE$, where all the quantities can be determined – and in principle measured – in an approximately flat region of the spacetime from the asymptotic form of the metric. Recall T. S. Eliot again: "There is only the fight to recover what has been lost / And found and lost again and again."

These types of generalized boost-rotation symmetric spacetimes ("generalized C-metrics") have been used by Hawking, Horowitz, Ross, and others [296] in the context of quantum gravity – to describe production of black hole pairs in strong background fields.

Recently, we have studied the spinning C-metric discovered by Plebański and Demiański [297]. Transformations can be found which bring this metric into the canonical form of spacetimes with boost-rotation symmetry [298]. The metric represents two uniformly *accelerated, rotating* black holes, either connected by conical singularity, or with conical singularities extending from each of them to infinity. The spacetime is radiative. No other spacetime of this type, with two Killing vectors which are not hypersurface orthogonal, is available in an explicit form.

12 The Cosmological Models

In light of Karl Popper's belief that "all science is cosmology", it appears unnecessary to justify the choice of solutions for this last section. As in the whole article, these will be primarily vacuum solutions. On the other hand, in light of the light coming from about 10^{11} galaxies, each with about 10^{11} stars, it may seem weird to consider *vacuum* models of the Universe. Indeed, it has become part of the present-day culture that spatially homogeneous and isotropic, expanding Friedmann-Robertson-Walker (FRW) models, filled with uniformly distributed matter, correspond well to basic observational data. In order to achieve a more precise correspondence, it appears sufficient to consider just perturbations of these "standard cosmological models". To explain some "improbable" features of these models such as their isotropy and homogeneity, one finds an escape in inflationary scenarios. These views of a "practical cosmologist" are, for example, embodied in one of the most comprehensive recent treatise on physical cosmology by Peebles [28].

Theoretical (or mathematical) cosmologists, however, point out that more general cosmological models exist which differ significantly from a FRW model at early times, approach the FRW model very closely for a certain epoch, and may diverge from it again in the future. Clearly, the FRW universes represent only a very special class of viable cosmological models, though the simplest and most suitable for interpretations of "fuzzy" cosmological observational data.

Simple exact solutions play a significant role in the evolution of more general models, either as asymptotic or intermediate states. By an "intermediate state" one means the situation when the universe enters and remains in a small neighbourhood of a saddle equilibrium point. A simple example is the Lemaître matter-filled, homogeneous and isotropic model with a nonvanishing cosmological constant (see e.g. [28]), which expands from a dense state (the big bang, or "primeval atom" in Lemaître's 1927 terminology), passes through a quasistatic epoch in which all parameters are close to those of the static Einstein universe (cf. Sect. 1.2), and then the universe expands again. An "asymptotic state" means close either to an initial big bang (or possibly a final big crunch) singularity, or the situation at late times in forever expanding universes. It is easy to see that at late times in indefinitely expanding universes the matter density decreases, and vacuum solutions may become important. However, as we shall discuss below, vacuum models play an important role also close to a singularity, when the matter terms in Einstein's equations are negligible compared to the "velocity terms" (given by the rate of change of scale factors) or to the curvature terms (characterizing the curvature of spacelike hypersurfaces). In particular, the pioneering (and still controversial) work started at the end of the 1950s by Lifshitz and Khalatnikov, and developed later on by Belinsky, Khalatnikov and Lifshitz, has shown that the fact that the presence of matter does not influence the qualitative behaviour of a cosmological model near a singularity has a very general significance (see [299] and [300] for the main original references, and [139] for a brief review).

In gaining an intuition in the analysis of general cosmological singularities, the class of spatially homogeneous anisotropic cosmological models have played a crucial role. These so called *Bianchi models* admit a simply transitive 3-dimensional homogeneity group. Among the Bianchi vacuum models there are special exact explicit solutions, in particular the Kasner and the Bianchi type II solutions, which exhibit some aspects of general cosmological singularities. The Bianchi models have also had an impact on other issues in general relativity and cosmology.

Much work, notably in recent years, has been devoted to the class of both vacuum and matter-filled cosmological solutions which are homogeneous only on 2-dimensional spacelike orbits. Thus they depend on time and on one spatial variable, and can be used to study spatial inhomogeneities as density fluctuations or gravitational waves. The vacuum cosmological models with two spacelike Killing vectors, sometimes called the *Gowdy models*,[22] are

[22] In fact, by Gowdy models, one more often means only the cases with closed group orbits, with two commuting spacelike othogonally-transitive Killing vectors (the surface elements orthogonal to the group orbits are surface-forming).

interpreted as gravitational waves in an expanding (or contracting) universe with compact spatial sections. We shall discuss these two classes separately.

12.1 Spatially Homogeneous Cosmologies

The simplest solutions, the Minkowski, de Sitter, and anti de Sitter spacetimes, which have also been used in cosmological contexts (cf. Sect. 1.3), are 4-*dimensionally homogeneous*. As noted in Sect. 8.1, the vacuum plane waves (see equations (51), (53), (61)) are also homogeneous spacetimes; and since they can be suitably sliced by spacelike hypersurfaces with expanding normal congruence, they can become asymptotic states in homogeneous expanding cosmologies. There exist several important non-vacuum homogeneous spacetimes, for example, the Einstein static universe (cf. Sect. 1.2), and Gödel's stationary, rotating universe (see e.g. [61,301]), famous for the first demonstration that Einstein's equations with a physically permissible matter source are compatible with the existence of closed timelike lines, i.e. with the violation of causality.

Here we shall consider models in which the symmetry group does not make spacetime a homogeneous space, but in which each event in spacetime is contained in a spatial hypersurface that is homogeneous. The standard FRW models represent a special case of such models (they admit, in addition, an isotropy group $SO(3)$ at each point). The general spatially homogeneous solutions comprise of the *Kantowski–Sachs universes* and a much wider class of Bianchi models. By definition, the Bianchi models admit a simply transitive 3-dimensional homogeneity group G_3. There exist special "locally rotationally symmetric" (LRS) Bianchi models which admit a 4-dimensional isometry group G_4 acting on homogeneous spacelike hypersurfaces, but these groups have a simply transitive subgroup G_3. In contrast to this, Kantowski–Sachs spacetimes admit G_4 (acting on homogeneous spacelike hypersurfaces) which does *not* have any simply transitive subgroup G_3; it contains a multiply transitive G_3 acting on 2-dimensional surfaces of constant curvature, $G_4 = \mathbb{R} \times SO(3)$. A special case of the vacuum Kantowski–Sachs universe is represented by the Schwarzschild metric inside the horizon (with t and r interchanged). There has been a continuing interest in the Kantowski–Sachs models since their discovery in 1966 [302], to which, as the authors acknowledge, J. Ehlers contributed by his advice. Some of these models had already appeared in the PhD thesis of Kip Thorne in 1965 (see also [303] for magnetic Kantowski–Sachs models). Here, however, we just refer the reader to [304,305] for their classical description, to [306] for a canonical and quantum treatment, and to [307] for the latest discussion of the Kantowski–Sachs quantum cosmologies.

Although the 3-dimensional Lie groups which are simply transitive on homogenous 3-spaces were classified by Bianchi in 1897, the importance of Bianchi's work for constructing vacuum cosmological models was only discovered by Taub in 1951 [197], when the Taub space (cf. Sect. 7) was first given. It is less known that at approximately the same time, if not earlier, the first explicit spatially homogeneous expanding and rotating cosmological models with matter (of the Bianchi type IX) were constructed by Gödel,[23] who first presented his results at the International Congress of Mathematics held at Cambridge (Mass.) from August 30 till September 5, 1950.

An exposition of Bianchi models has been given in a number of places: in the account on relativistic cosmology by Heckmann and Schücking [308] (complementing the chapter on exact solutions by Ehlers and Kundt [53]), in the monographs of Ryan and Shepley [304], and Zel'dovich and Novikov [309], in several comprehensive surveys by MacCallum (see e.g. [305] and [310] for his latest review containing a number of references), most recently, in the book on the dynamical system approach in cosmology (in the Bianchi models in particular) edited by Wainwright and Ellis [311]; and, first but not least, in the classics of Landau and Lifshitz [139]. The Hamiltonian approach initiated by Misner [312] in 1968, and used in, amongst other things, the construction of various minisuperspace models in quantum gravity, has been reviewed by Ryan [266]; for more recent accounts, see several contributions to Misner's Festschrift [313]. An interesting framework which unifies the Hamiltonian approach to the solutions which admit homogeneous hypersurfaces either spacelike (as Bianchi models) or timelike (as static spherical, or stationary cylindrical models) was recently developed by Uggla, Jantzen and Rosquist in [314] (with 115 references on many exact solutions). Herewith we shall only briefly introduce the Bianchi models, note their special role in understanding the character of an initial cosmological singularity, and mention some of the most recent developments not covered by the reviews cited above.

The line element of the Bianchi models can be expressed in the form

$$ds^2 = -dt^2 \ + \ g_{ab}(t) \ \omega^a \ \omega^b, \tag{97}$$

where the time-independent 1-forms ω^a $(= E^a_\alpha dx^\alpha)$, $a = 1, 2, 3$, are dual to time-independent[24] spatial frame vectors $\mathbf{E}_a$ (often an arbitrary time-variable $\tilde{t}$ is introduced by $dt = N(\tilde{t}) \ d\tilde{t}$, N being the usual lapse function). Both ω^a

[23] Gödel's profound ideas and results in cosmology, and their influence on later developments have been discussed in depth by G. Ellis in his lecture at the Gödel '96 conference in Brno, Czech Republic, where Gödel was born in 1906 (78 years before Gödel, Ernst Mach was born in a place which today belongs to Brno). In the extended written version of Ellis' talk [301] it is indicated that Gödel's work also initiated the investigation of Taub. This may well be true with Gödel's paper on the stationary rotating universe, but Taub's paper on Bianchi models was received by the Annals of Mathematics on May 15, 1959, i.e. before Gödel's lecture on expanding and rotating models at the Congress of Mathematics took place.

[24] The gravitational degrees of freedom are associated with the component (scalar) functions $g_{ab}(t)$ – the so called metric approach. Alternatively, in the orthonormal frame approach, one chooses $g_{ab}(t) = \delta_{ab}$ and describes the evolution by time-dependent forms ω^a. In still another approach one employs the automorphism

and $\mathbf{E}_a$ are group-invariant, commuting with the three Killing fields which generate the homogeneity group. They satisfy the relations

$$d\omega^a = -\frac{1}{2}C^a_{bc}\,\omega^b \wedge \omega^c, \tag{98}$$

$$[\mathbf{E}_a, \mathbf{E}_b] = C^c_{ab}\mathbf{E}^c, \tag{99}$$

where d is the exterior derivative and C^a_{bc} are the structure constants of the Lie algebra of the homogeneity group. The models are classified according to the possible distinct sets of the structure constants. They are first divided into two classes: in class A the trace $C^a_{ba} = 0$, and in class B, $C^a_{ba} \neq 0$. In class A one can choose $C^a_{bc} = n^{(a)}\epsilon_{abc}$ (no summation over a), and classify various symmetry types by parameters $n^{(a)}$ with values $0, \pm 1$. In class B, in addition to $n^{(a)}$, one needs the value of a constant scalar h (related to C^a_{ba}) to characterize types VI_h and VII_h (see e.g. [311]).

The simplest models are the Bianchi I cosmologies in class A with $n^{(a)} = 0$, i.e. $C^a_{bc} = 0$, so that all three Killing vectors (the group generators) commute. They contain the standard Einstein–de Sitter model with flat spatial hypersurfaces (curvature index $k = 0$). In the vacuum case, all Bianchi I models are given by the well-known 1-parameter family of *Kasner metrics* (found in 1921 by E. Kasner and in 1933 by G. Lemaître without considering the Bianchi groups)

$$ds^2 = -dt^2 \;+\; t^{2p_1}dx^2 \;+\; t^{2p_2}dy^2 \;+\; t^{2p_3}dz^2, \tag{100}$$

where

$$p_1 + p_2 + p_3 = 1,\;\; p_1^2 + p_2^2 + p_3^2 = 1. \tag{101}$$

These metrics were first used to investigate various effects in anisotropic cosmological models. For example, in contrast to standard FRW models with "point-like" initial singularities, the Kasner metrics can permit the so called *"cigar"* and *"pancake" singularities.* To be more specific, consider the congruence of timelike lines with unit tangent vectors n^α orthogonal to constant time hypersurfaces, and define the expansion tensor $\theta_{\alpha\beta}$ by $\theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}$, where $h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$ is a projection tensor, $\sigma_{\alpha\beta} = n_{(\alpha;\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$ is the shear, and $\theta = \theta_\alpha{}^\alpha$. Determining the three spatial eigenvectors of $\theta_{\alpha\beta}$ with the corresponding eigenvalues θ_i $(i = 1, 2, 3)$, one can define the scale factors l_i by the relation $\theta_i = (dl_i/dt)/l_i$, and the Hubble scalar $H = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3)$. In the FRW models, all $l_i \to 0$ at the big bang singularity. In the Kasner models at $t \to 0$ one finds that either two of the l_i go to zero, whereas the third unboundedly increases (a cigar); or one of the l_i tends to zero, while the other two approach a finite value (pancake). Also there is the *"barrel" singularity* in which the two of the l_i go to zero, and the third approaches

of the symmetry group to simplify the spatial metric g_{ab} (see [310,311] for more details).

a finite value. There is an open question as to whether some other possibilities exist [311]. Even in the perfect fluid Kasner model, the approach to the singularity is *"velocity-dominated"* – the "vacuum terms" given by the rates of change of the scale factors dominate the "matter terms" (curvature terms vanish since the Kasner models are spatially flat).

The general vacuum Bianchi type II cosmologies (with one $n^{(a)} = +1$, and the other two vanishing), discovered by Taub in [197], contain two free parameters:

$$ds^2 = -A^2 dt^2 + A^{-2}\, t^{2p_1}(dx + 4p_1 bz\, dy)^2 + A^2(t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2), \quad (102)$$

where

$$A^2 = 1 + b^2 t^{4p_1}, \; p_1 + p_2 + p_3 = 1, \; p_1^2 + p_2^2 + p_3^2 = 1. \quad (103)$$

If we put the parameter $b = 0$, the metrics (102) become the Kasner solutions (100). Near the big bang the general Bianchi type II solution is asymptotic to a Kasner model. In the future it is asymptotic again to a Kasner model, but with different values of parameters p_i (see e.g. [311]). This fact will be important in the following.

The general Bianchi type V vacuum solutions are also known – these are given by the 1-parameter family of Joseph solutions [311]. The type V models are the simplest metrics in class B (with all $n^{(a)} = 0$ but $C^a_{bc} = 2a_{[b}\delta^a_{c]}$, $a_b =$ constant), and are the simplest Bianchi models which contain the standard FRW open universes ($k = -1$). The Joseph solutions are asymptotic to the specific Kasner solution in the past, and tend to the *"isotropic Milne model"* in the future. This is intuitively understandable since open FRW models, as they expand indefinitely into the future with matter density decreasing, also approach the Milne model. As is well known, the Milne model is just an empty flat (Minkowski) spacetime in coordinates adapted to homogeneous spacelike hypersurfaces (the "mass hyperboloids"), with expanding normals (see e.g. [28]):

$$ds^2 = -d\tau^2 + \tau^2\left[(1+\rho^2)^{-1} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \quad (104)$$

with $\tau = t(1-u^2)^{1/2}$, $\rho = u(1-u^2)$, $u = r/t < 1$, where t, r, θ, φ are standard Minkowski (spherical) coordinates. Because of its significance as an asymptotic solution and its simplicity, the Milne model has been used frequently in pedagogical expositions of relativistic cosmology (see e.g. [28,214]) as well as in cosmological perturbation theory and quantization (see e.g. [315] and references therein). The Milne universe is also an asymptotic state of other Bianchi models such as, for example, the intriguing *Lukash vacuum* type VII$_h$ *solution* [316], which can be interpreted as two monochromatic, circularly polarized waves of time-dependent amplitude travelling in opposite directions on a FRW background, with flat or negative curvature spacelike sections. As was noticed earlier, some indefinitely expanding Bianchi models approach the homogeneous plane wave solutions. Barrow and Sonoda [317] studied the future asymptotic behaviour of the known Bianchi solutions in detail by using nonlinear stability techniques; and in [311] dynamical system methods were used.

From the late 1960s onwards the greatest amount of work was probably devoted to the Bianchi type IX vacuum models, baptized *the Mixmaster universe*[25] by Misner [312]. Type IX models are the most general class A models with all parameters $n^{(a)} = +1$. They are the only Bianchi universes which recollapse. If a perfect fluid is permitted as the matter source, the non-vacuum type IX solutions contain the closed FRW models ($k = +1$) with space sections having spherical topology. As a Bianchi I space admits a group isomorphic with translations in a 3-dimensional Euclidean space, the group of type IX spaces is isomorphic to the group of rotations. None of the pairs of three Killing vectors commute. A general Bianchi IX vacuum solution is not known, but a particular solution is available: the Taub-NUT spacetime, or rather, its spatially homogeneous anisotropic region – the Taub universe (see Sect. 7.2). This fact was, for example, employed in an attempt to understand the limitations of the minisuperspace methods of quantum gravity: by reducing the degrees of freedom to a general Mixmaster universe and then further to the Taub universe one can see what such restrictions imply [318].

The dynamics of general Bianchi cosmologies – and of the Mixmaster models in particular – close to the big bang singularity has been approached with essentially three methods [311]: (i) piecewise approximation methods, (ii) Hamiltonian methods, and (iii) dynamical system methods. In the first method, used primarily by Russian cosmologists (cf. [299,300]), the evolution is considered to be a sequence of periods in which certain terms in the Einstein equations dominate whereas other terms can be neglected. The Hamiltonian methods appeared first in the "Mixmaster paper" by Misner [312], were reviewed by Ryan [266], and more recently by Uggla in [311]. With the Hamiltonian (canonical) approaches, minisuperspace methods entered general relativity (cf. Sect. 9.2 on midisuperspace for cylindrical waves). In this approach, infinitely many degrees of freedom are reduced to a finite number: the state of the universe is described by a "particle" moving inside and reflecting instantaneously from the moving potential walls, which approximate the time-dependent potentials in the Hamiltonian. In the third method one employs the fact that Einstein's equations in the case of homogeneous cosmologies can be put into the form of an autonomous system of first-order (ordinary) differential equations on a finite dimensional space $\mathbb{R}^n$. This is of the form $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^n$ representing a state of the model (for example, the suitably normalized components of the shear σ, the Hubble

[25] The name comes from the fact that, in contrast to a standard FRW model, which has a horizon preventing the equalization of possible initial inhomogeneities over large scales, the horizon in a type IX universe is absent, so that mixing is in principle possible. However, as was shown e.g. in [309], "repeated circumnavigations of the universe by light are impossible in the Mixmaster model".

scalar θ, and parameters related to $n^{(a)}$, can serve as the "components" of $\mathbf{x}$). A study of the orbits $\mathbf{x}(t)$ indicates the behaviour of the model. Dynamical system methods are the focus of the book [311]. They also are the main tools of the monograph [319].

In the case of the Bianchi IX models (either vacuum or with perfect fluid), all three methods imply (though do not supply a rigorous proof) that an approach to the past big bang singularity is composed of an infinite sequence of intervals, in each of which the universe behaves approximately as a specific Kasner model (100). The transition "regimes" between two different subsequent Kasner epochs, in which the contraction proceeds along subsequently different axes, is approximately described by Bianchi type II vacuum solutions (102). This famous and enigmatic *"oscillatory approach to the singularity"* (or "Mixmaster behaviour") has rightly entered the classical literature (cf. e.g. [18,19,139]). It indicates that the big bang singularity (and, similarly, a singularity formed during a gravitational collapse) can be much more complicated than the "point-like" singularity in the standard FRW models. This oscillatory character has been suggested not only by the qualitative methods mentioned above, but also by extensive numerical work (see e.g. [311,320]). So far, however, it has resisted a rigorous proof.

In the "standard" picture of the Mixmaster model it is supposed that the evolution of the Bianchi type IX universe near the singularity can be approximated by a mapping of the so called *Kasner circle* onto itself. This is the unit circle in the $\Sigma_+\Sigma_-$ plane, where $\Sigma_\pm = \sigma_\pm/H$ describes the anisotropy in the Hubble flow (cf. e.g. Fig. 6.2 in [311]). Each point on the circle corresponds to a specific Kasner solution with given fixed values of parameters p_i satisfying the conditions (101). There are three exceptional points on the circle – those at which one of the $p_i = +1$, and the other two vanish. From each non-exceptional point P_1 on the Kasner circle there leads a 1-dimensional unstable orbit given by the vacuum Bianchi II solution (102), which joins P_1 to another point P_2 on the circle, then P_2 is mapped to P_3 , etc. This "Kasner map" in the terminology of [311], called frequently also the BKL (Belinsky-Khalatnikov-Lifshitz) map, describes subsequent changes of Kasner epochs during the oscillatory approach to a singularity. Recent rigorous results of Rendall [321] show that for any *finite* sequence generated by the BKL map, there exists a vacuum Bianchi type IX solution which reproduces the sequence with any required accuracy.[26]

The vacuum Bianchi IX models have been extensively analyzed in the context of deterministic chaos and their stochasticity, attracting the interest of leading experts in these fields [320,322]. Above all, it is the numerical work which strongly suggests that it is impossible to make long-time predictions of

[26] A. Rendall (private communication) reports that the main points of the BKL picture for homogeneous universes have been rigorously confirmed in a recent work of H. Ringström (to be published).

the evolution of the system from the initial data, which is the most significant property of a chaotic system.

Most recently, interest in the Bianchi cosmologies with (homogeneous) magnetic and scalar fields has been revived. Following his previous work with Wainwright and Kerr on magnetic Bianchi VI_0 cosmologies [323], LeBlanc [324] has shown that even in Bianchi I cosmologies one finds an oscillatory approach towards the initial singularity if a magnetic field in a general direction is present. (The points on the Kasner circle are now joined by Rosen magneto-vacuum solutions.) Hence, Mixmaster-like oscillations occur due to the magnetic field degrees of freedom, even in the absence of an anisotropic spatial curvature (present in the vacuum type IX models) – the result anticipated by Jantzen [325] in his detailed work on Hamiltonian methods for Bianchi cosmologies with magnetic and scalar fields. Similar conclusions have also been arrived at in [326] for magnetic Bianchi II cosmologies. (LeBlanc's papers contain some new exact magnetic Bianchi solutions and a number of references to previous work.) Interestingly, in contrast to the magnetic field, scalar fields in general *suppress* the Mixmaster oscillations when approaching the initial singularity [327,328].

The theory of spatially homogeneous, anisotropic models is an elegant, intriguing branch of mathematical physics. It has played an important role in general relativity. The classical monograph of Zel'dovich and Novikov [309], or the new volume of Wainwright and Ellis [311] analyze in detail the possible observational relevance of these models: they point out spacetimes close to FRW cosmologies (at least during an epoch of finite duration) which are compatible with observational data. For the most recent work on Bianchi VII_h cosmologies which are potentially compatible with the highly isotropic microwave background radiation, see [329] (and references therein). Nevertheless, the present status is such that, in contrast to for example the Kerr solution, which is becoming an increasingly strong attractor for practical astrophysicists (cf. Sect. 4.3), the anisotropic models have not really entered (astro)physical cosmology so far. Peebles, for example, briefly comments in [28]: "The homogeneous anisotropic solutions allowed by general relativity are a very useful tool for the study of departures from the Robertson-Walker line element. As a realistic model for our Universe, however, these solutions seem to be of limited interest, for they require very special initial conditions: if the physics of the early universe allowed appreciable shear, why would it not also allow appreciable inhomogeneities?"

An immediate reaction, of course, would be to point out that the FRW models require still more "special initial conditions". However, there appears to be a deeper reason why the oscillatory approach towards a singularity may be of fundamental importance. Belinsky, Khalatnikov and Lifshitz [299,300] employed their piecewise approximation method, and concluded 30 years ago that a singularity in a *general, inhomogeneous* cosmological model is spacelike and locally oscillatory: i.e. in their scenario, the evolution at different spatial point decouples. At each spatial point the universe approaches the singularity as a distinct Mixmaster universe. This view, often criticized by purists, appears now to be gaining an increasing number of converts, even among the most rigorous of relativists. As mentioned above, the homogeneous magnetic Bianchi type VI_0 models, investigated by LeBlanc et al., show Mixmaster behaviour. The Bianchi VI_0 models have, as do all Bianchi models, three Killing vectors, but two of them commute. The models can thus be generalized by relaxing the symmetry connected with the third Killing vector; one can so obtain effectively the inhomogeneous (in one dimension) Gowdy-type spacetimes. Weaver, Isenberg and Berger [330], following this idea of Rendall, analyzed these models numerically, and discovered that the Mixmaster behaviour is reached at different spatial points. The numerical evidence for an oscillatory singularity in a generic vacuum $U(1)$ symmetric cosmologies with the spatial topology of a 3-torus has been found still more recently by Berger and Moncrief [331].

Before turning to the Gowdy models, a last word on the "oscillatory approach towards singularity". I heard E. M. Lifshitz giving a talk on this issue a couple of times, with Ya. B. Zel'dovich in the audience. In discussions after the talk, Zel'dovich, who appreciated much this work (its detailed description is included in [309]), could not resist pointing out that the number of oscillations and Kasner epochs will be very limited (to only about ten) because of quantum effects which arise when some scale of a model is smaller than the Planck length $l_{Pl} \sim 10^{-33}$cm. This, however, seems to make the scenario still more intriguing. If this is confirmed rigorously within classical relativity, how will a future quantum gravity modify this picture?

12.2 Inhomogeneous Cosmologies

Among all of the known vacuum inhomogeneous models, the *Gowdy solutions* [332] have undoubtedly played the most distinct role. They belong to the class of solutions with two commuting spacelike Killing vectors. Within a cosmological context, they form a subclass of a wider class of G_2 *cosmologies* – as are now commonly denoted models which admit an Abelian group G_2 of isometries with orbits being spacelike 2-surfaces. A 2-surface with a 2-parameter isometry group must be a space of constant curvature, and since neither a 2-sphere nor a 2-hyperboloid possess 2-parameter subgroups, it must be intrinsically flat. If the 2-surface is an Euclidean plane or a cylinder, then one speaks about planar or cylindrical universes. Gowdy universes are compact – the group orbits are 2-tori T^2.

The metrics with two spacelike Killing vectors are often called the generalized Einstein–Rosen metrics as, for example, by Carmeli, Charach and Malin [333] in their comprehensive survey of inhomogeneous cosmological models of this type. In dimensionless coordinates (t, z, x^1, x^2), the line element can be written as $(A, B = 1, 2)$

$$ds^2/L^2 \;=\; e^F(-dt^2 + dz^2) + \gamma_{AB} dx^A dx^B, \qquad (105)$$

where L is a constant length, F and γ_{AB} depend on t and z only, and thus the spacelike Killing vectors are ${}^{(1)}\xi^\alpha = (0,0,1,0), {}^{(2)}\xi^\alpha = (0,0,0,1)$.

The local behaviour of the solutions of this form is described by the gradient of the "volume element" of the group orbits $W = (|\det(\gamma_{AB})|)^{1/2}$. Classical cylindrical Einstein–Rosen waves (cf. Sect. 9) are obtained if $W_{,\alpha}$ is globally spacelike. In Gowdy models, $W_{,\alpha}$ varies from one region to another.[27]

Considering for simplicity the polarized Gowdy models (when the Killing vectors are hypersurface orthogonal), the metric (105) can be written in diagonal form (cf. equations (62), (63), and (65), (66) in the analogous cases of plane and cylindrical waves)

$$ds^2/L^2 \;=\; e^{-2U}\left[e^{2\gamma}(-dt^2+dz^2)+W^2dy^2\right]+e^{2U}dx^2, \qquad (106)$$

in which $U(t,z)$ and $\gamma(t,z)$ satisfy wavelike dynamical equations and constraints following from the vacuum Einstein equations; the function $W(t,z)$, which determines the volume element of the group orbit, can be cast into a standard form which depends on the topology of t = constant spacelike hypersurfaces Σ.

As mentioned above, in Gowdy models one assumes these hypersurfaces to be compact. Gowdy [332] has shown that Σ can topologically be (i) a 3-torus $T^3 = S^1 \otimes S^1 \otimes S^1$ and $W = t$ (except for the trivial case when spacetime is identified as a Minkowski space), (ii) a 3-handle (or hypertorus, or "closed wormhole") $S^1 \otimes S^2$ with $W = \sin z \sin t$, or (iii) a 3-sphere S^3, again with $W = \sin z \sin t$. (For some subtle cases not covered by Gowdy, see [334].) As the form of W suggests, in the case of a T^3 topology, the universe starts with a big bang singularity at $t = 0$ and then expands indefinitely, whereas in the other two cases it starts with a big bang at $t = 0$, expands to some maximal volume, and then recollapses to a "big crunch" singularity at $t = \pi$. One can determine exact solutions for metric functions in all three cases in terms of Bessel functions [335]. Hence, for the first time *cosmological models closed by gravitational waves* were constructed. Charach found Gowdy universes with some special electromagnetic fields [336], and other generalized Gowdy models were obtained. We refer to the detailed survey [333] for more information, including the work on canonical and quantum treatments of these models, done at the beginning of the 1970s by Berger and Misner, and for extensive references.

Let us only add a few remarks on some more recent developments in which the Gowdy models have played a role. Gowdy-type models have been used to study the *propagation and collision of gravitational waves with toroidal wavefronts* (as mentioned earlier, 2-tori T^2 are the group orbits in the Gowdy

[27] The same is true in the boost-rotation symmetric spacetimes considered in Sect. 11: the part $t^2 > z^2$ of the spacetimes, where the boost Killing vector is spacelike, can be divided into four different regions, in two of which vector $W_{,\alpha}$ is spacelike, and in the other two timelike – see [288] for details.

cosmologies) in the FRW closed universes with a stiff fluid [337]. In the standard Gowdy spacetimes it is assumed that the "twists" associated with the isometry group on T^2 vanish. In [338] the generalized Gowdy models without this assumption are considered, and their global time existence is proved.

As both interesting and non-trivial models, the Gowdy spacetimes have recently attracted the attention of mathematical and numerical relativists with an increasing intensity, as indicated already at the end of the previous section. Chruściel, Isenberg and Moncrief [339] proved that Gowdy spacetimes developed from a dense subset in the initial data set cannot be extended past their singularities, i.e. in "most" Gowdy models the strong cosmic censorship is satisfied.

On cosmic censorship and spacetime singularities, especially in the context of compact cosmologies, we refer to a review by Moncrief [340], based on his lecture in the GR14 conference in Florence in 1995. The review shows clearly how intuition gained from such solutions as the Gowdy models or the Taub-NUT spaces, when combined with new mathematical ideas and techniques, can produce rigorous results with a generality out of reach until recently. To such results belongs also the very recent work of Kichenassamy and Rendall [341] on the sufficiently general class of solutions (containing the maximum number of arbitrary functions) representing unpolarized Gowdy spacetimes. The new mathematical technique, developed by Kichenassamy [342], the so called Fuchsian algorithm, enables one to construct singular (and nonsingular) solutions of partial differential equations with a large number of arbitrary functions, and thus provide a description of singularities. Applying the Fuchsian algorithm to Einstein's equations for Gowdy spacetimes with topology T^3, Kichenassamy and Rendall have proved that general solutions behave at the (past) singularity in a Kasner-like manner, i.e. they are asymptotically velocity dominated with a diverging Kretschmann (curvature) invariant. One needs an additional magnetic field not aligned with the two Killing vectors of the Gowdy unpolarized spacetimes in order to get a general oscillatory (Mixmaster) approach to a singularity, as shown by the numerical calculations [330] mentioned at the end of the previous section.

Much of the work on exact inhomogeneous vacuum cosmological models has been related to "large perturbations" of Bianchi universes. In [343] the authors confined attention to "plane wave" solutions propagating over Bianchi backgrounds of types I-VII. They found universes which are highly inhomogeneous and "chaotic" at early times, but are transformed into clearly "recognizable" gravitational waves at late times.

Other types of metrics can be considered as exact *"gravitational solitons"* propagating on a cosmological background. These are usually obtained by applying the inverse scattering or "soliton" technique of Belinsky and Zakharov [344] to particular solutions of Einstein's equations as "seeds". For example, Carr and Verdaguer [345] found gravisolitons by applying the technique to the homogeneous Kasner seed. Similarly to previous work [343], their models are very inhomogeneous at early times, but evolve towards homogeneity in a wavelike manner at late times.

More recently, Belinsky [346], by applying a two-soliton inverse scattering technique to a Bianchi type VI_0 solution as a seed, constructed an intriguing solution which he christened as a *"gravitational breather"*, in analogy with the Gordon breather in the soliton theory of the sine-Gordon equation. Gravisolitons and antigravisolitons, characterized by an opposite topological charge, can be heuristically introduced and shown to have an attractive interaction. The breather is a bound state of the gravisoliton and antigravisoliton. Belinsky suggests that a time oscillating breather exists; but a later discussion [347] indicates that the oscillations quickly decay. Alekseev, by employing his generalization of the inverse scattering method to the Einstein–Maxwell theory, obtained exact electrovacuum solutions generalizing Belinsky's breather (see his review [348], containing a general introduction on exact solutions).

Verdaguer [349] prepared a very complete review of solitonic solutions admitting two spacelike Killing vector fields, with the main emphasis on cosmological models. Among various aspects of such solutions, he has noted the role of the Bel-Robinson superenergy tensor in the interpretation of cosmological metrics. This tensor and its higher-order generalizations has also been significantly used in estimates in the proofs of long-time existence theorems [39,340]. Recently, differential conservation laws for large perturbations of gravitational field with respect to a given curved background have been fomulated [350], which found an application in solving equations for cosmological perturbations corresponding to topological defects [351]. They should bring more light also on various solitonic models in cosmology.

13 Concluding Remarks

It is hoped that the preceding pages have helped to elucidate at least one issue: that in such a complicated nonlinear theory as general relativity, it is not possible to ask relevant questions of a general character without finding and thoroughly analyzing specific exact solutions of its field equations. The role of some of the solutions in our understanding of gravity and the universe has been so many-sided that to exhibit this role properly on even more than a hundred pages is not really feasible ...

Although we have concentrated on only (electro)vacuum solutions, there remains a number of such solutions that have also played some role in various contexts, but, owing to the absence of additional space and time, or the presence of the author's ignorance, have not been discussed. Tomimatsu-Sato solutions and their generalizations, static plane and cylindrical metrics, and some algebraically special solutions are examples.

In his review of exact solutions, Ehlers [56] wrote 35 years ago that "it seems desirable to construct material sources for vacuum solutions", and 30 years later Bonnor [64], in his review, expressed a similar view. In the above we have noted only some of the thin disk sources of static and stationary spacetimes in Sect. 6. To find physically reasonable material sources for many of the known vacuum solutions remains a difficult open task. In order to make solutions of Einstein's equations with the right-hand side more tractable, one is often tempted to sacrifice realism and consider materials, again using Bondi's phraseology, which are not easy to buy in the shops. Nevertheless, there are solutions representing spacetimes filled with matter which would certainly belong in a more complete discussion of the role of exact solutions.

For example, one of the simplest, the spherically symmetric Schwarzschild interior solution with an incompressible fluid as matter source, modelling "a star of uniform density", gives surprisingly good estimates of an upper bound on the masses of neutron stars; on a more general level, it supplies an instructive example of relativistic hydrostatics [18]. Many other spherical perfect fluid solutions are listed in [61]. The proof of a very plausible fact that any equilibrium, isolated stellar model which is nonrotating must be spherically symmetric, was finally completed in [352] and [353]. Physically more adequate spherically symmetric static solutions with collisionless matter described by the Boltzmann (Vlasov) equation have been studied [101] (yielding, for example, arbitrarily large central redshifts); and some of their aspects have been recently reviewed from a rigorous, mathematical point of view [354]. Going over to the description of matter in terms of physical fields, we should mention the first spherically symmetric regular solutions of the Einstein–Yang–Mills equations ("non-Abelian solitons" discovered by Bartnik and McKinnon [355] in 1988), and non-Abelian black holes with "hair", which were found soon afterwards. They stimulated a remarkable activity in the search for models in which gravity is coupled with Yang–Mills, Higgs, and Skyrmion fields. Very recently these solutions have been surveyed in detail in the review by Volkov and Gal'tsov [356].

The role of the standard FRW cosmological models on the development of relativity and cosmology can hardly be overemphasized. As for two more recent examples of this influence let us just recall that the existence of cosmological horizons in these models was one of the crucial points which inspired the birth of inflationary cosmology (see e.g. [28]); and the very smooth character of the initial singularity has led Penrose [102] to formulate his Weyl curvature hypothesis, related to a still unclear concept of gravitational entropy. Homogeneous but anisotropic Bianchi models filled with perfect fluid are extensively analyzed in [311]. Very recent studies of Bianchi models with collisionless matter [357] reveal how the matter content can qualitatively alter the character of the model.

A number of Bianchi models approach self-similar solutions. Perfect fluid solutions admitting a homothetic vector, which in this case implies both geometrical and physical self-similarity, have been reviewed most recently by Carr and Coley [358]. In their review various astrophysical and cosmological applications of such solutions are also discussed. Self-similar solutions have played a crucial role in the critical phenomena in gravitational collapse. Since their discovery by Choptuik in 1993, they have attracted much effort, which has revealed quite unexpected facts. In [358] these phenomena are analyzed briefly. For a more comprehensive review, see [359].

Self-similar, spherically symmetric solutions have been very relevant in constructing examples of the formation of naked singularities in gravitational collapse (see [358] for a brief summary and references). In particular, the inhomogeneous, spherically symmetric Lemaître–Bondi–Tolman universes containing dust have been employed in this context. Solutions with null dust should be mentioned as well, especially the spherically symmetric Vaidya solutions: imploding spherical null-dust models have been constructed in which naked singularities arise at their centre (see [32] for summary and references).

The Lemaître–Bondi–Tolman models are the most frequently analyzed inhomogeneous cosmological models which contain the standard FRW dust models as special cases (see e.g. [28,32]). In his recent book Krasiński [360] has compiled and discussed most if not all of these exact inhomogeneous cosmological solutions found so far which can be viewed as "exact perturbations" of the FRW models.

Many solutions known already still wait for their role to be uncovered. The role of many others may forever remain just in their "being". However, even if new solutions of a "Kerr-like significance" will not be obtained in the near future, we believe that one should not cease in embarking upon journeys for finding them, and perhaps even more importantly, for revealing new roles of solutions already known. The roads may not be easy, but with todays equipment like Maple or Mathematica, the speed is increasing. Is there another so explicit way of how to learn more about the rich possibilities embodied in Einstein's field equations?

The most remarkable figure of Czech symbolism, Otokar Březina (1868-1929) has consoling words for those who do not meet the "Kerr-type" metric on the road: "Nothing is lost in the world of the spirit; even a stone thrown away may find its place in the hands of a builder, and a house in flames may save the life of someone who has lost his way...".

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