

## Notes on static cylindrical shells

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### Abstract

Static cylindrical shells made of various types of matter are studied as sources of the vacuum Levi-Civita metrics. Their internal physical properties are related to the two essential parameters of the metrics outside. The total mass per unit length of the cylinders is always less than  $\frac{1}{4}$ . The results are illustrated by a number of figures.

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### 1. Introduction

Cylindrically symmetric spacetimes and their sources have belonged to useful models in classical relativity for many years. Although recently attention was paid primarily to dynamical situations in which gravitational waves are present (see, e.g., [1, 2] and references therein) or to cylindrical fields motivated by string theories (e.g., [3]), unsolved questions persist even in the standard static cases.

In contrast to the Schwarzschild metric described completely by a single parameter—the Schwarzschild mass, the static vacuum Levi-Civita (LC) solution contains two essential constant parameters— $m$ , related to the local curvature, and  $C$ , determining the global conicity of the spacetime. Connecting these two parameters with physical properties of cylindrical sources turns out to be considerably more difficult than in the spherical case. Here, we do not wish to survey numerous contributions to the subject—we refer the reader to the recent review [4]; some more references, in particular on static cylindrical shells, are given in the following sections.

The purpose of this paper is to construct various types of physically plausible cylindrical shell sources and to connect their internal properties, especially their mass per unit length, with the two parameters of the external LC metrics. We consider not only the shells made of counter-rotating particles with non-zero rest mass, but also those made of photons. In addition, inspired by recent work on two counter-moving light beams [5], we study shells of counter-spiralling particles with both non-zero and zero rest masses. Next, we discuss

perfect fluids as well as shells made of anisotropic matter satisfying, respectively, the weak, strong and dominant energy conditions<sup>1</sup>. We determine permissible ranges of radii of the two-dimensional cylinders made of various types of matter. In all the cases considered, we conclude that the mass per unit length based on the definition by Marder [6] is restricted by  $M_1 \leq \frac{1}{4}$ . We find that even for small values of the external LC parameter  $m$ , it is important to realize that the conicity  $C$  enters into a relation between the internal parameter  $M_1$  and the external parameters (thus solving a puzzle in the literature). Although in most of the text we assume flat spacetime inside the shells, we make some remarks on the situations in which a ‘cosmic string’ is located along the axis of the cylinder and on the limits that can produce a cosmic string from a ‘shrinking’ cylinder.

At the end, we add a few notes on the geodesics outside the cylinder and, in the appendix, some basic properties of the Newtonian cylindrical shells and the relativistic spherical shells are summarized for comparison. In contrast to the literature on static cylinders we are aware of, we illustrate the results by a number of figures.

## 2. Cylindrical shells and their mass per unit length

If the singularity is located on the axis of symmetry, then the LC metric in the Weyl coordinates  $\tau \in \mathbb{R}$ ,  $r \in \mathbb{R}^+$ ,  $\zeta \in \mathbb{R}$ ,  $\varphi \in [0, 2\pi)$  can be written in terms of two constant dimensionless parameters  $m \in \mathbb{R}$  and  $C \in \mathbb{R}^+$  and constant  $r_0 \in \mathbb{R}^+$  with the dimension of length as follows:

$$ds^2 = -\left(\frac{r}{r_0}\right)^{2m} d\tau^2 + \left(\frac{r}{r_0}\right)^{2m(m-1)} (d\zeta^2 + dr^2) + r^2 \left(\frac{r}{r_0}\right)^{-2m} \frac{d\varphi^2}{C^2}. \quad (1)$$

Introducing dimensionless coordinates  $\rho \equiv \frac{r}{r_0}$ ,  $t \equiv \frac{\tau}{r_0}$ ,  $z \equiv \frac{\zeta}{r_0}$ , we obtain the metric in the standard dimensionless form (see, e.g., [4]):

$$d\tilde{s}^2 = \frac{ds^2}{r_0^2} = -\rho^{2m} dt^2 + \rho^{-2m} \left[ \rho^{2m^2} (dz^2 + d\rho^2) + \frac{1}{C^2} \rho^2 d\varphi^2 \right]. \quad (2)$$

In general, we assume two LC solutions to be given in two different sets of these coordinates, one inside the shell ( $\rho \leq \rho_-$ ), and the other outside ( $\rho \geq \rho_+$ ). Using Israel’s formalism [7], we readily find expressions for the energy–momentum tensor  $S_{ij}$  ( $i, j = T, Z, \Phi$ ; the coordinates on the shell are chosen in such a way that the induced 3-metric is flat) induced on the shell:

$$\begin{aligned} 8\pi S_{TT} &= \rho_-^{m_- - m_-^2 - 1} (1 - m_-)^2 - \rho_+^{m_+ - m_+^2 - 1} (1 - m_+)^2, \\ 8\pi S_{ZZ} &= \rho_+^{m_+ - m_+^2 - 1} - \rho_-^{m_- - m_-^2 - 1}, \\ 8\pi S_{\Phi\Phi} &= \rho_+^{m_+ - m_+^2 - 1} m_+^2 - \rho_-^{m_- - m_-^2 - 1} m_-^2, \end{aligned} \quad (3)$$

with non-diagonal components vanishing. We require the spacetime to be regular along the axis; thus, we put<sup>2</sup>  $m_- = 0$ ,  $C_- = 1$ . The proper length of a hoop with  $t, \rho, z$  constant must be the same from both sides of the cylinder:

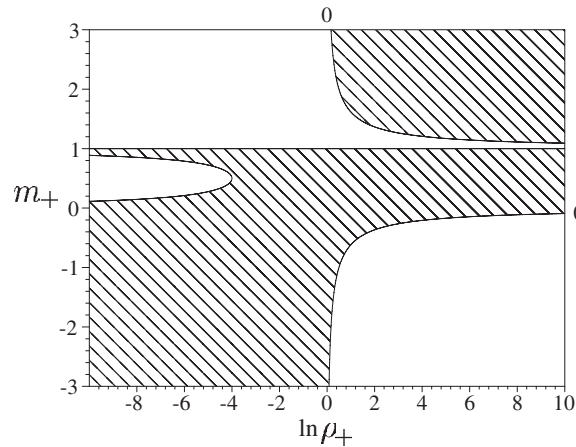
$$2\pi\rho_- = 2\pi\rho_+^{1-m_+}/C_+. \quad (4)$$

We define the mass per unit (coordinate) length of the cylinder, following [6], by

$$M_1 \equiv (\text{circumference}) S_{TT} = 2\pi\rho_- S_{TT}, \quad (5)$$

<sup>1</sup> As demonstrated recently [2], the particular type of energy condition plays an important role in the outcome of the gravitational collapse of cylindrical shells. (See [2] for a number of references on dynamical cylindrical spacetimes.)

<sup>2</sup> If we wish to admit a non-zero missing angle (a ‘cosmic string’) inside the shell, we need to replace  $C_+ \rightarrow C_+/C_-$  in the subsequent formulae. This does not have any major consequences.



**Figure 1.** The shaded regions indicate the intervals of  $\rho_+$  and  $m_+$  in which  $\partial M_1/\partial m_+ > 0$ , as expected on classical grounds. These regions are given as follows:  $[\rho_+ \in (0, e^{-4}) \times m_+ \in (-\infty, \frac{1}{2}(1 - \sqrt{1 + \frac{4}{\ln \rho_+}})) \cup (\frac{1}{2}(1 + \sqrt{1 + \frac{4}{\ln \rho_+}}), 1)] \cup [\rho_+ \in (e^{-4}, 1) \times m_+ < 1] \cup [\rho_+ \in [1, \infty) \times m_+ \in (\frac{1}{2}(1 - \sqrt{1 + \frac{4}{\ln \rho_+}}), 1) \cup (\frac{1}{2}(1 + \sqrt{1 + \frac{4}{\ln \rho_+}}), \infty)]$ .

where  $S_{TT}$  is given in equation (3). Equations (3) and (5) yield

$$M_1 = \frac{1}{4} \left( 1 - \frac{1}{C_+} \frac{(1 - m_+)^2}{\rho_+^{m_+^2}} \right). \quad (6)$$

It is immediately seen that  $M_1 \leq \frac{1}{4}$ . In the case of three-dimensional solid cylinders, an analogous restriction has been recently shown to hold for energy density per unit proper length of the cylinder (see [8], below equation (3.11)). Our result holds for *any* matter on the shell. It is quite in accord with the notion that a spacetime without singularities is free of horizons if the amount of matter within a given region is bounded by a certain finite value. This limiting value can be exceeded only at the expense of a singularity along the axis. Indeed, the case of  $M_1 > \frac{1}{4}$  requires  $m_- \neq 0$ , causing a singularity along the symmetry axis. Admitting  $m_- \neq 0$ , we find

$$M_1 = \frac{1}{4} \left[ (1 - m_-^2)(C_+/C_- \rho_+^{1-m_+})^{m_-^2/1-m_-} - (1 - m_+)^2 C_-/C_+ \rho_+^{m_+^2} \right],$$

which can attain both positive and negative values. A result similar to (6) can be derived for cylinders in asymptotically anti-de Sitter metric. Here, again  $M_1 \leq \frac{1}{4}$  if we demand that there be no singularity and some reasonable conditions are satisfied [9].

By examining expression (6), we find that it is an increasing function of  $C_+$  and  $\rho_+$ . Since the cylindrical shell can be regarded as a source of the outer LC metric, one expects the induced linear matter density (6) to increase with the outer ‘mass parameter’  $m_+$ . However, this is the case only for  $m_+$  in specific intervals depending on the radius  $\rho_+$  of the cylinder (see figure 1). For a finite  $\rho_+ > 0$  and sufficiently small  $m_+$ , we do find  $\partial M_1/\partial m_+ > 0$ , as expected on classical grounds. Note, however, that for small  $\rho_+$ , this ‘intuitive’ behaviour does not occur even for small  $m_+$  (the region around the origin in figure 1).

Expanding  $M_1$  into a series for small  $m_+$  we find

$$M_1 \sim \frac{1}{4} \left( 1 - \frac{1}{C_+} \right) + \frac{1}{C_+} \frac{m_+}{2} + O(m_+^2). \quad (7)$$

The first term represents the contribution from the angular deficit present in the exterior metric, while the second term results from the influence of the local curvature of the spacetime. With no angular defect ( $C_+ = 1$ ), we obtain the Newtonian limit as  $M_1 \sim m_+/2$ . It is, however, clear that even if  $m_+ = 0$ , we can have a non-zero mass per unit length of the source. This corresponds to a locally flat outer metric with a missing angle. We only obtain a flat spacetime everywhere if we put  $m_+ = 0$  and  $\rho_- = \rho_+$ . Otherwise,  $C_+ \neq 1$  ('compensating' the unequal lengths of the hoops in locally flat inner and outer spacetimes), and expansion (7) does not start with  $m_+/2$ . This case does not have a Newtonian limit: one has to consider the global properties of the outer metric ( $C_+$ ) since we are examining total mass per unit length of the cylinder which is not a local quantity. Our conclusion here is that Marder's definition [6] of mass per unit length of the source is appropriate in the case of a shell.

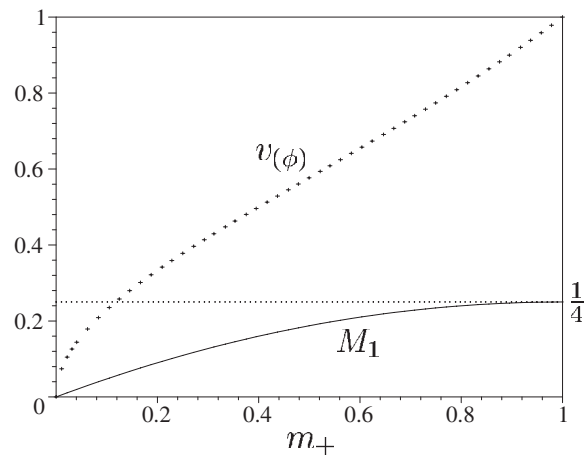
Wang *et al* constructed some cylindrical shell models [10] for which they discuss (in sections 3 and 5) various definitions of the mass per unit length. In particular, they claim that in the case of their (somewhat contrived but satisfying energy conditions) shell of an anisotropic fluid (see their equations (12) and (30), respectively), Marder's definition gives an incorrect Newtonian limit. To change to their notation we have to do the following replacements:  $C_+ \rightarrow C$ ,  $m_+ \rightarrow 2\sigma$ ,  $\rho_+ \rightarrow (Ar_0)^{1/A}$ , where

$$A = m_+^2 - m_+ + 1 = 4\sigma^2 - 2\sigma + 1. \quad (8)$$

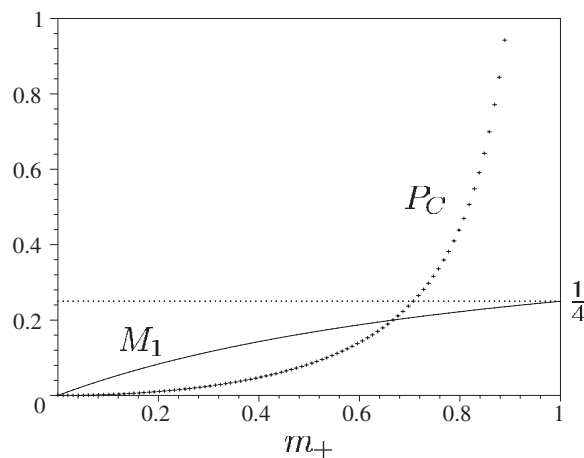
Now, for the shell they consider, we have  $C = A^{(1-2\sigma)/A} r_0^{-4\sigma^2/A} \sim 1 - 2\sigma$  which does not approach 1 fast enough (quadratically in  $\sigma$ ) and thus, the parameter  $C$  influences the limiting value of  $M_1$ . Therefore, the actual value of  $M_1$  must be different from  $\sigma = m_+/2$  as follows from our expansion (7). If in any of these cases we do require  $C_+ = 1$ , we always obtain the correct Newtonian limit. For Wang *et al*, this means  $r_0 = (A(r_0 - a))^{(1-2\sigma)/A}$  (see their equations (5) and (22), respectively) and then  $M_1 = \frac{1}{4}(1 - (1 - 2\sigma)^2 r_0^{-4\sigma^2/(1-2\sigma)})$ , which indeed gives  $M_1 \sim \sigma + O(\sigma^2)$ . It is essential that for the conicity to be 1 up to the second order in  $\sigma$ , one has to have the radial shift  $a$ , introduced in [10], non-vanishing. This is just the case excluded in [10] (see also [11]).

Stachel [12] also considered cylindrical shells; however, he was interested in the physical meaning of the metric parameters rather than in the structure of the shell itself. Langer [13], using Israel's formalism, worked out the case of a cylinder composed of two streams of free particles counter-orbiting in  $\pm\Phi$ -directions and found agreement of his results with those of Raychaudhuri and Som [14] who did not use the Israel formalism but a limiting procedure starting from a shell of a finite thickness. It turns out that in this case the parameter  $m_+ \in [0, 1)$ . The lower limit is due to the condition of a non-negative matter density whereas the upper one is because the geodesics become null for  $m_+ = 1$ . Further, we find  $C_+ = \rho_+^{-m_+^2}$ , and the mass per unit length  $M_1 = \frac{1}{4}m_+(2 - m_+)$ , independent of other parameters, is always greater than or equal to 0. It has a supremum of  $\frac{1}{4}$  at  $m_+ = 1$ . The formula for  $M_1$  gives the correct weak-field limit. The velocity of the particles, as measured by static observers, is  $v_{(\Phi)} = \sqrt{\frac{m_+}{2-m_+}}$  (see figure 2). If we fix the outer parameters, the inner radius is also fixed at  $\rho_- = C_+^{-A/m_+^2}$ , with  $A$  given in equation (8). By letting  $C_+$  increase from 0 to  $\infty$ ,  $\rho_-$  decreases from  $\infty$  to 0.

Langer [13] also considered the case of an ideal fluid. Here  $M_1 = \frac{1}{2} \frac{m_+}{m_+ + 1}$  is also independent of other parameters and gives the right Newtonian limit. The admissible range of  $m_+$  is  $m_+ \in [0, 1)$ ; if we require  $\mu \geq p$  ( $\mu$  is the surface energy density), then  $m_+ \in [0, \frac{2}{3}]$ . The lower limit is due to the non-negativity of the energy density, the upper one to requiring a finite pressure. Again,  $M_1 \geq 0$ , and it has the supremum of  $\frac{1}{4}$  for  $m_+ = 1$ . We have  $C_+ = (1 - m_+^2)/\rho_+^{m_+^2}$  and  $p \equiv S_{ZZ} = S_{\Phi\Phi} = m_+^2/8\pi\rho_-(1 - m_+^2)$ . In figure 3, the mass per



**Figure 2.** Static cylindrical shells of counter-rotating massive particles: the mass per unit length,  $M_1$ , and the velocity  $v_{(\phi)}$  of the particles (measured by static observers) as functions of the Levi-Civita external-field parameter  $m_+$ . With increasing  $m_+$ , both  $M_1$  and  $v_{(\phi)}$  increase monotonically up to the limiting values  $v_{(\phi)} \rightarrow 1$ ,  $M_1 \rightarrow \frac{1}{4}$ .

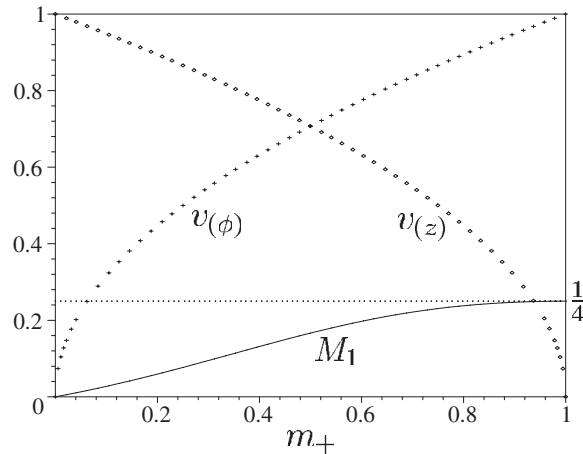


**Figure 3.** Static cylindrical shells of perfect fluid: the mass per unit length,  $M_1$ , and the magnitude of the surface pressure integrated along a ring,  $P_C \equiv (2\pi\rho_-)p$ , as functions of the Levi-Civita external-field parameter  $m_+$ . With increasing  $m_+$ , both  $M_1$  and  $P_C$  increase monotonically up to the limiting values  $P_C \rightarrow +\infty$ ,  $M_1 \rightarrow \frac{1}{4}$ .

unit length and the pressure are illustrated. Using the outer parameters, we can express the inner radius as  $\rho_- = C_+^{-A/m_+^2} (1 - m_+^2)^{(1-m_+)/m_+^2}$ , where  $A$  is given in equation (8). Increasing  $C_+$  means decreasing  $\rho_-$ , as in the case of counter-rotating streams.

### 3. Generalized models of cylindrical shells

The shells from counter-rotating particles in  $\pm\Phi$ -directions can be generalized in the following ways:



**Figure 4.** Static cylindrical shells of counter-spiralling photons with physical velocities  $v_{(\phi)}$ ,  $v_{(z)}$  ( $v_{(\phi)}^2 + v_{(z)}^2 = 1$ ) as measured by static observers. The mass per unit length,  $M_1$ , and both velocities as functions of the Levi-Civita parameter  $m_+$  are illustrated. With  $m_+$  increasing,  $M_1$  and  $v_{(\phi)}$  monotonically increase, whereas  $v_{(z)}$  decreases. The limiting point  $m_+ = 1$ ,  $v_{(\phi)} = 1$ ,  $v_{(z)} = 0$  corresponds to the point  $m_+ = 1$ ,  $v_{(\phi)} = 1$  in figure 2.

- (i) We admit counter-streaming particles to move along the  $\pm Z$ -directions as well ('counter-spiralling motion'). Their velocity can be expressed as  $v_{(z)}^2 \equiv \left(\frac{dz}{dt}\right)^2 = (1 - \frac{\rho_+^A}{\rho_-}) / (\frac{\rho_+^A}{\rho_-} - (1 - m_+)^2)$  and  $v_{(\phi)}^2 = m_+^2 / (\frac{\rho_+^A}{\rho_-} - (1 - m_+)^2)$ . We find the mass (6) to read

$$M_1 = \frac{1}{4} \left( 1 - \frac{\rho_-}{\rho_+^A} (1 - m_+)^2 \right). \quad (9)$$

Necessary conditions for the velocities  $v_{(\phi)}$ ,  $v_{(z)}$  to be real read  $\rho_+^A / \rho_- \in [A, 1]$  and  $(m_+ - 1)^2 \leq \rho_+^A / \rho_-$ . From this we obtain  $m_+ \in [0, 1]$  and  $M_1 \in [\frac{m_+}{4A}, \frac{m_+(2-m_+)}{4}]$ , so that  $M_1 \in [0, \frac{1}{4}]$ . Again,  $M_1 = \frac{1}{4}$  represents the maximum possible mass per unit length. We can rewrite these conditions in terms of the inner radius as

$$\rho_- \in \left[ C_+^{-\frac{A}{m_+}} A^{\frac{1-m_+}{m_+}}, C_+^{-\frac{A}{m_+}} \right]. \quad (10)$$

The lower limit for the inner radius corresponds to the case of photons counter-rotating just in  $\pm\Phi$ -directions. The upper limit is given by  $\mu, p_Z \geq 0$  and  $C_+ = \rho_+^{1-m_+} / \rho_-$ .

- (ii) The shell is composed of photons counter-rotating in  $\pm\Phi$ -directions (instead of massive particles). We then get  $m_+ = 1$  and, consequently,  $\rho_- = \rho_+ = 1/C_+$ . (In contrast to spherical shells of photons (see the appendix), the cylindrical shells can have arbitrary radii depending on  $C_+$ .) Therefore,  $M_1 = \frac{1}{4}$  and the orbiting velocity is equal to 1.
- (iii) We can also consider counter-streaming photons with each stream being in a spiral motion like the massive particles in case (i) above. We find  $m_+ \in [0, 1]$ ,  $M_1 = \frac{1}{4} \frac{m_+}{A}$ . The angular velocity is  $v_{(\phi)}^2 = m_+$  and the velocity along  $\pm Z$ -directions reads  $v_{(z)}^2 = 1 - m_+$ . The inner radius is at  $\rho_- = C_+^{-A/m_+} A^{(1-m_+)/m_+}$ . When  $m_+ = 1$ , this case reduces to case (ii). In figure 4, the physical velocities and mass  $M_1$  are plotted as functions of  $m_+$ .

Let us now investigate a general case of two-dimensional matter with a diagonal stress tensor (see equation (3)) satisfying various types of energy conditions (see, e.g., [15, 16]):

- (a) Denoting  $S_{TT} = \mu$ ,  $S_{\Phi\Phi} = p_\Phi$ ,  $S_{ZZ} = p_Z$ , the weak energy condition,  $\mu \geq 0$ ,  $\mu + p_Z \geq 0$ ,  $\mu + p_\Phi \geq 0$ , for  $m_+ \in [0, 1]$  implies

$$\rho_- \geq C_+ \frac{-\frac{A}{m_+^2}}{(1-m_+)^{\frac{2(1-m_+)}{m_+^2}}}, \quad (11)$$

and for  $m_+ \in [1, 2]$ , we get

$$\rho_- \leq C_+ \frac{-\frac{A}{m_+^2}}{(1-m_+)^{\frac{2(1-m_+)}{m_+^2}}}, \quad (12)$$

the boundary values of  $\rho_-$ , corresponding to the equality signs in (11) and (12), are given by the condition  $\mu \geq 0$ . We find  $\rho_+ \geq \left(\frac{(1-m_+)^2}{C_+}\right)^{1/m_+^2}$ ,  $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{(1-m_+)^2}$  in both intervals of  $m_+$ .

- (b) The strong energy condition requires  $\mu + p_Z + p_\Phi \geq 0$ ,  $\mu + p_Z \geq 0$ ,  $\mu + p_\Phi \geq 0$ . For  $m_+ \in [0, \frac{1}{2}]$ , this implies that

$$\rho_- \geq C_+ \frac{-\frac{A}{m_+^2}}{(1-2m_+)^{\frac{1-m_+}{m_+^2}}}, \quad (13)$$

the boundary value of  $\rho_-$  is now given by  $\mu + p_\Phi \geq 0$ . Here, we find  $\rho_+ \geq \left(\frac{1-2m_+}{C_+}\right)^{1/m_+^2}$ ,  $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1-2m_+}$ . For  $m_+ \in [\frac{1}{2}, 2]$ ,  $\rho_-$  and  $\rho_+$  can be arbitrary.

- (c) The dominant energy condition,  $\mu \geq |p_Z|$ ,  $\mu \geq |p_\Phi|$ , for  $m_+ \in [0, \frac{2}{3}]$  implies that

$$\rho_- \geq C_+ \frac{-\frac{A}{m_+^2}}{\left[\frac{m_+^2 - 2m_+ + 2}{2}\right]^{\frac{1-m_+}{m_+^2}}}, \quad (14)$$

the boundary value of  $\rho_-$  being determined by  $\mu \geq p_Z$ . We find  $\rho_+ \geq \left(\frac{m_+^2 - 2m_+ + 2}{2C_+}\right)^{1/m_+^2}$ ,  $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1-m_+ + m_+^2/2}$ . For  $m_+ \in [\frac{2}{3}, 1]$ , we get

$$\rho_- \geq C_+ \frac{-\frac{A}{m_+^2}}{(2m_+^2 - 2m_+ + 1)^{\frac{1-m_+}{m_+^2}}}, \quad (15)$$

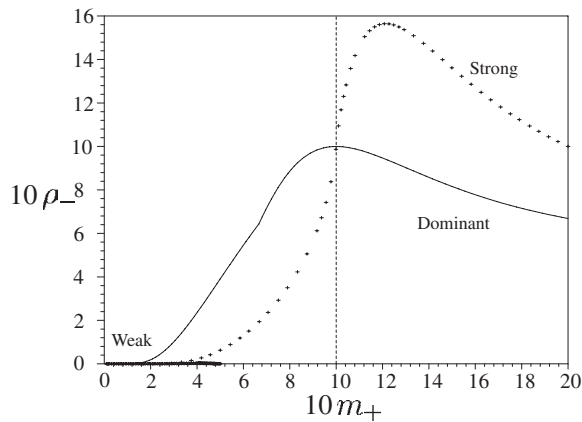
and for  $m_+ \in [1, 2]$ , we finally find

$$\rho_- \leq C_+ \frac{-\frac{A}{m_+^2}}{(2m_+^2 - 2m_+ + 1)^{\frac{1-m_+}{m_+^2}}}, \quad (16)$$

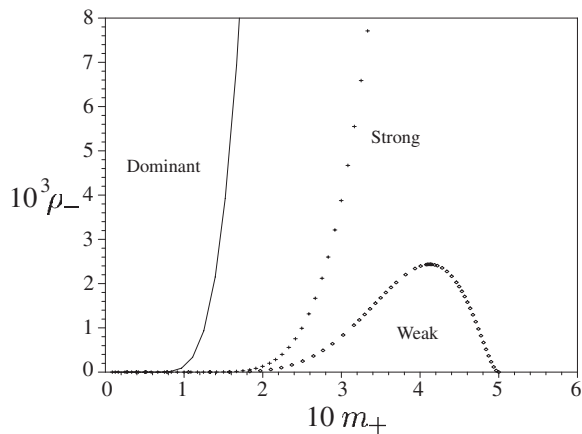
where the boundary value of  $\rho_-$  is given by  $\mu \geq p_\Phi$ . In the last two intervals, we get  $\rho_+ \geq \left(\frac{2m_+^2 - 2m_+ + 1}{2C_+}\right)^{1/m_+^2}$ ,  $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1-2m_+ + 2m_+^2}$ .

Therefore, if  $m_+$  is not large ( $m_+ \leq \frac{1}{2}$ ), there exists a lower limit for possible values of the inner radius  $\rho_-$  of the cylinder in the case of all three energy conditions. In all three cases, we allowed  $m_+ \in [0, 2]$  which follows from the condition  $\mu + p_Z \geq 0$  (in agreement with Wang *et al* [10]). In figures 5 and 6 the above results are illustrated graphically. Let us also note that if we let  $C_+$  change from  $\infty$  to 0, we find  $\rho_-$  changing from 0 to  $\infty$  in all three cases.

What are the limiting values of the mass per unit length of the cylinders? We can rewrite the expression for the mass as  $M_1 = \frac{1}{4}\left(1 - \frac{\rho_-}{\rho_+^A}(m_+ - 1)^2\right)$ . From the above inequalities (11)–(16), we obtain  $M_1 \in [0, \frac{1}{4}]$  for the weak and dominant energy conditions, where the lower limit is reached at the maximum possible value of  $\rho_-/\rho_+^A$ , whereas the upper bound results from  $\rho_-/\rho_+^A \rightarrow 0$ . In the case of the strong energy condition and  $m_+ \in [0, \frac{1}{2}]$  we find that  $\frac{1}{4} \geq M_1 \geq -\frac{m_+^2}{4(1-2m_+)}$ . However, for  $m_+ \in [\frac{1}{2}, 2]$  there is *no* lower bound on  $M_1$ , the upper bound remaining the same. The mass diverges to  $-\infty$  as  $\rho_+ \rightarrow 0^+$ . This can happen since the strong energy condition does not restrict the value of the density to be non-negative. It is important that in the interval in which it is appropriate to speak about a Newtonian limit ( $m_+ \in [0, \frac{1}{2}]$ ), the range of  $M_1$  is still restricted from below.



**Figure 5.** The radii  $\rho_-$  of the cylinders satisfying the weak, strong and dominant energy conditions. For  $m_+ \in [0, 1]$ , the admissible range of radii is *above* the corresponding curve. For  $m_+ = 1$ , one has  $\rho_- = 1/C_+$ . (Here, we fixed  $C_+ = 1$ .) For  $m_+ \in (1, 2]$ ,  $\rho_-$  must lie *below* the curves. For the details around the origin, see figure 6.

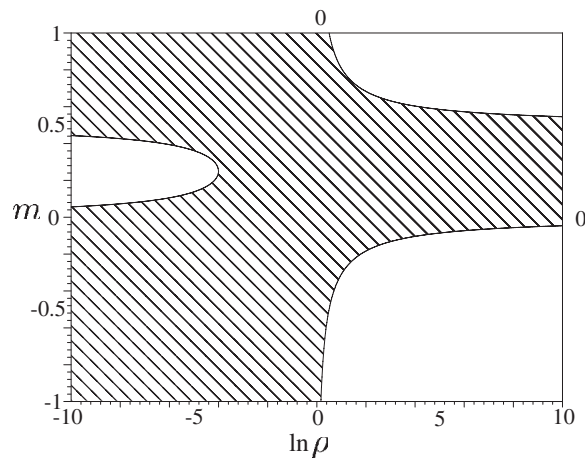


**Figure 6.** Possible radii  $\rho_-$  for small  $m_+$  under various energy conditions (see also the text to figure 5).

Summarizing, we see that in the above examples the mass per unit length (equations (6) and (7)) of the cylindrical shell is typically equal neither to  $\frac{1}{4}(1 - 1/C_+)$  nor to  $m_+/2$ , but rather it is a combination of both contributions. As pointed out recently [8], if we keep the parameters of the outer metric fixed while decreasing the outer radius  $\rho_+$  of the shell, we inevitably end up with mass per unit length,  $M_1$ , of the cylinder diverging,  $M_1 \rightarrow -\infty$ . The limit  $\rho_+ \rightarrow 0$  is permitted only under the strong energy condition for values of  $m_+ \in [\frac{1}{2}, 2]$ , and there is no lower bound on  $M_1$  in this case. Then, however, all other energy conditions are violated except the strong one.

The only two possible candidates for finite-radius models of a cosmic string with finite mass per unit length then are: (i) a shell with  $m_+ = 1$ ,  $M_1 = \frac{1}{4}$  and  $\rho_- = \frac{1}{C_+}$  which satisfies the weak and strong energy conditions (this is not the case of photons); (ii) a shell with  $m_+ = 0$ ,  $C_+ > 1$ ,  $M_1 = \frac{1}{4}(1 - \frac{1}{C_+})$  that satisfies all three energetic conditions, its circumference





**Figure 7.** The behaviour of the acceleration of free test particles at fixed radius  $\rho$  under the changes of the mass parameter  $m$ . In the shaded regions there is  $\partial|\frac{d^2\rho}{d\tau^2}|/\partial|m| > 0$ .

being equal to zero. Indeed, this is a standard model of a cosmic string [8, 17]—all its effects are caused by a deficit angle in a locally flat spacetime.

In the appendix we present corresponding results for classical cylindrical shells and for relativistic spherical shells.

#### 4. Test particles outside cylinders

Let us look at a radial force exerted upon a free particle at rest in the coordinate system of equation (2). The geodesic equation gives

$$\frac{d^2\rho}{d\tau^2} = -\frac{m}{\rho^{2m^2-2m+1}}. \quad (17)$$

The axis is attractive for  $m > 0$ ; for  $m < 0$ , it is repulsive. The ‘magnitude’ of the radial acceleration (the absolute value of expression (17)) decreases with increasing  $\rho$  (and thus also with the proper distance from the axis as this is an increasing function of  $\rho$ ) for any non-zero  $m$ . However, the behaviour due to the changes in  $m$  with constant  $\rho$  is more subtle (see figure 7). One would expect that a bigger  $m$  means a stronger influence of the centre. This, in general, is not the case and, moreover, the behaviour depends on the distance from the axis. For very small distances,  $\rho \in (0, e^{-4}]$ , and for  $m \in (\frac{1}{4}(1 - \sqrt{1 + 4/\ln \rho}), \frac{1}{4}(1 + \sqrt{1 + 4/\ln \rho}))$ , the magnitude of the acceleration decreases with increasing  $m$ . This is against classical intuition since  $m$  is positive in the given interval. For  $\rho \in (e^{-4}, 1]$ , the magnitude is an increasing function of  $|m|$  for any  $m$ . In the last interval  $\rho \in (1, \infty)$ , the acceleration behaves intuitively for  $m \in (\frac{1}{4}(1 - \sqrt{1 + 4/\ln \rho}), \frac{1}{4}(1 + \sqrt{1 + 4/\ln \rho}))$ , while it is counterintuitive in the remaining range. In [18], the author claims (see p 1218) that the acceleration decreases with  $m$  for  $m \in (\frac{1}{2}, 1)$  (increasing magnitude). However, this is not the case for sufficiently large  $\rho$ . (Also, he relates the acceleration of particles falling radially from rest to the existence of circular geodesics, which does not appear to be very telling if we recall, for example, the case of the Schwarzschild metric.)

For any value of  $\rho$ , there exists an interval of  $m$ , containing 0, where the acceleration behaves in accordance with classical intuition. The same is true for  $\rho > e^{-4}$  and  $m \in [0, \frac{1}{2}]$ .

The first point saves the Newtonian limit, the second provides a classical intuition for this interval of  $m$ .

Finally, we study geodesics with  $\rho = \text{constant}$ . We write the 4-velocity as  $U^\alpha = U^t(1, v_z, \omega, 0)$ . For photons performing different types of motion we obtain, subsequently

$$\begin{aligned} z\text{-direction:} & \quad v_z = \pm 1, \quad \omega = 0, \quad m = 2 \quad \text{or} \quad m = 0, \quad U^t, \quad \rho \text{ arbitrary} \\ \varphi\text{-direction:} & \quad v_z = 0, \quad \omega = \pm C, \quad m = \frac{1}{2}, \quad U^t, \quad \rho \text{ arbitrary} \\ \text{spiral motion:} & \quad v_z = \pm \rho^{m(2-m)} \sqrt{\frac{2m-1}{m^2-1}}, \quad \omega = \pm C \rho^{2m-1} \sqrt{\frac{m(m-2)}{m^2-1}}, \\ & \quad m \in \left(0, \frac{1}{2}\right) \cup (2, \infty), \quad U^t, \quad \rho \text{ arbitrary.} \end{aligned}$$

For particles with non-zero rest mass, we find

$$\begin{aligned} z\text{-direction:} & \quad v_z = \pm \frac{\rho^{m(2-m)}}{\sqrt{m-1}}, \quad \omega = 0, \quad U^t = \rho^{-m} \sqrt{\frac{m-1}{m-2}}, \quad m > 2, \quad \rho \text{ arbitrary} \\ & \quad \text{and } v_z = \pm \sqrt{1 - \frac{1}{(U^t)^2}}, \quad m = 0, \quad U^t \geq 1, \quad \rho \text{ arbitrary} \\ \varphi\text{-direction:} & \quad v_z = 0, \quad \omega = \pm C \rho^{2m-1} \sqrt{\frac{m}{1-m}}, \quad U^t = \rho^{-m} \sqrt{\frac{m-1}{2m-1}}, \\ & \quad m \in \left(0, \frac{1}{2}\right), \quad \rho \text{ arbitrary} \\ \text{spiral motion:} & \quad v_z = \pm [\rho^{m(1-m)} / U^t] [(1-m + (U^t)^2 \rho^{2m} (2m-1)) / (m^2-1)]^{1/2}, \\ & \quad \omega = \pm C [\rho^{m-1} / U^t] [m(1-m + (U^t)^2 \rho^{2m} (m-2)) / (m^2-1)]^{1/2}. \end{aligned}$$

In the last case of spiral motion, the parameters  $(m, U^t, \rho)$  are related by the condition that  $v_z, \omega$  are real and  $v_z^2 + \omega^2 < 1$ . There is a solution to these inequalities only as long as  $m \in (-\infty, \frac{1}{2}) \cup (2, \infty)$ . In other words, we can choose two of these parameters and find the admissible range of the remaining parameter. The admissible intervals of  $m$  found above in the cases more general than just circular test particle motion, correspond to a possible interchange of the roles of coordinates  $\varphi$  and  $z$ , suggested in [11].

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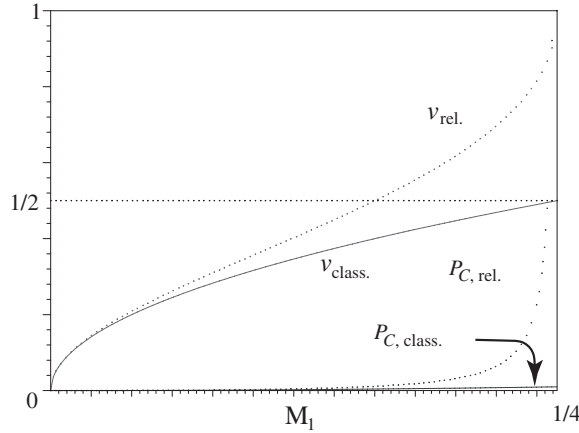
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## Appendix

Here, we summarize some basic properties of Newtonian cylindrical shells and relativistic spherical shells.

### A.1. Newtonian cylindrical shells

The angular velocity of test particles orbiting a cylinder at  $\rho = \text{constant}$  is  $\omega = \frac{\sqrt{2M_1}}{r}$ . The same is true for particles spiralling parallel to the axis. The angular velocity of the particles making up the cylinder of radius  $R$  and mass per unit length  $M_1$  is  $\omega = \frac{\sqrt{M_1}}{R}$ . These particles may rotate in one direction only and move parallel to the axis as well. With a relativistic cylinder, we have  $\omega = \frac{1}{\rho_-} \sqrt{\frac{1-\sqrt{1-4M_1}}{1+\sqrt{1-4M_1}}}$ . If the thin cylinder consists of an ideal fluid, the pressure reads  $p = \frac{M_1^2}{2\pi r}$ . In relativity, we get  $p = \frac{M_1^2}{2\pi \rho_-(1-4M_1)}$ . In figure 8, the behaviour of these characteristic functions is illustrated.



**Figure 8.** Classical versus relativistic cylindrical shells. We plot  $v_{\text{class}} = r\omega$ ,  $v_{\text{rel}} = \rho\omega$ , and ‘circumferential’ pressures  $P_{C, \text{class}} = (2\pi r)p$  and  $P_{C, \text{rel}} = (2\pi\rho)p$  for the physically appropriate range of  $M_1$ .

### A.2. Relativistic thin spherical shells

Let the outer Schwarzschild mass parameter be  $m$ . There is a flat spacetime inside. The shell is located at  $r_- = r_+ \equiv r$ . The mass of the shell is defined as  $M \equiv 4\pi r^2 \mu$ , with  $\mu \equiv S_{TT}$ . We get  $S_{TT} = \frac{1-\sqrt{F}}{4\pi r}$ ,  $S_{\Theta\Theta} = \frac{\sqrt{F}-1+\frac{m}{r\sqrt{F}}}{8\pi r}$  and  $S_{\Phi\Phi} = \frac{\sqrt{F}-1+\frac{m}{r\sqrt{F}}}{8\pi r}$ , with  $ds^2 = -Fdt^2 + \frac{dr^2}{F} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$ . The principal pressures are the same due to spherical symmetry. We require  $r \geq 2m$ . For  $m \geq 0$ , we have  $\mu \geq 0$ . Regardless of the parameters we find  $p \equiv S_{\Theta\Theta} = S_{\Phi\Phi} \geq 0$ .

Different energy conditions lead to the following results:

- (i) *Weak energy condition.* One gets  $m \geq 0$ , if  $\mu \geq 0$ . This is satisfied for any  $r \geq 2m$ . We find  $M_{\text{max}} = 2m$  for  $r = 2m$ ; however, the pressure then diverges.
- (ii) *Strong energy condition.* One gets  $m \geq 0$ , if  $\mu + 2p \geq 0$ . This is satisfied for any  $r \geq 2m$ . The case of  $M_{\text{max}}$  is the same as in (i).
- (iii) *Dominant energy condition.* We need  $r \geq \frac{25}{12}m$ , if  $\mu \geq p$ . (In this case  $m$  may be negative but then also  $\mu < 0$  and  $p < 0$ , with  $|p| \geq |\mu|$ .) We find  $M_{\text{max}} = \frac{5}{3}m$ .
- (iv) If the shell is made of photons, we find  $r = \frac{9}{4}m$ ,  $M = \frac{3}{2}m$ .
- (v) For a shell of particles with non-zero rest mass, we obtain  $r > \frac{9}{4}m$ , while  $M_{\text{max}} = \frac{3}{2}m$ . For this limiting value of  $M$ , the trajectories become null.

Maximal  $M$  of the shell is again achieved in the case of photons. The dominant energy condition is the most restrictive as in the cylindrical case.

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