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Robinson–Trautman spacetimes with an electromagnetic field in higher dimensions

Marcello Ortaggio[1], **Jiří Podolský**[2] **and Martin Žofka**[2]

[1] Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain

[2] Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

E-mail: ortaggio'AT'ffn.ub.es, podolsky'AT'mbox.troja.mff.cuni.cz and zofka'AT'mbox.troja.mff.cuni.cz

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Abstract

We investigate higher dimensional Robinson–Trautman spacetimes with an electromagnetic field aligned with the hypersurface orthogonal, non-shearing, expanding geodesic null congruence. After integrating the system of Einstein–Maxwell equations with an arbitrary cosmological constant, we present the complete family of solutions. In *odd* spacetime dimensions they represent (generalized) Reissner–Nordström–de Sitter black holes. The event horizon (more generically, the transverse space) may be any Einstein space, and the full metric is specified by three independent parameters related to mass, electric charge and cosmological constant. These solutions also exhaust the class of Robinson–Trautman spacetimes with an aligned Maxwell–Chern–Simons field (the CS term must vanish because of the alignment assumption and Einstein equations). In *even* dimensions an additional magnetic 'monopole-like' parameter is also allowed provided now the transverse space is an (almost-) Kähler Einstein manifold. The Weyl tensor of all such solutions is of algebraic type D. We also consider the possible inclusion of aligned pure radiation.

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1. Introduction

In general relativity, the study of ray optics has played a major role in the construction, interpretation and invariant classification of exact solutions (see, e.g., [1] for a review and for original references). This applies, in particular, to solutions representing gravitational radiation. During the golden age of theoretical studies of exact radiative spacetimes, Robinson and Trautman introduced and investigated $D = 4$ dimensional Lorentzian geometries that admit a geodesic, non-twisting, non-shearing, expanding null congruence [2, 3]. The Robinson–Trautman family is by now one of the fundamental classes of exact solutions to Einstein's field equations in vacuum and with principal matter fields such as pure radiation or an electromagnetic field [1]. It includes a number of well-known spacetimes ranging from static black holes and the Vaidya solution to the C-metric and other radiative solutions. Noticeably, the Goldberg–Sachs theorem [1] implies that Robinson–Trautman geometries are algebraically special (at least in vacuum and with 'sufficiently aligned' matter fields), since they are non-shearing. In fact, explicit vacuum solutions of all special Petrov types are known [1–3].

The geometric optics approach was naturally developed in the framework of $D = 4$ general relativity. On the other hand, in recent years string theory and specific extra-dimension scenarios have stimulated the investigation of gravity in more than four spacetime dimensions. It is thus now interesting to consider possible extensions of the above concepts to arbitrary (higher) dimensions, and their relation to the $D > 4$ classification of the Weyl tensor [4]. In [5–9], various general aspects of geometric optics in $D > 4$ dimensions (which evades the standard $D = 4$ Goldberg–Sachs theorem in many ways) have been analyzed. In [10], the Robinson–Trautman family of solutions has been extended to higher dimensions in the case of empty space possibly with a cosmological constant and in the case of aligned pure radiation. The authors pointed out important differences with respect to the $D = 4$ case for vacuum spacetimes (see also [9, 11]). However, from a higher dimensional perspective one would also be interested in theories that incorporate electromagnetic fields. It is thus the purpose of this paper to study Robinson–Trautman spacetimes in the higher dimensional Einstein–Maxwell theory (for any value of the cosmological constant). For simplicity, we will focus on aligned fields. In $D \geqslant 5$ odd dimensions, we shall also consider the inclusion of an additional Chern–Simons term, which gives rise, e.g., to the bosonic sector of five-dimensional minimal (gauged) supergravity.

The paper is organized as follows. In section 2, we present the line element of generic Robinson–Trautman spacetimes [10] and study purely algebraic properties of an aligned Maxwell field. In section 3, we proceed by integrating systematically the full set of Einstein–Maxwell equations within such a setting. We summarize the resulting spacetimes and discuss some special cases in section 4. Concluding remarks are given in section 5, and some technical details in appendix A. Throughout the paper, we focus on $D > 4$ dimensions, and well-known results in the special case $D = 4$ are summarized in appendix B.

2. Robinson–Trautman geometry and aligned Maxwell fields

As shown in [10], the general line element for any D-dimensional spacetime which admits a non-twisting, non-shearing but expanding congruence [5, 6] generated by the geodesic null vector field $\boldsymbol{k}$ can be written as

$$\mathrm{d}s^2 = g_{ij}(\mathrm{d}x^i + g^{ri}\mathrm{d}u)(\mathrm{d}x^j + g^{rj}\mathrm{d}u) - 2\,\mathrm{d}u\,\mathrm{d}r - g^{rr}\mathrm{d}u^2. \tag{1}$$

Here, $u = \text{const}$ are the null hypersurfaces to which $\boldsymbol{k}$ is normal, r is the affine parameter along the geodesics generated by $\boldsymbol{k} = \partial_r$, and $x \equiv (x^i) \equiv (x^1, x^2, \ldots, x^{D-2})$ are spatial coordinates on a 'transverse' $(D-2)$-dimensional Riemannian manifold $\mathcal{M}_{(D-2)}$. The metric functions

$$g^{ri} = g^{ij}g_{uj}, \qquad g^{rr} = -g_{uu} + g^{ij}g_{ui}g_{uj}, \qquad \text{and} \qquad g_{ui} = g^{rj}g_{ij}, \tag{2}$$

may depend arbitrarily on (x, u, r), while the spatial components g_{ij} have the factorized form $g_{ij} = p^{-2}(x, u, r)h_{ij}(x, u)$, and $g_{rr} = 0 = g_{ri}$ (note that $\det g_{ij} = -\det g_{\alpha\beta}$). The expansion of $\boldsymbol{k}$ is given by $\theta \equiv k^{\alpha}_{;\alpha}/(D-2) = -(\ln p)_{,r}$, which we assume to be non-vanishing. The above metric is invariant under the coordinate transformations

$$x^i = x^i(\tilde{x}, \tilde{u}), \qquad u = u(\tilde{u}), \qquad r = r_0(\tilde{x}, \tilde{u}) + \tilde{r}/\dot{u}(\tilde{u}). \tag{3}$$

The next step is to impose Einstein's equations with a suitable energy–momentum tensor in the above Robinson–Trautman class. In the present paper we concentrate on spacetimes with Maxwell fields *aligned with the geometrically privileged null vector field* $\boldsymbol{k}$, characterized by

$$F_{\alpha\beta}k^{\beta} = \mathcal{N}k_{\alpha}, \tag{4}$$

where $\mathcal{N}$ is an arbitrary function. In the coordinate system introduced above this means

$$F_{ri} = 0 = F^{ui}, \qquad F_{ru} = \mathcal{N} = F^{ur}, \tag{5}$$

with components F_{ij}, F_{ui} (or $F^{ij} = g^{ik}g^{jl}F_{kl}$, $F^{ir} = -\mathcal{N}g^{ri} + g^{ij}F_{uj} - g^{rk}g^{ij}F_{kj}$) still arbitrary. Consequently, $F^{u}{}_{r} = F^{u}{}_{i} = F^{i}{}_{r} = 0 = F_{r}{}^{u} = F_{r}{}^{i} = F_{i}{}^{u}$ and $F^{r}{}_{r} = -F^{u}{}_{u} = \mathcal{N} = -F_{r}{}^{r} = F_{u}{}^{u}$. For the corresponding energy–momentum tensor of the electromagnetic field

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\mu}F_{\beta}{}^{\mu} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right), \tag{6}$$

we find

$$T_{rr} = T_{ri} = 0, \tag{7}$$

with the remaining components $T_{ij}, T_{ur}, T_{ui}, T_{uu}$ in principle non-trivial and specified below. Note that the trace

$$T^{\mu}_{\mu} = \frac{4-D}{16\pi}F_{\mu\nu}F^{\mu\nu} = \frac{4-D}{16\pi}(F_{ij}F^{ij} - 2\mathcal{N}^2) \tag{8}$$

is generally non-zero unless $D = 4$. The field equations $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$ including an arbitrary *cosmological constant* Λ thus take the form

$$R_{\alpha\beta} = \frac{2}{D-2}\Lambda g_{\alpha\beta} + 8\pi T_{\alpha\beta} + \frac{1}{2}\frac{D-4}{D-2}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}, \tag{9}$$

which will now be solved together with source-free Maxwell equations $F_{[\alpha\beta;\gamma]} = 0$ and $F^{\mu\nu}{}_{;\nu} = 0$, and their Chern–Simons modification in odd dimensions.

3. Integration of the Einstein–Maxwell field equations

3.1. Equations $R_{rr} = 0$ and $R_{ri} = 0$

Due to (7) and (1), the Einstein equations (9) for R_{rr} and R_{ri} are exactly the same as in the vacuum case [10]. Consequently, for the Robinson–Trautman class of spacetimes, we obtain $p = r^{-1}$ (up to a trivial rescaling of h_{ij} by a function of (x, u)) [10], i.e.

$$g_{ij} = r^2 h_{ij}(x, u), \qquad g^{ri} = \mathrm{e}^i(x, u) + r^{1-D}f^i(x, u), \tag{10}$$

where h_{ij}, which is the transverse spatial part of the metric, and e^i, f^i are arbitrary functions of x and u. The r-dependence of the metric functions g_{ij}, g^{ri} is now fixed. Thanks to (10), we can write

$$-\det g_{\alpha\beta} = r^{2(D-2)}h, \tag{11}$$

where

$$h = h(x, u) \equiv \det h_{ij}(x, u). \tag{12}$$

We further note that the expansion of the congruence $\boldsymbol{k}$ is now given by $\theta = 1/r$.

3.2. Maxwell equations (step one)

To determine the r-dependence of the components $F_{\mu\nu}$, we now employ Maxwell's equations. With equation (5), the 'geometrical' equations $F_{[\alpha\beta;\gamma]}=0$, equivalent to $F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0$, imply

$$F_{ij,r}=0, \tag{13}$$

$$F_{ui,r}=-\mathcal{N}_{,i}, \tag{14}$$

$$F_{ij,u}=F_{uj,i}-F_{ui,j}, \tag{15}$$

$$F_{[ij,k]}=0. \tag{16}$$

In view of (11), the 'dynamical' equations $F^{\mu\nu}{}_{;\nu}=(-\det g_{\alpha\beta})^{-\frac{1}{2}}\left((-\det g_{\alpha\beta})^{\frac{1}{2}}F^{\mu\nu}\right)_{,\nu}=0$ are

$$(r^{D-2}\mathcal{N})_{,r}=0, \tag{17}$$

$$\sqrt{h}(r^{D-2}F^{ir})_{,r}=-r^{D-2}(\sqrt{h}F^{ij})_{,j}, \tag{18}$$

$$(\sqrt{h}F^{ir})_{,i}=-(\sqrt{h}\mathcal{N})_{,u}. \tag{19}$$

From (13) we observe that the components F_{ij} are independent of r,

$$F_{ij}=F_{ij}(x,u). \tag{20}$$

Using (17), we find

$$F_{ru}=\mathcal{N}=r^{2-D}Q(x,u), \tag{21}$$

with $Q(x,u)$ arbitrary. Using this result and (14), we obtain

$$F_{ui}=r^{3-D}\frac{Q_{,i}}{D-3}-\xi_i(x,u), \tag{22}$$

with $\xi_i(x,u)$ being some functions of x and u. Thus we found the r-dependence of all electromagnetic field components. In particular, the invariant $F_{\mu\nu}F^{\mu\nu}$ of the Maxwell field is

$$F_{\mu\nu}F^{\mu\nu}=r^{-4}F^2-r^{2(2-D)}2Q^2, \tag{23}$$

where we have defined

$$F^2(x,u)\equiv F_{ik}F_{jl}h^{ij}h^{kl}, \tag{24}$$

and (from now on) h^{ij} denotes the inverse of h_{ij}. We always have $F^2\geqslant 0$ (with $F^2=0\Leftrightarrow F_{ij}=0$) because, in an orthonormal frame, $F^2=\sum_{i,j}F^2_{(i)(j)}$.

Substituting (22) into (15), we get

$$F_{ij,u}=\xi_{i,j}-\xi_{j,i}, \tag{25}$$

while equation (16) is unchanged. Finally, if expanded in the powers of r using the previous results, the remaining Maxwell equations (18) and (19) yield the following set of relations in $D>4$:

$$Qf^i=0, \tag{26}$$

$$F_{jk}f^k=0, \tag{27}$$

$$Q_{,j}=0, \tag{28}$$

$$\xi_j-F_{jk}\,\mathrm{e}^k=0, \tag{29}$$

$$(\sqrt{h}\, h^{ik} h^{jl} F_{kl})_{,j} = 0, \tag{30}$$

$$(\sqrt{h} Q)_{,u} - (\sqrt{h} Q\, \mathrm{e}^i)_{,i} = 0. \tag{31}$$

Relations (25)–(31) and (16) place restrictions on the admissible electromagnetic fields (20)–(22) in Robinson–Trautman spacetimes. We shall return to the implications of these constraints after we employ the next Einstein equation in subsection 3.4. For the special case $D = 4$, see appendix B.

The above results already have an important consequence. Namely, one of the necessary conditions for having a *null Maxwell field* reads $F_{\mu\nu}F^{\mu\nu} = 0$. In view of the r-dependence specified by (23), one finds immediately that for $D > 4$ this requires $F_{ij} = 0 = Q$ and thus $F_{ru} = 0$. Substituting into (18), this also gives $F_{ui} = 0$, that is, a vanishing electromagnetic field. Hence *higher dimensional Robinson–Trautman spacetimes do not admit aligned null Maxwell fields*, as opposed to the $D = 4$ case [1, 3]. This is an explicit example of the result of [11] that higher dimensional null Maxwell fields cannot have expanding rays with vanishing shear.

3.3. Chern–Simons term

In theories which include a Chern–Simons term, formulated in odd spacetime dimensions ($D = 2n + 1$), the set of geometrical equations (13)–(16) is unchanged ($\mathrm{d}\boldsymbol{F} = 0$). On the other hand, the dynamical set now contains an additional term on the rhs (cf, e.g., [12])

$$(\sqrt{-\det g_{\alpha\beta}}\, F^{\mu\nu})_{,\nu} = -\lambda \epsilon^{\mu\gamma\delta\ldots\sigma\tau} \underbrace{F_{\gamma\delta}\ldots F_{\sigma\tau}}_{n \text{ times}}, \tag{32}$$

where λ is a coupling constant. Note that $F_{ri} = 0$ thanks to the alignment condition (5), so that the Chern–Simons term does not affect equation (32) with $\mu = u$, which thus takes again the form (17). In fact, this is the only dynamical equation we used in the discussion above, so that equations (20)–(25) apply also in the Chern–Simons case. Moreover, if one assumes that also $F_{ij} = 0$ (no 'magnetic' field), the Chern–Simons term then vanishes identically for any odd D. In particular, null fields are thus ruled out again. Further analysis will be simpler after looking at the next Einstein equation.

3.4. Equation $R_{ij} = \frac{2}{D-2}\Lambda g_{ij} + 8\pi T_{ij} + \frac{1}{2}\frac{D-4}{D-2} g_{ij} F_{\mu\nu}F^{\mu\nu}$

The T_{ij} components of the energy–momentum tensor are $T_{ij} = \frac{1}{8\pi} r^{2(3-D)} Q^2 h_{ij} + r^{-2}\frac{1}{4\pi}\left(F_{ik}F_{jl}h^{kl} - \frac{1}{4}F^2 h_{ij}\right)$. Using (20) and (23), one can separate terms with different r-dependences and thus proceed to integrate the corresponding field equation. As detailed in appendix A, one finds the important simplification

$$f^i = 0 = \mathrm{e}^i, \tag{33}$$

so that $g^{ri} = 0$. Then, the Robinson–Trautman metric is simplified considerably and reads

$$\mathrm{d}s^2 = r^2 h_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j - 2\,\mathrm{d}u\, \mathrm{d}r - g^{rr}\, \mathrm{d}u^2, \tag{34}$$

where (cf equations (A.2) and (A.3)) the coefficient g^{rr} is explicitly given by

$$g^{rr} = \frac{\mathcal{R}}{(D-2)(D-3)} + \frac{2(\ln\sqrt{h})_{,u}}{D-2} r - \frac{2\Lambda}{(D-2)(D-1)} r^2 - \frac{\mu}{r^{D-3}} + \frac{2Q^2}{(D-2)(D-3)}\frac{1}{r^{2(D-3)}} - \frac{F^2}{(D-2)(D-5)}\frac{1}{r^2}. \tag{35}$$

The function $\mu(x, u)$ (which renames the c_6 of appendix A) is arbitrary.

The $(D-2)$-dimensional spatial metric h_{ij} is constrained by (A.4) and (A.5) (with $\mathrm{e}^i = 0$):

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D-2} h_{ij}, \tag{36}$$

$$h_{ij,u} = \frac{2}{D-2} h_{ij} (\ln \sqrt{h})_{,u}. \tag{37}$$

As in [10], relation (36) tells us that at any given $u = u_0 = \text{const}$, the spatial metric $h_{ij}(x, u_0)$ must describe an Einstein space $(\mathcal{M}_{(D-2)}, h_{ij})$. For $D > 4$ this implies [10] that the spatial Ricci scalar $\mathcal{R}$ can *only* depend on the coordinate u (and that in the particular case $D = 5$ the metric $h_{ij}(x, u_0)$ corresponds to a 3-space of constant curvature). Equation (37) 'controls' the parametric dependence of $h_{ij}(x, u)$ on u, and can easily be integrated to obtain $h_{ij} = h^{1/(D-2)} \gamma_{ij}(x)$. Consequently, $h \equiv \det h_{ij} = h \det \gamma_{ij}$, so that the matrix γ_{ij} must be unimodular. Considering equation (12),

$$h_{ij} = \frac{\gamma_{ij}(x)}{P^2(x,u)} \qquad \text{where} \quad \det \gamma_{ij} = 1, \qquad P^{-2} = h^{1/(D-2)}. \tag{38}$$

The spatial metric $h_{ij}(x, u)$ can thus depend on the coordinate u only via the conformal factor P^{-2}.

It follows from (A.8) and the subsequent discussion that the Maxwell field is constrained by

$$F^2 = 0 \qquad\qquad (D = 2n+1 \text{ odd}), \tag{39}$$

$$h_{ij} F^2 = (D-2) F_{ik} F_{jl} h^{kl} \qquad (D = 2n+2 \text{ even}). \tag{40}$$

3.5. Maxwell equations (step two)

Let us now return to the Maxwell equations. As noted above, for any odd D we have $F^2 = 0$, i.e. $F_{ij} = 0$. Thanks to this significant simplification in odd dimensions, *the Chern–Simons term in equation* (32) *vanishes identically* (cf the discussion in subsection 3.3), and from now on we can thus study both Maxwell and Maxwell–Chern–Simons theories in a unified way.

Since now $\mathrm{e}^i = 0 = f^i$, cf (33), the dynamical Maxwell equations (26)–(29) simplify to $Q_{,j} = 0$, $\xi_j = 0$. In view of (22), (21) and (25), we see that for $D > 4$

$$F_{ui} = 0, \qquad F_{ru} = \frac{Q(u)}{r^{D-2}}, \qquad F_{ij} = F_{ij}(x). \tag{41}$$

The only remaining Maxwell equations (16), (30) and (31) read

$$F_{[ij,k]} = 0, \tag{42}$$

$$(\sqrt{h}\, h^{ik} h^{jl} F_{kl})_{,j} = 0, \tag{43}$$

$$(\sqrt{h}\, Q)_{,u} = 0. \tag{44}$$

Note that in even dimensions (cf (39)) relations (42) and (43) are effective source-free Maxwell equations for the $(D-2)$-dimensional 'spatial' (magnetic) field F_{ij} *in the Riemannian geometry of* h_{ij}. That is, the 2-form

$$\tilde{\boldsymbol{F}} \equiv \tfrac{1}{2} F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \tag{45}$$

must be closed ($\mathrm{d}\tilde{\boldsymbol{F}} = 0$) and coclosed ($\mathrm{d}^*\tilde{\boldsymbol{F}} = 0$) in $(\mathcal{M}_{(D-2)}, h_{ij})$. However, $\tilde{\boldsymbol{F}}$ must also obey the extra constraint (40), that is the last remnant of the Einstein equation for R_{ij}.

Recalling the block-diagonal canonical form $\tilde{\boldsymbol{F}} = c_{12}\boldsymbol{m}^{(1)} \wedge \boldsymbol{m}^{(2)} + c_{34}\boldsymbol{m}^{(3)} \wedge \boldsymbol{m}^{(4)} + \cdots$ of a generic even-dimensional antisymmetric matrix in an adapted orthonormal coframe $(\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(D-2)})$ of h_{ij}, the condition (40) requires that in such a coframe one has in fact

$$\tilde{\boldsymbol{F}} = \frac{F}{\sqrt{D-2}}(\boldsymbol{m}^{(1)} \wedge \boldsymbol{m}^{(2)} + \boldsymbol{m}^{(3)} \wedge \boldsymbol{m}^{(4)} + \cdots + \boldsymbol{m}^{(D-3)} \wedge \boldsymbol{m}^{(D-2)})$$
$$(D = 2n + 2 \text{ even}). \tag{46}$$

This special form of $\tilde{\boldsymbol{F}}$ implies

$$^*\tilde{\boldsymbol{F}} = \frac{(2n)^{(n-2)/2}}{(n-1)!} F^{-(n-2)} \underbrace{\tilde{\boldsymbol{F}} \wedge \tilde{\boldsymbol{F}} \wedge \ldots \wedge \tilde{\boldsymbol{F}}}_{(n-1)\text{ times}}, \tag{47}$$

where the *-duality and $\wedge$-product are (in this paragraph only) those of $(\mathcal{M}_{(D-2)}, h_{ij})$. Hence, for $n > 2$ $(D > 6)$ imposing that the 2-form $\tilde{\boldsymbol{F}}$ is simultaneously closed and coclosed requires $F_{,i} = 0$. For $n = 2$ $(D = 6)$, instead, $\tilde{\boldsymbol{F}}$ is self-dual, therefore if it is closed it is also automatically coclosed, without any restriction on F. We will recover the same results explicitly also below using the Einstein equations, cf equation (51).

Note also that if $\tilde{\boldsymbol{F}}$ is supposed to be regular and non-zero on $\mathcal{M}_{(D-2)}$, then equation (40) requires that the Einstein space $(\mathcal{M}_{(D-2)}, h_{ij})$ is an almost-Hermitian (possibly, Hermitian) manifold [13] with the almost-complex structure $J^i_j = |F|^{-1}(D-2)^{1/2}F^i_j$. In view of the previous comments, for $D = 2n + 2 > 6$ the Maxwell equations imply that the 2-form $J_{ij} = h_{ik}J^k_j$ associated with the almost-complex structure is closed, so that the transverse space is not only almost-Hermitian but actually almost-Kähler (possibly, Kähler).

3.6. Equation $R_{ur} = -\frac{2}{D-2}\Lambda + 8\pi T_{ur} - \frac{1}{2}\frac{D-4}{D-2}F_{\mu\nu}F^{\mu\nu}$

The Ricci tensor component R_{ur} for metric (34) reads $R_{ur} = \frac{1}{2}r^{2-D}\left(r^{D-2}g^{rr}_{,r}\right)_{,r} - r^{-1}(\ln\sqrt{h})_{,u}$, see [10]. Substituting expression (35) we obtain $R_{ur} = -\frac{2\Lambda}{D-2} + r^{2(2-D)}\frac{D-3}{D-2}2Q^2 + r^{-4}\frac{1}{D-2}F^2$. Using (23) and $T_{ur} = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi}r^{2(2-D)}Q^2$ we observe that the corresponding field equation is automatically satisfied in any dimension.

3.7. Equation $R_{ui} = 8\pi T_{ui}$

For the energy–momentum tensor, using (6) and (41), we find $T_{ui} = 0$. The Ricci tensor component R_{ui} for metric (34) and (35), using $Q_{,i} = 0$ and relation (37), is

$$R_{ui} = r^{-1}\frac{(D-4)\mathcal{R}_{,i}}{2(D-2)(D-3)} + r^{2-D}\frac{\mu_{,i}}{2} - r^{-3}\frac{(D-6)(F^2)_{,i}}{2(D-2)(D-5)}. \tag{48}$$

Comparing the coefficients of different powers of r, we obtain immediately the following conditions:

$$(D-4)\mathcal{R}_{,i} = 0, \tag{49}$$
$$\mu_{,i} = 0, \tag{50}$$
$$(D-6)(F^2)_{,i} = 0. \tag{51}$$

Therefore, (for $D > 4$) the functions $\mathcal{R}$ and μ must be independent of the spatial coordinates,

$$\mathcal{R} = \mathcal{R}(u), \qquad \mu = \mu(u), \tag{52}$$

and we further find

$$F^2 = F^2(u) \qquad \text{for} \quad D \neq 6, \text{even}, \tag{53}$$

with $F^2 = 0$ in any odd D. For $D = 6$, equation (51) is satisfied identically, with F^2 remaining a function of both x and u. In $D = 4$, corresponding to the standard general relativity, equation (49) is an identity, so that one can have a much more general function $\mathcal{R}(x, u)$; equations (50) and (51) are also modified, cf appendix B.

3.8. Equation $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi T_{uu} + \frac{1}{2}\frac{D-4}{D-2}F_{\mu\nu}F^{\mu\nu}g_{uu}$

As explained in appendix A, after some calculations this field equation requires

$$\mu_{,u} = (D-1)(\ln P)_{,u}\mu \qquad (D \neq 4, 6), \tag{54}$$

or

$$\mu_{,u} = 5(\ln P)_{,u}\mu - \tfrac{1}{16}\Delta(F^2) \qquad (D = 6), \tag{55}$$

in the two distinct cases $D \neq 6$ and $D = 6$. We analyze these separately in the next section.

4. Summary and discussion

Starting from the general Robinson–Trautman geometric ansatz (1), in the preceding section we have imposed all the constraints following from the Einstein–Maxwell equations. The resulting metric takes a simplified form (34), which is fully specified by the single function g^{rr} in equation (35), along with the transverse Einstein geometry $(\mathcal{M}_{(D-2)}, h_{ij})$, as determined by (36). The specific form of the parameters and functions entering equation (35) and possible constraints on the Einstein metrics h_{ij} depend on the number of spacetime dimensions, as we will discuss below.

4.1. Even dimensions: the generic case $(D \neq 6)$

For an arbitrary even $D > 4$ such that $D \neq 6$, by differentiating any of equations (A.12), (A.13), (A.15) with respect to the spatial coordinates, we obtain (recall that $Q = Q(u)$, $F = F(u)$, $\mu = \mu(u)$)

$$(\ln P)_{,ui} = 0, \tag{56}$$

unless $\mu = 0$ and $Q = 0 = F^2$, which is the exceptional vacuum spacetime discussed in [10, 11], and [9]. Equation (56) can be integrated immediately, yielding the factorized form $P(x, u) = P(x)U(u)$, where P and U are arbitrary functions. Without loss of generality, we can set $U = 1$ by a suitable coordinate transformation of the form $u = u(\tilde{u})$, $r = \tilde{r}/\dot{u}(\tilde{u})$, under which the form of the metric (34), (35) is invariant and the individual metric functions are reparametrized as follows:

$$\tilde{P} = P\dot{u}, \qquad \tilde{\mathcal{R}} = \mathcal{R}\dot{u}^2, \qquad \tilde{\mu} = \mu\dot{u}^{D-1}, \qquad \tilde{F}^2 = F^2\dot{u}^4, \qquad \tilde{Q} = Q\dot{u}^{D-2}. \tag{57}$$

Choosing $\dot{u} = 1/U$ and dropping tildes, we obviously achieve

$$P(x, u) = P(x), \tag{58}$$

and considering equations (A.12), (A.13), and (A.15), we thus have

$$\mu = \text{const}, \qquad Q = \text{const}, \qquad F^2 = \text{const}. \tag{59}$$

Considering (52) and the fact that $\mathcal{R}$ is the Ricci scalar associated with the spatial metric $h_{ij} = h_{ij}(x) = P^{-2}(x)\gamma_{ij}(x)$, cf (38), which now does not involve u, we conclude

$$\mathcal{R} = \text{const.} \tag{60}$$

In addition, we can now always set the constant term $K \equiv \mathcal{R}/(D-2)(D-3)$ in the metric (35) to $K = \pm 1, 0$ using the remaining scaling freedom (57), namely $u \to \mathcal{C}u, r \to r/\mathcal{C}, x^i \to \mathcal{C}^{1/(D-2)}x^i$.

To summarize, the explicit form of even-dimensional ($D \neq 6$) Robinson–Trautman spacetimes with an aligned electromagnetic field and possibly a cosmological constant is

$$\mathrm{d}s^2 = r^2 h_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j - 2\,\mathrm{d}u\,\mathrm{d}r - 2H(r)\,\mathrm{d}u^2. \tag{61}$$

The function $2H \equiv g^{rr} = -g_{uu}$ and the Maxwell field are given by

$$2H = K - \frac{2\Lambda}{(D-2)(D-1)}r^2 - \frac{\mu}{r^{D-3}} + \frac{2Q^2}{(D-2)(D-3)}\frac{1}{r^{2(D-3)}} - \frac{F^2}{(D-2)(D-5)}\frac{1}{r^2}, \tag{62}$$

$$\boldsymbol{F} = \frac{Q}{r^{D-2}}\,\mathrm{d}r \wedge \mathrm{d}u + \tfrac{1}{2}F_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \qquad (D \neq 6, \text{even}), \tag{63}$$

where $K = \pm 1, 0$, and μ, Q, F are constants.[3] The transverse manifold $(\mathcal{M}_{(D-2)}, h_{ij})$ is a *Riemannian Einstein space*, see (36), with the Ricci scalar normalized as $\mathcal{R} = K(D-2)(D-3)$. If this is taken to be compact, these solutions admit a *black hole interpretation*.[4] Obviously, Λ is the cosmological constant, μ parametrizes the mass, and Q is the electric charge. If the magnetic term F_{ij} is non-zero, $\left(\mathcal{M}_{(D-2)}, h_{ij}, J^i_j\right)$ must be an *almost-Kähler Einstein* manifold (cf subsection 3.5). The almost-complex structure gives F_{ij} (up to a constant factor), which thus satisfies the 'effective' $(D-2)$-dimensional Maxwell equations (42) and (43).

Note that when F is non-zero and $D > 4$, $(\mathcal{M}_{(D-2)}, h_{ij})$ cannot be a sphere of constant curvature,[5] as one would require, e.g., for an asymptotically flat spacetime. By contrast, spherically symmetric magnetic monopole solutions of the Einstein–Yang–Mills equations have been recently found in [14]. The line element given in [14] coincides with our equations (61), (62) in the special subcase $K = 1, Q = 0$, except that $(\mathcal{M}_{(D-2)}, h_{ij})$ is a round sphere there. Note, however, that even in that case the large-r behavior of the F term in (62) does spoil the standard 'good properties' of an asymptotically simple spacetime [14] (in asymptotically flat spaces in D-dimensions the 'mass term' behaves as $1/r^{D-3}$, cf [21]—terms with a slower fall-off give infinite Komar integrals).

From the above form (63) of the Maxwell field, it is clear that it is of type D [22] with principal null directions given by

$$\boldsymbol{k} = \partial_r, \qquad \boldsymbol{l} = \partial_u - H\partial_r. \tag{64}$$

[3] In the special case $D = 4$ (see also appendix B) the electric and magnetic monopole terms in Q and F become indistinguishable in the metric. This corresponds to the well-known fact that in $D = 4$ Einstein–Maxwell gravity all solutions are determined only up to a constant duality rotation of the electromagnetic field.

[4] There is clearly a curvature singularity at $r = 0$, as one can see, e.g., from the Ricci scalar $R = \frac{2D}{D-2}\Lambda + \frac{D-4}{D-2}F^{\mu\nu}F_{\mu\nu}$ (with equation (23)). In general, however, this is hidden behind an event horizon. Various Killing horizons are in fact possible, as determined by the roots of the function $H(r)$. The appropriate parameter range for the existence (and number) of such roots has been discussed in detail in [14] (cf also [15] for the case $F = 0$). For $\Lambda \leqslant 0$ the asymptotic region is static, whereas for $\Lambda > 0$ the Killing vector ∂_u becomes spacelike at large r (like in de Sitter space).

[5] More generally, it is an old result that Kähler manifolds of constant (Riemannian) curvature must be flat in $2n > 2$ real dimensions [16]. It has been demonstrated more recently that this applies also to almost-Kähler manifolds (see [17] and references therein). In this context, it is also worth mentioning that the celebrated conjecture of [18] that almost-Kähler, Einstein, compact manifolds must be Kähler has been proven in the case of non-negative scalar curvature [19]. See, e.g., [17, 20] for some more general properties of almost-Kähler Einstein manifolds and for more references.

In addition, in view of (59) the line element (61) and (62) is a warped product of $(\mathcal{M}_{(D-2)}, h_{ij})$ with a two-dimensional Lorentzian factor. For such a type of warped spacetimes, the Weyl tensor is necessarily of type D, unless zero (type O) [9]. However, the latter case cannot occur here since, e.g., the Weyl component C_{ruru} reads

$$C_{ruru} = -(D-2)(D-3)\frac{\mu}{2r^{D-1}} + \frac{(2D-5)}{(D-1)}\frac{2Q^2}{r^{2(D-2)}} - \frac{(D-3)}{(D-1)(D-2)(D-5)}\frac{6F^2}{r^4}. \tag{65}$$

The above Robinson–Trautman spacetimes in $D > 4$ are thus of type D with WANDs given again by (64) (cf [9]). Conformal flatness requires $\mu = 0$ and $Q = 0 = F$, in particular vacuum spacetimes [10], so that the only possible conformally flat metrics are of constant curvature.

Note finally that when $F_{ij} = 0$ (i.e., $F = 0$), these solutions are *electrically charged black holes*. In the simplest case when $(\mathcal{M}_{(D-2)}, h_{ij})$ is a round sphere, one obtains the well-known asymptotically flat/(A)dS spacetimes of [23]. However, $(\mathcal{M}_{(D-2)}, h_{ij})$ can now be *any* Einstein space (cf also [24], and see, e.g., [25, 26] for related discussions in the vacuum case $Q = 0 = F$). Stability properties of these black holes have been studied in [15].

4.1.1. An explicit example. For the sake of definiteness, as a simple example with $F \neq 0$ we can consider $(\mathcal{M}_{(D-2)}, h_{ij})$ as the Riemannian analog of Nariai-like solutions with geometries $S^2 \times S^2 \times \cdots$ or $H^2 \times H^2 \times \cdots$, namely

$$\begin{aligned} h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j &= \sum_{I=1}^{n}\left[\left(1-\epsilon\frac{\rho_I^2}{a^2}\right)\mathrm{d}\psi_I^2 + \left(1-\epsilon\frac{\rho_I^2}{a^2}\right)^{-1}\mathrm{d}\rho_I^2\right], \\ \tfrac{1}{2}F_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j &= \frac{F}{\sqrt{D-2}}\sum_{I=1}^{n}\mathrm{d}\psi_I\wedge\mathrm{d}\rho_I \qquad (D = 2n+2) \end{aligned} \tag{66}$$

where $\epsilon = +1$ or $\epsilon = -1$ (or $\epsilon = 0$, which gives a flat h_{ij}), a and F are constants, the scalar curvature is given by $K = \epsilon a^{-2}(2n-1)^{-1}$ (normalizable to $K = \epsilon$ if desired) and $D = 2n+2$ is the number of spacetime dimensions. Note that F_{ij} is covariantly constant in $(\mathcal{M}_{(D-2)}, h_{ij})$.

4.2. Odd dimensions

For odd D, as above one can reduce the line element to the form (61), (62). Since in odd dimensions $F = 0$ identically (i.e. $\tilde{\boldsymbol{F}} = 0$, see (45)), a complete solution of the Maxwell equations is now simply given by a purely electric 'radial' field

$$\boldsymbol{F} = \frac{Q}{r^{D-2}}\,\mathrm{d}r\wedge\mathrm{d}u \qquad (D\ \text{odd}). \tag{67}$$

As in even D with $F = 0$, these are again a generalization of the familiar Reissner–Nordström–de Sitter spacetimes [23, 24]: the standard Schwarzschild-type form

$$\mathrm{d}s^2 = -2H(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{2H(r)} + r^2h_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j, \qquad \boldsymbol{F} = \frac{Q}{r^{D-2}}\,\mathrm{d}r\wedge\mathrm{d}t, \tag{68}$$

is achieved via the transformation[6] $\mathrm{d}u = \mathrm{d}t - \mathrm{d}r/2H$. Recall also that these represent the only Robinson–Trautman solutions with an aligned electromagnetic field which obeys either the Maxwell or the Maxwell–Chern–Simons equations.

4.3. The special case $D = 6$

The even-dimensional $D = 6$ case is special in that F^2 may depend also on the spatial x coordinates, see equation (51). This fact has two consequences. First, one has to solve the more complicate equation (55). In addition, when $Q = 0$, one cannot conclude now that $P(x, u)$ takes the factorized form $P(x, u) = P(x)U(u)$. Let us discuss the two possible cases separately.

4.3.1. Factorized $P(x, u)$ (generic transverse space). Because of equations (A.12), this corresponds to the generic situation with $Q \neq 0$. In such a case we can arrive again at (58) so that $h_{ij} = h_{ij}(x)$, and equations (A.12) and (A.13) lead to

$$Q = \text{const}, \qquad F^2 = F^2(x). \tag{69}$$

In addition, equation (55) simplifies to $\mu_{,u}(u) = -\frac{1}{16}\Delta(F^2)(x)$ which requires *both terms to be constant*. By integration we obtain $\mu(u) = \mu_0 + c_0 u$ and $\Delta(F^2)(x) = -16c_0$, where μ_0, c_0 are constants.

If we restrict to the case when $(\mathcal{M}_{(4)}, h_{ij})$ is compact, as for black hole solutions, by standard results (cf, e.g., [27], and [13] on p 338) the only regular solution is

$$\mu = \text{const}, \qquad F^2 = \text{const}, \tag{70}$$

as in the $D > 6$ even-dimensional case. Therefore the results of subsection 4.1 apply, and $\left(\mathcal{M}_{(4)}, h_{ij}, J^i_j\right)$ is again (almost-)Kähler Einstein (e.g., flat, or $S^2 \times S^2$, etc).

4.3.2. Non-factorized $P(x, u)$ (transverse space of constant curvature). From (A.12) we observe that this case is possible only for $Q = 0$, so that we can assume $F \neq 0$ (otherwise the Maxwell field would be identically zero). When $P(x, u)$ is non-factorized, as in [11] one can argue that the Riemannian metric $h_{ij}(x, u)$ describes a family of conformal four-dimensional Einstein spaces parametrized by u. It is well known that four-dimensional Riemannian Einstein spaces which admit a conformal (non-homothetic) map on Einstein spaces *must be of constant curvature* [28]. Since $h_{ij} = P^{-2}(x, u)\gamma_{ij}(x)$, this means that we can always find suitable x coordinates such that

$$h_{ij} = P^{-2}\delta_{ij}, \qquad P = a(u) + b_i(u)x^i + c(u)\delta_{ij}x^i x^j. \tag{71}$$

Here $i, j = 1, \ldots, 4$, and $a(u), b_i(u), c(u)$ are arbitrary functions of u related to the constant curvature K by $K = 4ac - \sum_{i=1}^4 b_i^2$ [10]. Recall also that $\left(\mathcal{M}_{(4)}, h_{ij}, J^i_j\right)$ must be almost-Hermitian. The self-dual 'spatial' Maxwell field (cf (47) with $n = 2$) is proportional to the almost-complex structure and must satisfy the Maxwell equations (42) (or, now equivalently, (43)). In addition, there is constraint (55).

Using (24), the (analog of the) Ricci identity applied to the 2-form F_{ij}, the effective Maxwell equations (42), (43), and the constant curvature equation $\mathcal{R}_{ijkl} = K(h_{ik}h_{jl} - h_{il}h_{jk})$,

[6] This transformation (which also applies in the even-dimensional case with $F \neq 0$) explicitly shows that the two WANDs (64) are related by 'time reflection', as observed for arbitrary algebraically special static spacetimes in [9]. This also implies that the two WANDs must have equivalent optical properties (e.g., geodeticity). Note, in particular, that while $\boldsymbol{k}$ is a principal null direction of the Maxwell 2-form by construction, it turned out that also $\boldsymbol{l}$ shares this property.

we can write (cf, e.g., [27], for detailed calculations) $\Delta(F^2) = 2(4KF^2 + h^{mi}h^{nj}h^{pk}F_{ij\|k}F_{mn\|p})$ or, by (24) and (71),

$$\Delta(F^2) = 2P^4\left(4K\sum_{i,j=1}^{4}F_{ij}F_{ij} + P^2\sum_{i,j,k=1}^{4}F_{ij\|k}F_{ij\|k}\right). \tag{72}$$

The right-hand side of equation (72) is non-negative for $K \geqslant 0$. Therefore, if we again restrict to the case of a *compact* $(\mathcal{M}_{(4)}, h_{ij})$, for $K \geqslant 0$ standard results [13, 27] imply that F^2 does not depend on the x coordinates, and that $F_{ij\|k} = 0$. In particular, the case $K > 0$ requires also $F^2 = 0$, i.e. $F_{ij} = 0$ and there is no electromagnetic field (we had already $Q = 0$). For $K = 0$, as in subsection 4.1 thanks to $F = F(u)$ we can achieve $P = P(x)$ (and $F = \text{const}$, $\mu = \text{const}$). But now one can rescale and shift the spatial coordinates to fix $P = 1$, i.e. $h_{ij} = \delta_{ij}$ is manifestly flat and $F_{ij\|k} = 0$ becomes $F_{ij,k} = 0$ (this is the solution of subsubsection 4.1.1 with $\epsilon = 0$, $D = 6$).

The exceptional case $F^2 = F^2(x, u)$, $P = P(x, u)$ (non-factorized) can thus possibly arise only when the transverse space $(\mathcal{M}_{(4)}, h_{ij})$ is non-compact, or of constant negative curvature $K = -1$ (in which case equation (72) does not prevent it from being compact, in principle, provided now $F_{ij\|k} \neq 0$). We do not investigate further this very special case here. Let us only observe that ∂_u is no longer a Killing vector field since the metric depends on u.

4.4. *Inclusion of pure radiation*

It is not difficult to generalize these results to include a pure radiation field aligned with the null vector $\boldsymbol{k}$. In this case, the total energy–momentum tensor to insert into the Einstein equations is given by the sum of the electromagnetic energy–momentum tensor (6) and the pure radiation contribution $\tilde{T}_{\alpha\beta} = \Phi^2 k_\alpha k_\beta$. In the coordinate system introduced above this means that only the $\tilde{T}_{uu} = \Phi^2$ component is non-vanishing. Moreover, since the covariant divergence of the electromagnetic energy–momentum tensor (6) vanishes, the Bianchi identities imply $\tilde{T}^{\alpha\beta}{}_{;\beta} = 0$. For the Robinson–Trautman family of spacetimes this leads to (cf [10])

$$\Phi^2 = r^{2-D}n^2(x, u), \tag{73}$$

where n is an arbitrary function of x and u.

This additional term modifies the field equation of subsection 3.8. Instead of (54), in the generic case we obtain the equation

$$(D-1)\mu(\ln P)_{,u} - \mu_{,u} = \frac{16\pi n^2}{D-2} \qquad (D \neq 6). \tag{74}$$

It is thus possible to *prescribe* the 'mass function' $\mu(u)$, and relation (74) then uniquely determines the corresponding null matter profile $n^2(x, u)$, provided its left-hand side is positive. In the exceptional case $D = 6$ equation (55) becomes

$$5\mu(\ln P)_{,u} - \mu_{,u} - \tfrac{1}{16}\Delta(F^2) = 4\pi n^2 \qquad (D = 6). \tag{75}$$

Again, when the left-hand side is positive, this may be considered as the definition of the function n. We do not study further details of pure radiation spacetimes here. Let us just observe that purely electric solutions (such that $F = 0$) contain generalized charged Vaidya spacetimes, cf [29].

5. Conclusions

We have derived systematically all higher dimensional spacetimes that contain a hypersurface orthogonal, non-shearing, and expanding congruence of null geodesics, together with an aligned electromagnetic field. These are solutions of the coupled Maxwell(–Chern–Simons) and Einstein equations (for any value of the cosmological constant). As already noted in the vacuum case [10], there arise important differences with respect to the standard $D = 4$ family of Robinson–Trautman solutions [1–3]. In particular, for $D > 4$ there is no analog of radiative spacetimes such as the charged C-metric, and the aligned null Maxwell field is not permitted. After integrating the full set of equations, one is essentially left only with (a variety of) static black holes (exceptional subcases possibly arise in $D = 6$). These are characterized by mass, electric charge and cosmological constant, and by the topology and geometry of the horizon, which must be an Einstein space. In even spacetime dimensions an additional magnetic parameter is permitted provided the horizon is not only Einstein but also (almost-)Kähler. Some of the presented solutions were already known (see the references mentioned above), but we have obtained them systematically as elements of the Robinson–Trautman class, which was the purpose of our work.

Our contribution also makes contact with recent studies of the algebraic classification of the Weyl tensor and of geometric optics in higher dimensions. For instance, it has been recently shown [9] that arbitrary $D > 4$ *static* spacetimes can be only of the algebraic types G, I_i, D or O. Using another result of [9], we have demonstrated that our specific static solutions are restricted to the type D, and we have also given the corresponding WANDs with no need to compute the Weyl tensor. In addition, along with various previous results [5, 6, 8–10], the new features pointed out above for $D > 4$ indicate that in some cases the shear-free assumption might be too strong for expanding solutions in higher dimensions. In future work it would thus be worth investigating spacetimes with shear and expansion, at least with some alternative simplifying assumptions.

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Appendix A. Some technical details

Equation $R_{ij} = \frac{2}{D-2}\Lambda g_{ij} + 8\pi T_{ij} + \frac{1}{2}\frac{D-4}{D-2} g_{ij} F_{\mu\nu}F^{\mu\nu}$

The Ricci tensor component R_{ij} for metric (1) was calculated in [10]. With (10) this reads

$$R_{ij} = \mathcal{R}_{ij} - r^{4-D}(r^{D-3}g^{rr})_{,r}h_{ij} - r^{2(2-D)}\frac{(D-1)^2}{2}h_{ik}h_{jl}f^k f^l - r\left[\frac{D-2}{2}\left(2h_{k(i}e^k{}_{,j)} + e^k h_{ij,k} - h_{ij,u}\right) + \left(e^k_{,k} + e^k(\ln\sqrt{h})_{,k} - (\ln\sqrt{h})_{,u}\right)h_{ij}\right] + r^{2-D}\left[\frac{1}{2}\left(2h_{k(i}f^k{}_{,j)} + f^k h_{ij,k}\right) - \left(f^k_{,k} + f^k(\ln\sqrt{h})_{,k}\right)h_{ij}\right], \tag{A.1}$$

where $\mathcal{R}_{ij}$ is the Ricci tensor associated with the spatial metric h_{ij}, and indices in small round brackets are symmetrized. Using (20) and (23), one can separate terms in the field equation with different r-dependences. By contracting with h^{ij}, we obtain a differential equation for g^{rr} which can be integrated immediately. For $D > 5$ this yields

$$g^{rr} = c_1 + c_2 r + c_3 r^2 + c_4 r^{2-D} + c_5 r^{2(2-D)} + c_6 r^{3-D} + c_7 r^{-2} + c_8 r^{2(3-D)}, \tag{A.2}$$

where $c_1, \ldots, c_8$ are functions of (x, u) as follows,

$$\begin{aligned}
c_1 &= \frac{\mathcal{R}}{(D-2)(D-3)}, & c_2 &= \frac{2}{D-2}\big[(\ln\sqrt{h})_{,u} - \mathrm{e}^k_{,k} - \mathrm{e}^k(\ln\sqrt{h})_{,k}\big],\\
c_3 &= -\frac{2\Lambda}{(D-1)(D-2)}, & c_4 &= \frac{D-3}{D-2}\big[f^k_{,k} + f^k(\ln\sqrt{h})_{,k}\big],\\
c_5 &= \frac{1}{2}\frac{D-1}{D-2}h_{kl}f^k f^l, & c_6 &\ \text{arbitrary},\\
c_7 &= -\frac{F^2}{(D-2)(D-5)}, & c_8 &= \frac{2Q^2}{(D-2)(D-3)},
\end{aligned} \tag{A.3}$$

where $\mathcal{R} = h^{ij}\mathcal{R}_{ij}$. For $D = 5$ the only difference is that in (A.2) one should replace $c_7 r^{-2}$ with the term $-\frac{1}{3}F^2 r^{-2}\ln(c_7 r)$ where c_7 is an arbitrary function of (x, u) with the dimension of an inverse length. Next, substituting the above expressions back into the Einstein equation for R_{ij}, we determine for *any* $D > 4$ the following constraints on the metric h_{ij} and the functions e^i and f^i:

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D-2}h_{ij}, \tag{A.4}$$

$$2h_{k(i}\mathrm{e}^k{}_{,j)} + \mathrm{e}^k h_{ij,k} - h_{ij,u} = \frac{2}{D-2}\big[\mathrm{e}^k_{,k} + \mathrm{e}^k(\ln\sqrt{h})_{,k} - (\ln\sqrt{h})_{,u}\big]h_{ij}, \tag{A.5}$$

$$2h_{k(i}f^k{}_{,j)} + f^k h_{ij,k} = \frac{2}{D-2}\big[f^k_{,k} + f^k(\ln\sqrt{h})_{,k}\big]h_{ij}, \tag{A.6}$$

$$(h_{kl}f^k f^l)h_{ij} = (D-2)(h_{ik}f^k)(h_{jl}f^l), \tag{A.7}$$

$$h_{ij}F^2 = (D-2)F_{ik}F_{jl}h^{kl}. \tag{A.8}$$

As we note, (A.7) is identical to the vacuum case discussed in [10] and it requires

$$f^i = 0. \tag{A.9}$$

In analogy to [10], we also use the coordinate freedom (3) to achieve

$$\mathrm{e}^i = 0, \tag{A.10}$$

so that $g^{ri} = 0 = g_{ui}$ and equations (A.6) and (A.7) are now satisfied identically.

In addition, constraint (A.8) requires $F^2 = 0$ for *any odd* D. Indeed, taking the determinant of (A.8), we obtain $(F^2)^{D-2}h^2 = (2-D)^{D-2}(\det F_{ij})^2$, but $\det F_{ij} = 0$ for any antisymmetric matrix F_{ij} and odd dimension $D-2$ since $\det F_{ij} = \det(-F_{ij}) = (-1)^{D-2}\det F_{ij}$. Consequently, the logarithmic term in $D = 5$ is zero, and we need not treat the $D = 5$ case separately.

Equation $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi T_{uu} + \frac{1}{2}\frac{D-4}{D-2}F_{\mu\nu}F^{\mu\nu}g_{uu}$

First, we evaluate the Ricci tensor component R_{uu}. Using the general expression (31) of [10], relation (35) for $g^{rr} = -g_{uu}$, equations (37), (38) implying $\sqrt{h} = P^{2-D}$, and (52), we obtain

$$
\begin{aligned}
R_{uu} = {} & \frac{2}{D-2}\Lambda g_{uu} - \left(r^{-4}\frac{F^2}{D-2} + r^{2(2-D)}\frac{D-3}{D-2}2Q^2\right) g_{uu} \\
& + r^{2-D}\frac{D-2}{2}[(D-1)\mu(\ln P)_{,u} - \mu_{,u}] - r^{5-2D}\frac{2Q}{D-3}[(D-2)Q(\ln P)_{,u} - Q_{,u}] \\
& + \frac{r^{-3}}{2(D-5)}[4(F^2)(\ln P)_{,u} - (F^2)_{,u}] - \frac{r^{-4}\Delta(F^2)}{2(D-2)(D-5)},
\end{aligned} \tag{A.11}
$$

where Δ is the covariant Laplace operator with respect to the spatial metric h_{ij}, i.e. $\Delta(F^2) \equiv (F^2)^{\|j}_{\|j} = [(F^2)_{,i}h^{ij}]_{,j} + (2-D)h^{ij}(F^2)_{,i}(\ln P)_{,j}$. Note that we also dropped the term proportional to r^{-1}, which vanishes identically, see [10] (equations (33) and (B4) therein).

Now, the coefficient of the r^{5-2D} term vanishes provided the Maxwell equation (44) is satisfied, and the coefficient of r^{-3} is zero thanks to $F_{ij} = F_{ij}(x)$ and equations (24) and (38)—indeed these conditions can be re-expressed as

$$Q_{,u} = (D-2)(\ln P)_{,u}Q, \tag{A.12}$$

$$(F^2)_{,u} = 4(\ln P)_{,u}(F^2). \tag{A.13}$$

Moreover, using (23) and (41), we have

$$8\pi T_{uu} + \frac{1}{2}\frac{D-4}{D-2}F_{\mu\nu}F^{\mu\nu}g_{uu} = -\left(r^{-4}\frac{F^2}{D-2} + r^{2(2-D)}\frac{D-3}{D-2}2Q^2\right) g_{uu}. \tag{A.14}$$

We thus now only need to make sure that in (A.11) the coefficients of r^{2-D} and of the last term in r^{-4} vanish. Note that, by (53), $(F^2)_{,i} = 0$ for $D \neq 6$ so that the latter is automatically zero. On the other hand, in the special case $D = 6$ both terms are non-zero and they combine in a single expression. The field equations thus require

$$\mu_{,u} = (D-1)(\ln P)_{,u}\mu \qquad (D \neq 4, 6), \tag{A.15}$$

or

$$\mu_{,u} = 5(\ln P)_{,u}\mu - \tfrac{1}{16}\Delta(F^2) \qquad (D = 6), \tag{A.16}$$

in the two distinct cases $D \neq 6$ and $D = 6$.

Appendix B. The special case of $D = 4$

For comparison, we will present here a summary of the results in the familiar case $D = 4$ [1, 3]. We first note that the trace (8) of the energy–momentum tensor of the electromagnetic field is now zero, $T^{\mu}_{\mu} = 0$. Maxwell's equations still imply (20)–(22), which now read

$$F_{ij} = F_{ij}(x,u), \qquad F_{ru} = r^{-2}Q(x,u), \qquad F_{ui} = r^{-1}Q_{,i} - \xi_i(x,u), \tag{B.1}$$

where $i, j = 1, 2$, so that F_{12} is the only independent F_{ij} component. The invariants of the Maxwell field are thus

$$F_{\mu\nu}F^{\mu\nu} = r^{-4}(F^2 - 2Q^2), \qquad F_{\mu\nu}{}^{*}F^{\mu\nu} = 4r^{-4}P^2F_{12}Q, \tag{B.2}$$

so that there can be null Maxwell fields when $Q = 0 = F$ (i.e., $F_{ru} = 0 = F_{ij}$ while $F_{ui} = -\xi_i$). The source-free equation (16) is now an identity. We further have the relation (25). Finally, the remaining Maxwell equations (18), (19), when expanded in powers of r using previous results, yield the following set of relations:

$$F_{jk}f^k = Qh_{jk}f^k, \tag{B.3}$$

$$\sqrt{h}\, h^{ij} Q_{,j} = (\sqrt{h}\, h^{ik} h^{jl} F_{kl})_{,j}, \tag{B.4}$$

$$(\sqrt{h} Q)_{,u} - (\sqrt{h} Q \mathrm{e}^i)_{,i} = (\sqrt{h}\, h^{ij} (\xi_j - F_{jk}\, \mathrm{e}^k))_{,i}. \tag{B.5}$$

Note that the remaining condition $(\sqrt{h} h^{ij} Q_{,j})_{,i} = 0$ is satisfied identically as a consequence of (B.4) and the antisymmetry of F_{kl}.

Applying now the field equation for R_{ij}, we observe that the powers of r in (A.2) coincide in the terms corresponding to c_4, c_7 and c_8. The expansion of g^{rr} then only contains the first of these terms, yet (A.7) remains unchanged so we obtain $f^i = 0$, and we can again set $\mathrm{e}^i = 0$. The expression for c_4 is thus modified to $c_4 = Q^2 + F^2/2$. We further find that (A.4) and (A.8) remain unchanged. However, for $D = 4$, they are both identically satisfied so they do not provide additional constraints on h_{ij} and on the electromagnetic field. Thus the expansion of g^{rr} is the same as in (35) but the last two terms are combined (cf also footnote 3).

Let us also emphasize that in the $D = 4$ case the spatial metric h_{ij} is two-dimensional, so that it can always be written in the conformally flat form $h_{ij} = P^{-2}(x, u)\delta_{ij}$, with $\sqrt{h} = P^{-2}$. In fact, for $D = 4$ we can achieve this by a transformation $x^i = x^i(\tilde{x})$ involving only the spatial coordinates x, since the u-dependence is factorized out as in equation (38). Consequently, $\mathcal{R} = 2\Delta \ln P = 2P^2[(\ln P)_{,11} + (\ln P)_{,22}]$.

We can thus summarize that the Robinson–Trautman metric in $D = 4$ can be cast in the form

$$\mathrm{d}s^2 = r^2 P^{-2}(x, u)((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2) - 2\,\mathrm{d}u\,\mathrm{d}r - 2H\,\mathrm{d}u^2, \tag{B.6}$$

and the aligned electromagnetic field is given by

$$\boldsymbol{F} = \frac{Q}{r^2}\,\mathrm{d}r \wedge \mathrm{d}u + \left(\frac{Q_{,1}}{r} - \xi_1\right) \mathrm{d}u \wedge \mathrm{d}x^1 + \left(\frac{Q_{,2}}{r} - \xi_2\right) \mathrm{d}u \wedge \mathrm{d}x^2 + F_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{B.7}$$

The various functions and parameters above are constrained by the conditions

$$2H = \frac{\mathcal{R}}{2} - 2r(\ln P)_{,u} - \frac{\Lambda}{3}r^2 - \frac{\mu}{r} + \frac{Q^2 + \frac{1}{2}F^2}{r^2}, \tag{B.8}$$

where $\mu = \mu(x, u)$, $\frac{1}{\sqrt{2}}F = P^2 F_{12}$, and (from (B.4), (B.5) and (25))

$$Q_{,1} = \left(\tfrac{1}{\sqrt{2}}F\right)_{,2}, \qquad Q_{,2} = -\left(\tfrac{1}{\sqrt{2}}F\right)_{,1}, \tag{B.9}$$

$$(QP^{-2})_{,u} = \xi_{1,1} + \xi_{2,2}, \qquad \left(\frac{1}{\sqrt{2}}FP^{-2}\right)_{,u} = \xi_{1,2} - \xi_{2,1}. \tag{B.10}$$

Unlike in higher dimensions (cf equations (28) and (29)), in $D = 4$ we have $Q(x, u)$ depending on the spatial coordinates x, and $\xi_i(x, u) \neq 0$.

The field equation for R_{ur} in now satisfied. Also equation (49) for R_{ui} is satisfied identically, so that one can have a much more general function $\mathcal{R}(x, u)$. Using (B.9), R_{ui} yields only two remaining equations

$$\mu_{,1} = 4\left(Q\xi_1 - \tfrac{1}{\sqrt{2}}F\xi_2\right), \qquad \mu_{,2} = 4\left(Q\xi_2 + \tfrac{1}{\sqrt{2}}F\xi_1\right). \tag{B.11}$$

Finally, the field equation R_{uu} gives

$$\left(Q^2 + \tfrac{1}{2}F^2\right)_{,11} + \left(Q^2 + \tfrac{1}{2}F^2\right)_{,22} = 4\left(Q_{,1}^2 + Q_{,2}^2\right), \tag{B.12}$$

$$P^2(\mu_{,11} + \mu_{,22}) + 8(\ln P)_{,u}\left(Q^2 + \tfrac{1}{2}F^2\right) - 2\left(Q^2 + \tfrac{1}{2}F^2\right)_{,u} = 8P^2(Q_{,1}\xi_1 + Q_{,2}\xi_1), \tag{B.13}$$

and

$$\Delta\mathcal{R} + 12\mu(\ln P)_{,u} - 4\mu_{,u} = 4P^2\left(\xi_1^2 + \xi_2^2\right), \tag{B.14}$$

where Δ is the covariant Laplace operator on a 2-space with metric h_{ij}, i.e. $\Delta\mathcal{R} = P^2(\mathcal{R}_{,11} + \mathcal{R}_{,22})$.

We have thus recovered the well-known results summarized in theorems 28.3 and 28.7 of [1] with the identification $\zeta = \frac{1}{\sqrt{2}}(x^1 + \mathrm{i}x^2)$, $h(\zeta, \bar{\zeta}, u) = \frac{1}{\sqrt{2}}(\xi_1 + \mathrm{i}\xi_2)$, and the complex function $Q(\zeta, u)$ related to $Q(x, u)$ and $\frac{1}{\sqrt{2}}F(x, u)$ as its real and imaginary parts, respectively. Indeed, (B.9) are the Cauchy–Riemann conditions so that $Q(\zeta, u)$ must be analytic in ζ. Consequently, $Q_{,11} + Q_{,22} = 0 = F_{,11} + F_{,22}$, and (B.12) is an identity. Equations (B.10) and (B.11) correspond to equations (28.37e) in reference [1], (B.13) leads to (28.37d), and (B.14) is exactly the equation (28.37c) in [1].

Recall that for $D = 4$ electrovacuum Robinson–Trautman solutions can be of the Petrov types II, D or III [1].

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