

Gravitational waves and dragging effects

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Abstract

Linear and rotational dragging effects of gravitational waves on local inertial frames are studied in purely vacuum spacetimes. First, the linear dragging caused by a simple cylindrical pulse is investigated. Surprisingly strong transverse effects of the pulse are exhibited. The angular momentum in cylindrically symmetric spacetimes is then defined and confronted with some results in the literature. In the main part, a general procedure is developed for studying weak gravitational waves with translational but not axial symmetry which can carry angular momentum. After a suitable averaging the rotation of local inertial frames due to such rotating waves can be calculated explicitly and illustrated graphically. This is done in detail in the accompanying paper. Finally, the rotational dragging is given for strong cylindrical waves interacting with a rotating cosmic string with a small angular momentum.

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1. Introduction

Bets have been won by those who prognosticated that gravitational waves will not be positively detected in an Earth-bound laboratory until 1999, 2001, . . . , 2008. But it would be hard to find a reader of *Classical and Quantum Gravity* who does not believe that eventually gravitational waves will be detected through their effects on test laboratory masses. From purely theoretical perspectives, from the year 2000 at least six papers appeared in this journal in which dragging effects even on gyroscopes were studied: for example, the precession of a small test gyroscope induced by a weak plane gravitational wave [1, 2], the behavior of a gyroscope located at smooth and polyhomogeneous null infinity [3, 4], the motion of spinning test particles in plane gravitational waves [5, 6].

However, a truly ‘Machian’ problem associated with gravitational waves is not concerned with the behavior of a gyroscope immersed directly in a gravitational wave. A collapsing,

slowly rotating spherical massive shell drags into rotation inertial frames inside its *flat* interior [7, 8]. In linearly perturbed Friedmann–Robertson–Walker universes, the local inertial frames can be seen to be determined instantaneously via the perturbed Einstein field equations even from the distributions of energy, momentum and angular momentum beyond a cosmological horizon [9, 10]. Correspondingly, it is important to demonstrate whether energy and angular momentum of gravitational waves in *purely vacuum* spacetimes can cause the local inertial frames to rotate. If, indeed, this is so, it is of interest to see what such effects look like explicitly.

Thanks to progress in ‘analytic gluing techniques’ in geometric analysis and differential geometry over the past 25 years, and their recent applications to general relativity, the question of the existence of the effect of the angular momentum of gravitational waves on the rotation of inertial frames can now be answered with full rigor. In our context the important *physical* statement following from these developments, in particular from the work of Corvino [11] and Corvino and Schoen [12], is that asymptotically flat initial vacuum data, i.e., data which represent pure gravitational waves can be deformed so that in the far region they are exactly Kerrian, i.e., stationary. Since the Kerr metric is one of the best prototypes to demonstrate the dragging of local inertial frames, there is now clear theoretical evidence available that angular momentum due to gravitational waves causes local inertial frames to rotate with respect to the local inertial frames at infinity. Moreover, as in the cosmological perturbation theory, or in the examples with rotating shells, this effect is global and instantaneous (all the construction occurs on a given spatial hypersurface). From the perspective of Mach’s principle, one would prefer to demonstrate the rotation of local inertial frames in a *closed* universe filled only with gravitational waves since then boundary conditions (like asymptotic flatness) play no role. Such a study lies in the future. However one would also like to see the dragging effects caused by the ‘rotating waves’ explicitly. To construct explicit examples, we turn to the waves with a symmetry along Oz in cylindrical polar coordinates.

In the following section, the metric with a translational symmetry is introduced in suitable coordinates and the vacuum field equations in four dimensions are rewritten as Einstein’s equations in three dimensions with a scalar field as a source, following [13]. A rotational symmetry is then assumed in addition, and the equations for standard Einstein–Rosen waves are written, including one of their simplest explicit solutions, the Bonnor/Weber–Wheeler time symmetric pulse. Section 3 is devoted to the effects this pulse exerts on test particles. Specifically, we calculate the force acting on a particle which is at rest on a cylinder with either fixed proper circumference or a fixed proper radial distance from the axis. Interestingly, it can happen that *no* physical force can keep particles at fixed circumferences in the ‘strong region’ of the pulse. However, the particle can be forced to stay fixed at a given proper distance from the axis or at a given radial coordinate (defined geometrically in Einstein–Rosen spacetimes). The ‘linear stretching’ due to the cylindrical wave in the directions transverse to its propagation thus appears to be stronger than the ‘linear dragging’ in the radial direction.

In the following sections we turn to the ‘rotational dragging’. First, however, we analyze the angular momentum in cylindrically symmetric spacetimes (section 4). Those spacetimes possess the rotational Killing vector which can be normalized at the axis of symmetry and used to define the total angular momentum density, contained in a two-dimensional cylinder of a unit length, by adopting the Komar expression. The Komar-type expression yields meaningful results even if a rotating cosmic string is present along the axis. We show that among cylindrical metrics which appeared recently in the literature the angular momentum is non-vanishing only in those cases which represent cylindrical gravitational waves interacting with a rotating cosmic string.

An interplay between a strong gravitational wave and a rotating string is of interest but a physically ‘clean’ explicit demonstration of dragging effects due to gravitational waves should not involve any matter like a string. A simple procedure for studying gravitational waves with a net angular momentum is presented in section 5. We assume the waves to have the translational symmetry but not axial symmetry. Then it is convenient to make dimensional reduction after which the vacuum problem in four dimensions becomes that of a scalar field in three-dimensional spacetime [13]. The scalar field satisfies the wave equation in flat spacetime provided that it is weak. We can construct ‘ φ -dependent’, rotating solutions of this flat-space wave equation and obtain the effective energy–momentum tensor—which is of the second order—on the right-hand side of the Einstein field equations in three dimensions. The resulting metric deviates from a three-dimensional Minkowski metric by terms of second order. In order to find the rotation of inertial frames at the axis we focus on the axially symmetric terms in the Fourier expansion of the angular velocity as a function of φ by averaging over φ . We discuss the structure of all the field equations, their consistency and suitable boundary conditions within this approach. The same function in the metric which approaches a constant at spatial infinity and represents the total energy in the case of the Einstein–Rosen waves, also yields the energy of rotating waves. The explicit solution of these equations based on a non-axially symmetric generalization of a weak Bonnor/Weber–Wheeler pulse is constructed in the accompanying paper [14]. Therein the dragging effects of the resulting gravitational waves on local inertial frames are analyzed in detail and graphically illustrated. In the appendix, the Ricci tensor for the averaged rotating gravitational waves in 2+1 dimensions is given.

The rotational dragging in the spacetime of the exact (in general strong) Bonnor/Weber–Wheeler cylindrical pulse perturbed by a rotating cosmic string with a small angular momentum is considered in section 6.

2. The metric and field equations for axially rotating gravitational waves

At the core of this work and the following paper [14] are source-free metrics with at least one hypersurface orthogonal Killing vector of spacelike translations ζ . In this case coordinates exist⁴

$$\{x^\mu\} = \{x^a, x^3 = z\}, \quad \text{where } \zeta = \{0, 0, 0, 1\}, \quad (2.1)$$

and the metric can be written in the form introduced by Ashtekar *et al* [13],

$$ds^2 = e^{-2\psi} g_{ab} dx^a dx^b - e^{2\psi} dz^2, \quad (2.2)$$

where ψ and g_{ab} are functions of x^c only. The source-free Einstein’s equations $R_{\mu\nu} = 0$ take the following interesting form in terms of the Ricci tensor \mathcal{R}_{ab} of the 3-space $d\sigma^2 = g_{ab} dx^a dx^b$:

$$R_{ab} = 0 \quad \Rightarrow \quad \mathcal{R}_{ab} = 2\partial_a \psi \partial_b \psi, \quad (2.3)$$

$$R_{33} = 0 \quad \Rightarrow \quad g^{ab} \nabla_a \nabla_b \psi = 0. \quad (2.4)$$

These equations can be interpreted as Einstein’s equations in three dimensions with a scalar field $\mathcal{F} = \psi/\sqrt{4\pi G}$ as a source.

⁴ Greek indices $\lambda, \mu, \nu, \rho, \dots$ run over the four spacetime coordinate labels 0, 1, 2, 3; Latin indices a, b, c, \dots run over the time and two spatial coordinate labels 0, 1, 2. The metric $g_{\mu\nu}$ has signature +--- and g is its determinant. Covariant derivatives are indicated by a D , partial derivatives by a ∂ and covariant derivatives in a three subspace by ∇_a . The permutation symbol in four dimensions is $\epsilon_{\mu\nu\rho\sigma}$ with $\epsilon_{0123} = 1$, $\eta_{\mu\nu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$.

The Einstein–Rosen metrics [16] represent a particular class of metrics of the form (2.2) which admits an additional hypersurface orthogonal Killing field of rotations η . In this case, coordinates are generally chosen such that $\eta = \{0, 0, 1, 0\}$ and are denoted by $x^0 = t, x^1 = \rho, x^2 = \varphi$. The metric then reads

$$ds^2 = e^{-2\psi} [e^{2\gamma} (dt^2 - d\rho^2) - \rho^2 d\varphi^2] - e^{2\psi} dz^2, \tag{2.5}$$

where ψ and γ are functions of t and ρ . Note that the radial variable ρ is a geometric object [17]. In this case the non-trivial equations (2.3) and (2.4) can be written as follows (in the formulae, $\partial_0 X = \partial_t X = \dot{X}$ and $\partial_1 X = \partial_\rho X = X'$ for any $X(t, \rho)$):

$$\mathcal{R}_{00} = \gamma'' + \frac{1}{\rho}\gamma' - \ddot{\psi} = 2\dot{\psi}^2, \tag{2.6}$$

$$\mathcal{R}_{11} = -\gamma'' + \frac{1}{\rho}\gamma' + \ddot{\psi} = 2\dot{\psi}'^2, \tag{2.7}$$

$$\mathcal{R}_{01} = \frac{1}{\rho}\dot{\gamma} = 2\dot{\psi}\psi' \tag{2.8}$$

and

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0. \tag{2.9}$$

The last equation is that of cylindrical waves in cylindrical coordinates in flat space. Given a solution ψ satisfying appropriate boundary conditions, the function γ is entirely defined by the following combinations of equations (2.6)–(2.8):

$$\gamma' = \rho(\dot{\psi}'^2 + \dot{\psi}^2) \quad \text{and} \quad \dot{\gamma} = 2\rho\dot{\psi}\psi'. \tag{2.10}$$

The wave equation (2.9) guarantees that the integrability conditions of the last two equations are satisfied.

Of particular interest are the Bonnor [18] and Weber–Wheeler [19] time symmetric incoming and outgoing waves which are smooth and finite everywhere at all time. They have also been discussed in some detail in Weber’s book [20]. That metric is not flat at spatial infinity. Its asymptotic behavior received special attention in [15]. The explicit forms of ψ and γ given below are taken from Ashtekar *et al* [15]; we shall write them with non-dimensional variables,

$$\tilde{\rho} = \frac{\rho}{a}, \quad \tilde{t} = \frac{t}{a}, \quad b = \frac{\sqrt{2}C}{a}, \tag{2.11}$$

where C and a are two constants of integration: a is a measure of the width and C is a measure of the amplitude of the waves⁵ [19, 20],

$$\psi = b \left\{ \frac{1 + \tilde{\rho}^2 - \tilde{t}^2 + [(1 + \tilde{\rho}^2 - \tilde{t}^2)^2 + 4\tilde{t}^2]^{1/2}}{(1 + \tilde{\rho}^2 - \tilde{t}^2)^2 + 4\tilde{t}^2} \right\}^{1/2}, \quad \tilde{\psi} = \frac{\psi}{b}, \tag{2.12}$$

$$\gamma = \frac{b^2}{4} \left\{ 1 - 2\tilde{\rho}^2 \frac{(1 + \tilde{\rho}^2 - \tilde{t}^2)^2 - 4\tilde{t}^2}{[(1 + \tilde{\rho}^2 - \tilde{t}^2)^2 + 4\tilde{t}^2]^2} - \frac{1 - \tilde{\rho}^2 + \tilde{t}^2}{[(1 + \tilde{\rho}^2 - \tilde{t}^2)^2 + 4\tilde{t}^2]^{1/2}} \right\}, \quad \tilde{\gamma} = \frac{\gamma}{b^2}. \tag{2.13}$$

Figure 1 illustrates $\tilde{\psi}(t, \rho)$ and $\tilde{\gamma}(t, \rho)$ as functions of $\tilde{\rho}$ for various values of \tilde{t} . Note that $\tilde{\gamma} \rightarrow \frac{1}{2}$ for $\rho \rightarrow \infty$.

⁵ We shall refer to them simply as ‘the width’ and ‘the amplitude’.

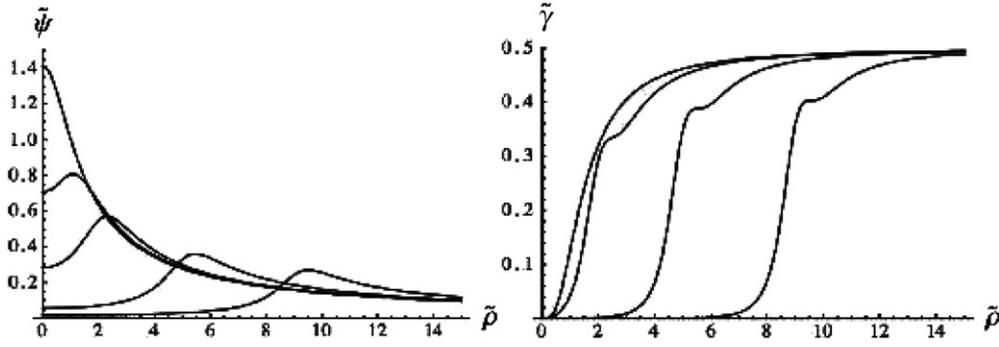


Figure 1. The plots of $\tilde{\psi}(\tilde{t}, \tilde{\rho})$ for $|\tilde{r}| = 0, 1, 2, 5, 9$ from the left to the right, and of $\tilde{\gamma}(\tilde{t}, \tilde{\rho})$ as functions of $\tilde{\rho}$ for $|\tilde{r}| = 0, 2, 5, 9$ from the left to the right.

3. The linear dragging in a Bonnor and Weber–Wheeler wave

One way to calculate the acceleration due to gravity on Earth is to calculate the force per unit mass necessary to oppose it and prevent the body from falling. To a relativist that force has the spatial components of the 4-acceleration necessary to keep the body at rest. Here we evaluate the force necessary to keep a particle in various geometrically defined positions subject to a Bonnor/Weber–Wheeler pulse. Surprisingly, we find it sometimes impossible to keep the particle on a cylinder of a fixed proper circumference length. However, it is possible to keep the particle at a fixed proper distance from the axis or at fixed ρ . These worldlines are always timelike.

Let $v^\mu = dx^\mu/d\tau$, $g_{\mu\nu}v^\mu v^\nu = 1$ represent the 4-velocity of a particle with proper time τ which is not falling freely. Note that the momenta of a *freely falling* particle in the φ or z directions are constants of motion because of the global symmetries of spacetime. Thus any acceleration in the φ or z directions can only be due to non-gravitational forces. Although the wave can stretch significantly distances between points in transverse directions, gravitational forces act only in the ρ direction. The interesting cases here are thus those of particles forced to stay or move in the ρ direction. Their 4-velocity has two nonzero components v^0 and v^1 . Following (2.5),

$$g_{\mu\nu}v^\mu v^\nu = e^{2(\gamma-\psi)}[(v^0)^2 - (v^1)^2] = 1, \quad (3.1)$$

while $v_0 = e^{2(\gamma-\psi)}v^0$ and $v_1 = -e^{2(\gamma-\psi)}v^1$. So equation (3.1) can also be written as

$$(v_0)^2 - (v_1)^2 = e^{2(\gamma-\psi)}. \quad (3.2)$$

The momenta per unit mass $v_2 = v_3 = 0$. The 4-acceleration is $a_\mu = Dv_\mu/D\tau = dv_\mu/d\tau + \frac{1}{2}v_\rho v_\sigma \partial_\mu g^{\rho\sigma}$. It has two components,

$$a_0 = \frac{dv_0}{d\tau} - (\dot{\gamma} - \dot{\psi}) \quad \text{and} \quad a_1 = \frac{dv_1}{d\tau} - (\gamma' - \psi'). \quad (3.3)$$

However, if the particle is forced to move along the worldline $[t(\tau), \rho(\tau)]$, so that $v_\mu = v_\mu[t(\tau), \rho(\tau)]$, we have successively

$$\frac{dv_0}{d\tau} = \dot{v}_0 v^0 + v_0' v^1 \quad \text{and} \quad \frac{dv_1}{d\tau} = \dot{v}_1 v^0 + v_1' v^1. \quad (3.4)$$

On the other hand, the derivatives of (3.2) give respectively

$$v_0 \dot{v}_0 - v_1 \dot{v}_1 = (\dot{\gamma} - \dot{\psi}) e^{\gamma-\psi} \quad \text{and} \quad v_0 v_0' - v_1 v_1' = (\gamma' - \psi') e^{\gamma-\psi}. \quad (3.5)$$

We may eliminate for instance \dot{v}_0 and v_1' from the pair of equations (3.4) and (3.5). The new expressions for $dv_0/d\tau$ and $dv_1/d\tau$ may be substituted into (3.3) which becomes

$$a_0 = v^1(v_0' - \dot{v}_1) \quad \text{and} \quad a_1 = -v^0(v_0' - \dot{v}_1). \quad (3.6)$$

This makes it obvious that $a_\mu v^\mu = 0$.

The frame component of the force f per unit mass or the acceleration exerted on the particle is

$$f = \sqrt{-g_{11}}a^1 = -\sqrt{-g^{11}}a_1 = -e^{-(\gamma-\psi)}a_1. \quad (3.7)$$

The magnitude of the acceleration is

$$\sqrt{-a_\mu a^\mu} = e^{-2(\gamma-\psi)}|v_0' - \dot{v}_1|. \quad (3.8)$$

Suppose now a particle sits at $\varphi = \varphi_0$ and $z = z_0$ and its worldline is given by

$$F(t, \rho) = \tilde{C}, \quad \tilde{C} \text{ is constant.} \quad (3.9)$$

If $F' \neq 0$, we may use \tilde{C} instead of ρ as a spatial coordinate. Differentiating (3.9), we obtain $d\tilde{C} = \dot{F} dt + F' d\rho$, from which follows that $d\rho = (1/F')(d\tilde{C} - \dot{F} dt)$. Substituting this $d\rho$ into (2.5) we obtain

$$ds^2 = \frac{e^{2(\gamma-\psi)}}{F'^2} [(F'^2 - \dot{F}^2) dt^2 + 2\dot{F} d\tilde{C} dt - d\tilde{C}^2] - e^{-2\psi} \rho^2 d\varphi^2 - e^{2\psi} dz^2. \quad (3.10)$$

The particle in question is at rest in $\{t, \tilde{C}, \varphi, z\}$ coordinates. The proper time elapsing after an interval dt is thus $d\tau = (e^{\gamma-\psi}/|F'|)\sqrt{F'^2 - \dot{F}^2} dt$, and *the particle can only be at rest if*

$$F'^2 - \dot{F}^2 > 0. \quad (3.11)$$

Otherwise $\tilde{C} = F(t, \rho)$ represents a spacelike worldline.

Assuming the inequality (3.11) is satisfied, we may calculate the force per unit mass that must act on a particle to be kept at fixed \tilde{C} . Since

$$\frac{dF}{d\tau} = \dot{F}v^0 + F'v^1 = e^{-2(\gamma-\psi)}(\dot{F}v_0 - F'v_1) = 0, \quad (3.12)$$

equation (3.12) with (3.2) defines v_0 and v_1 . One readily finds that if $v_0 > 0$ which is always allowed,

$$v_0 = \epsilon_{F'} F' H, \quad v_1 = \epsilon_{F'} \dot{F} H \quad \text{with} \quad \epsilon_{F'} = \text{sign}(F'), \quad H = \frac{e^{\gamma-\psi}}{\sqrt{F'^2 - \dot{F}^2}}, \quad (3.13)$$

and the force per unit mass is given by (3.7) or

$$f = e^{\psi-\gamma} v^0 (v_0' - \dot{v}_1). \quad (3.14)$$

Now the particle will stay on a circumference of length l if $F = \rho e^{-\psi} = l/2\pi$, or in dimensionless form, $L = \tilde{\rho} e^{-\psi} = l/2\pi a$. Then, $\partial_{\tilde{\rho}} L = (1 - \rho\psi') e^{-\psi}$ and $\partial_{\tilde{t}} L = -\rho\dot{\psi} e^{-\psi}$, and condition (3.11) amounts to having $(\partial_{\tilde{\rho}} L)^2 - (\partial_{\tilde{t}} L)^2 > 0$, or

$$U(\tilde{t}, \tilde{\rho}) = (1 - \rho\psi')^2 - (\rho\dot{\psi})^2 > 0. \quad (3.15)$$

We are interested in the sign of $U(\tilde{t}, \tilde{\rho})$ as a function of $L(\tilde{t}, \tilde{\rho})$. We consider first an amplitude $b = 1$. We shall see later what happens when b is not equal to 1.

$L(\tilde{t}, \tilde{\rho} = 0) = 0$ and $L(\tilde{t}, \tilde{\rho} = \infty) = \infty$. L increases monotonically for any $|\tilde{t}|$ less than about 6. For $|\tilde{t}| \gtrsim 6$, L decreases in some interval say $\{\tilde{\rho}_1, \tilde{\rho}_2\}$ which depends on $|\tilde{t}|$ and is always around $\tilde{\rho} = |\tilde{t}|$; see figure 2.

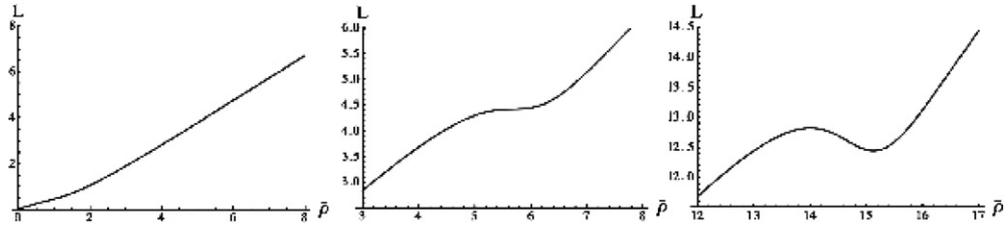


Figure 2. From left to right L as a function of $\tilde{\rho}$ for $|\tilde{t}| = 1, 6$ and 15 .

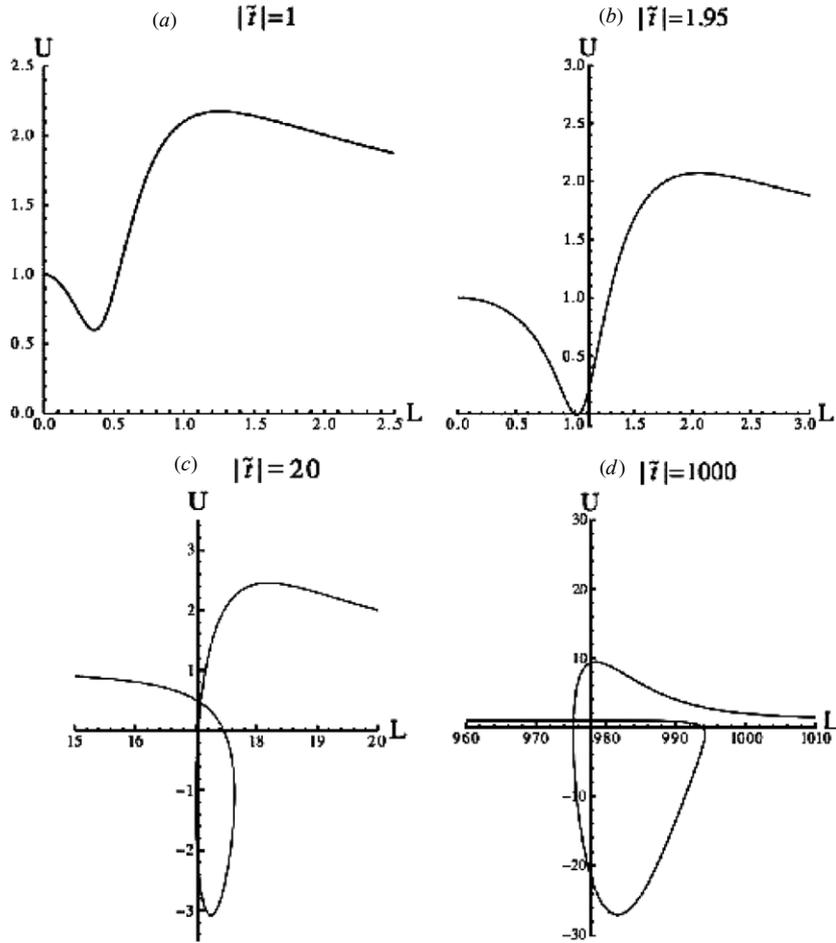


Figure 3. U as a function of L . A particle can stay at a given L only if $U > 0$ (see the text for details).

Now consider U . $U(\tilde{t}, \tilde{\rho} = 0) = 1$ and $U(\tilde{t}, \tilde{\rho} = \infty) = 1$. Thus U is positive for small and big L 's. What happens for intermediate L 's? Consider U as a function of L for various values of the dimensionless time \tilde{t} . For $|\tilde{t}| \lesssim 2$, $U > 0$. U varies monotonically when ρ nears 0 and ∞ . A typical example is given in figure 3(a) where $|\tilde{t}| = 1$. Note that in the graphs of figure 3 the U axis cuts the L axis where $|\tilde{t}| = \tilde{\rho}$. This is done for reasons

appearing later. Figure 3(b) shows how $U(L)$ varies when $|\tilde{t}| \simeq 1.95$; this is a limiting case. It becomes impossible to keep a particle on some of the cylinders at $|\tilde{t}| \gtrsim 1.95$. For instance for $|\tilde{t}| = 6$, when L has a horizontal inflexion, $U < 0$ for $4.22 \lesssim L \lesssim 4.56$ corresponding to an interval $4.83 \lesssim \tilde{\rho} \lesssim 6.32$. For $|\tilde{t}| \gtrsim 6$, $U(L)$ has a loop because L does no more increase monotonically; see for instance figure 3(c) in which $|\tilde{t}| = 20$. At that time it becomes impossible to keep a particle at $17.05 \lesssim L \lesssim 17.4$ corresponding to $18.1 \lesssim \tilde{\rho} \lesssim 20$. The loop persists at any later ($\tilde{t} \gtrsim 6$) or earlier ($\tilde{t} \lesssim -6$) times and the knot is always where $U > 0$ while U is always negative for $|\tilde{t}| = \tilde{\rho}$. See for instance figure 3(d) for $|\tilde{t}| = 1000$.

What if the amplitude b is different from 1? For $b \neq 1$ the phenomena is essentially the same. Loops appear at bigger L 's for $b < 1$ and at small L 's for $b > 1$. It may be noted that the U axis, which cuts the L axis where $|\tilde{t}| = \tilde{\rho}$ also cuts the loop in the region where $U < 0$. It is therefore interesting to evaluate the upper bound on b resulting from condition (3.15) for $|\tilde{t}| = \tilde{\rho}$. In this way we find that U is always negative when $|\tilde{t}| = \tilde{\rho}$ if

$$b > b_M = \frac{\sqrt{(1+4\tilde{\rho}^2)\sqrt{1+(1+4\tilde{\rho}^2)}}}{2\tilde{\rho}^2}. \quad (3.16)$$

Thus, the phenomena described for $b = 1$ exists at any amplitude however small since for $\tilde{\rho} \rightarrow \infty$, $b_M \rightarrow 2^{5/4}/\tilde{\rho}^{1/2} \rightarrow 0$.

If a particle is kept at a fixed proper distance $\lambda = \text{const}$ from the axis, then condition (3.11) becomes

$$-e^{\gamma-\psi} < \partial_t \left(\int_0^\rho e^{\gamma-\psi} d\rho \right) < e^{\gamma-\psi}. \quad (3.17)$$

The situation is even clearer, and much easier to manage numerically, if the particle is kept at fixed⁶ ρ . Since ρ is a spatial coordinate there is manifestly no condition like (3.11). In this case, $v^1 = 0$, $v_0 = e^{\gamma-\psi}$, and the dimensionless force is given by $\tilde{f} = af = -\partial_\rho e^{-(\gamma-\psi)}$. In real value, $f = \frac{c^2}{a} \tilde{f}$. The particle can always be kept at fixed radial coordinate ρ .

As an example, let $\tilde{f} \simeq 0.3$, a characteristic value for $|\tilde{t}| \gtrsim 20$ and $b = 1$. Let f be barely bearable, say $f = 5g \simeq 5 \times 10^3 \text{ cm s}^{-2}$. Then, $a = \frac{c^2 \tilde{f}}{f} \simeq 3 \times 10^3 \text{ AU}$. The amplitude $\sqrt{2}C = ab$ is of the same order as a . The characteristic time t_C during which the acceleration must be withstood is $t_C \simeq \frac{a}{c} \simeq 2 \times 10^6 \text{ s} \simeq 3 \text{ weeks}$.

4. Angular momentum of cylindrically symmetric spacetimes

Here we consider spacetimes with two spacelike Killing fields ζ and η . ζ is in general hypersurface orthogonal but rotational η is not, so that the two-dimensional isometry group is not orthogonally transitive [21]. The spacetime may admit an additional Killing vector ξ which is timelike and may thus represent a stationary field of a rotating massive cylinder or an infinite straight rotating cosmic string.

Although in the main part, section 5, we require the axis of symmetry to be regular, in section 6 we also discuss some aspects of cylindrical spacetimes containing an infinitely thin string in the ‘wire approximation’. It is associated with specific conical singularities along the axis which, though being ‘quasi-regular’, do not strictly speaking, belong to the spacetime. An extensive literature on cylindrical waves interacting nonlinearly with cosmic strings is available⁷. The wave solutions *in vacuo* are considered by some authors as too restrictive. The reason is a generalization of Papapetrou’s theorem from stationary axisymmetric vacuum

⁶ See [17] where it is shown that ρ is a geometrically well-defined quantity.

⁷ See the review [22] and references below.

spacetimes to the cylindrical case: if the spacetime contains at least part of a regular axis of cylindrical symmetry, the isometry group must be orthogonally transitive, so Killing orbits admit orthogonal surfaces [23–25]. The orthogonal transitivity thus excludes the possibility of a global rotation. In other words *in vacuo* there can be no ‘rotating cylindrical waves’. With a material source present as in the case of the rigidly rotating dust cylinder [26], for example, the spacetime can, of course, be regular everywhere with a non-vanishing angular momentum per unit length. Bondi [27] studied general changes in time of such systems which can lead to radiation. As he noted, the conservation of angular momentum occurs even if gravitational waves are emitted by the cylinder since the cylindrical symmetry of the waves precludes their carrying angular momentum.

In all the cases mentioned, the spacetimes possess the rotational Killing vector η with closed orbits. This can be normalized at the axis of symmetry, with the corresponding modification in the case of a string present along the axis. When the rotational Killing vector is available we can define a total angular momentum density contained inside a two-dimensional cylinder \mathcal{C} of ‘unit coordinate height’, generated by orbits of the isometry group, by adapting the Komar [28] expression⁸ to cylindrical symmetry,

$$J(\mathcal{C}) = \frac{1}{2\kappa} \int_{\mathcal{C}} D^{[\mu} \hat{\eta}^{\nu]} d\Sigma_{\mu\nu}, \quad \text{where} \quad d\Sigma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma, \\ \hat{\eta}^\nu = \sqrt{-g} \eta^\mu, \quad \kappa = \frac{8\pi G}{c^2}. \quad (4.1)$$

Since

$$\hat{j}^\mu = \frac{1}{2\kappa} \partial_\nu (D^{[\mu} \hat{\eta}^{\nu]}) \quad (4.2)$$

is conserved, $\partial_\mu \hat{j}^\mu = 0$, in vacuo the integral (4.1) is independent of the choice of \mathcal{C} ; the integral over ‘bottom’ and ‘top’ of the solid cylinder formed by two cylinders \mathcal{C} and \mathcal{C}' vanishes due to cylindrical symmetry. This corresponds to the well-known fact that the Komar energy, defined as (4.1) with η being replaced by the timelike Killing field ξ in stationary spacetimes, is independent of the choice of the closed 2-surface as it is moved continuously through a vacuum region [30]. Hence, we can define the total angular momentum per unit length in the ζ direction by

$$J = \frac{1}{2\kappa} \int_{\mathcal{C}_\infty} D^{[\mu} \hat{\eta}^{\nu]} d\Sigma_{\mu\nu}, \quad (4.3)$$

where \mathcal{C}_∞ denotes the cylinder of unit length at infinity (there the proper and coordinate lengths along the ζ direction coincide).

Let us express the angular momentum (4.3) (‘density’ will often be omitted) explicitly for some specific metrics. In all the cases the coordinates adapted to the symmetry are those defined in section 2 (equation (2.4)). The 2-surface element

$$d\Sigma_{01} = -d\Sigma_{10} = d\varphi dz. \quad (4.4)$$

Hence the angular momentum (4.3) reads

$$J = \frac{1}{\kappa} \int_{\mathcal{C}_\infty} D^{[0} \hat{\eta}^{1]} d\varphi dz. \quad (4.5)$$

The integrand does not depend on φ and z . Integrating we thus get⁹

$$J = \frac{2\pi}{\kappa} [D^{[0} \hat{\eta}^{1]}]_{\rho \rightarrow \infty}. \quad (4.6)$$

⁸ For using Komar’s expression in the case of the angular momentum, see [29] and [30, p 296].

⁹ Let us remark that with z replaced by $\theta \in [0, \pi]$ and $\{t, \rho, \theta, \varphi\}$ denoting the Boyer–Linqvist coordinates in the Kerr metric (e.g. [30]) with the mass M and the angular momentum per unit mass a , expression (4.6) yields indeed $J = Ma$.

Mashhoon *et al* [31] considered a class of ‘rotating gravitational waves’ which should represent ‘radiation that propagates outward or inward and at the same time has non-trivial azimuthal motion’. Their metrics satisfying vacuum field equations have the form

$$ds^2 = e^{2\gamma-2\psi} (dt^2 - d\rho^2) - W^2 e^{-2\psi} (d\varphi + \omega dt)^2 - e^{2\psi} dz^2; \quad (4.7)$$

the functions γ , ψ , W and ω depend on t and ρ only (in [31] $\rho = R$ and $W = \mu$). If $\omega = 0$, the field equations imply that W can be chosen equal to ρ , and the metrics (4.7) become the well-known metrics of Einstein–Rosen waves (e.g. [15, 21]). The calculation of the angular momentum (4.6) yields

$$J = \frac{\pi}{\kappa} W^3 e^{-2\gamma} \omega' \Big|_{\rho \rightarrow \infty}. \quad (4.8)$$

In section 6 we shall especially need the angular momentum for small ω . One of the field equations (cf [31] or the appendix where the Ricci tensor in the reduced three-dimensional spacetime is given) then implies $W = \rho$ (like in the case of Einstein–Rosen waves), and $\omega = \varpi_\infty / \rho^2 + \mathcal{O}(\rho^{-3})$. The angular momentum then becomes

$$J = \frac{\pi}{\kappa} \rho^3 e^{-2\gamma} \omega' \Big|_\infty = -\frac{2\pi}{\kappa} e^{-2\gamma_\infty} \varpi_\infty \quad \text{or} \quad \varpi_\infty = -\frac{\kappa}{2\pi} J e^{2\gamma_\infty}. \quad (4.9)$$

Next, consider the case of infinitely thin cosmic strings. When they possess an angular momentum J , the corresponding metric can be written as (e.g. [32, 33])

$$ds^2 = \left(dt + \frac{\kappa}{2\pi} J d\varphi \right)^2 - (\bar{\rho})^{-\kappa\mu/\pi} (d\bar{\rho}^2 + \bar{\rho}^2 d\varphi^2) - dz^2, \quad (4.10)$$

with μ being the mass per unit length. After putting $(\bar{\rho})^{-4\mu} d\bar{\rho} = d\rho$, the metric can be rewritten as

$$ds^2 = \left(dt + \frac{\kappa}{2\pi} J d\varphi \right)^2 - d\rho^2 - \left(1 - \frac{\kappa\mu}{2\pi} \right)^2 \rho^2 d\varphi^2 - dz^2. \quad (4.11)$$

In both metrics $\varphi \in [0, 2\pi)$.¹⁰ Comparing (4.10) and (4.11) with the metric (4.7) and the angular momentum (4.9) for small ω , resp. J , it is seen immediately that the Komar expression gives indeed J appearing in the metric.

Various authors [34–37] constructed new families of exact solutions of the vacuum Einstein (and Einstein–Maxwell) equations for cylindrically symmetric non-stationary spacetimes which they interpreted as gravitational (and electromagnetic) cylindrical waves interacting with a rotating cosmic string. In their solutions, however, the cross-term $dt d\varphi$ is missing. Applying our expression (4.6) for the angular momentum to their metrics, we get $J = 0$. More recently, Manojlović and Marugán [38] considered thoroughly cylindrical waves interacting with spinning cosmic strings within the Hamiltonian formalism. In their work the non-diagonal terms proportional to $dt d\varphi$ are admitted—their value of the angular momentum coincides with that following from (4.9). Let us also state that our expression (4.6), taken at fixed ρ , when applied to Bondi’s cylindrical metric (equation (1) in [27]) given in the Weyl–Papapetrou form, yields precisely the Bondi expression for the angular momentum.

To the end of this section let us mention one plausible feature of the angular momentum defined above. Imagine we transform the metric (4.7) into rotating axes by introducing $\bar{\varphi} = \varphi + \Omega t$, in which Ω may depend on time. Then $d\bar{\varphi} = d\varphi + \Omega(t) dt$, but $\bar{\omega}' = \omega'$ so that the total angular momentum (4.9) remains unchanged. It thus characterizes an intrinsic property of the field.

¹⁰ The transformation $\bar{\varphi} = (1 - \frac{\kappa\mu}{2\pi})\varphi$ accompanied with the change of time $\bar{t} = t + \frac{\kappa}{2\pi} J\varphi$ leads to the flat metric with the range $\bar{\varphi} \in [0, (1 - \frac{\kappa\mu}{2\pi})2\pi)$, corresponding to conical geometry and the time coordinate ‘jumping’ by κJ when the string is circumrotated.

5. Rotating waves in the symmetry reduced general relativity

To construct rotating gravitational waves with a non-vanishing angular momentum we can keep the translational symmetry but we have to drop the assumption of axial symmetry. In general we would have to solve the coupled system of equations (2.3) and (2.4) with ψ and g_{ab} depending on all three coordinates $x^0 = t$, $x^1 = \rho$, $x^2 = \varphi$. In three-dimensional spacetimes the Riemann tensor

$$\mathcal{R}_{abcd} = 2 \left[(\mathcal{R}_{a[c} - \frac{1}{4} g_{a[c} \mathcal{R}) g_{d]b} - (\mathcal{R}_{b[c} - \frac{1}{4} g_{b[c} \mathcal{R}) g_{d]a} \right]. \quad (5.1)$$

The Ricci tensor can be determined directly from the field ψ by equations (2.3).

Clearly to tackle such a problem is a formidable task. However, the above form of Einstein's equations suggests naturally the following approximation procedure. Assume the field ψ and its derivatives to be small, $\psi = \epsilon \Psi(x^a)$, ϵ is a small dimensionless parameter. Then the field equations (2.3) show that $\mathcal{R}_{ab} \sim \mathcal{O}(\epsilon^2)$ and, according to (5.1), the Riemann tensor can be calculated in terms of $\partial_a \psi$ up to $\mathcal{O}(\epsilon^2)$. We may thus write the three-dimensional metric in the form

$$g_{ab} = \eta_{ab} + \epsilon^2 \gamma_{ab}(x^c) \quad \text{or} \quad g^{ab} = \eta^{ab} - \epsilon^2 \gamma^{ab}(x^c). \quad (5.2)$$

Therefore we can construct a *rotating* (' φ -dependent') solution of the wave equation in *flat space* in cylindrical coordinates $\{t, \rho, \varphi\}$ and still satisfy the field equations (2.4) in terms $\sim \mathcal{O}(\epsilon^2)$.

Next, we can write the Ricci tensor on the left-hand side of equations (2.3) for a general φ -dependent metric, $g_{ab} = g_{ab}(t, \rho, \varphi)$, and solve the equations in terms which are $\sim \mathcal{O}(\epsilon^2)$. However, we are primarily interested in the rotation of inertial frames at the axis. φ -dependent terms in ω do not affect the rotation there. We therefore concentrate on the axially symmetric terms in the Fourier expansion of ω as a function of φ . We may calculate such terms from equations *averaged* over φ . Since we are interested in the averaged effects we consider the right-hand side of (2.3) averaged over φ , i.e., we assume the effective matter source to be of the form

$$S_{ab} = \langle \partial_a \psi \partial_b \psi \rangle = \int_0^{2\pi} \partial_a \psi \partial_b \psi \, d\varphi. \quad (5.3)$$

Since the source becomes then axisymmetric, we consider the metric g_{ab} , resp. γ_{ab} axisymmetric as well. However, the Killing vector η is not in general hypersurface orthogonal, and we cannot use the vacuum equations to simplify the choice of coordinates. A sufficiently general axisymmetric 3-metric has the form

$$d\sigma^2 = e^{2\gamma} (dt^2 - d\rho^2) - W^2 (d\varphi - \omega dt)^2, \quad (5.4)$$

where γ , W and ω are functions of t and ρ . Note that this is precisely the 4-metric (4.7) reduced to the three dimensions and rescaled by the norm of the Killing vector ζ . The metric (5.4) satisfies the relations $g_{00} = -g_{11}$, $g_{01} = g_{12} = 0$, which can be achieved by a suitable choice of coordinates. The Ricci tensor components for the exact metric (5.4) are listed in the appendix. Within our approximation scheme it is sufficient to consider only terms $\sim \mathcal{O}(\epsilon^2)$. Since for $\epsilon = 0$ the metric (5.4) becomes flat, we can write

$$W(t, \rho) = \rho + \epsilon^2 w(t, \rho), \quad (5.5)$$

where $w \sim \mathcal{O}(\rho^2)$ at $\rho \rightarrow 0$, but it will be convenient to work with W . Henceforth, ϵ^2 is absorbed in the corresponding symbols, so it is assumed that ψ is of the order $\mathcal{O}(\epsilon)$, and γ , w and ω and their derivatives are all of $\mathcal{O}(\epsilon^2)$; their products can be neglected. From the

expressions for \mathcal{R}_{ab} in the appendix we get, to order $\mathcal{O}(\epsilon^2)$, the following set of equations:

$$\mathcal{R}_{00} = -\ddot{\gamma} + \gamma'' + \frac{1}{\rho}\gamma' - \frac{1}{\rho}\ddot{W}, \quad \mathcal{R}_{01} = \frac{1}{\rho}\dot{\gamma} - \frac{1}{\rho}\dot{W}', \quad \mathcal{R}_{02} = \frac{1}{2\rho}(\rho^3\omega')', \quad (5.6)$$

$$\mathcal{R}_{11} = \dot{\gamma} - \gamma'' + \frac{1}{\rho}\gamma' - \frac{1}{\rho}W'', \quad \mathcal{R}_{12} = \frac{1}{2}\rho^2\dot{\omega}', \quad \mathcal{R}_{22} = \rho(\ddot{W} - W''). \quad (5.7)$$

The wave equation (2.4) for the field $\psi(t, \rho, \varphi)$ becomes, in our approximation, the flat space wave equation in cylindrical coordinates,

$$\ddot{\psi} - \psi'' - \frac{1}{\rho}\psi' - \frac{1}{\rho^2}\partial_\varphi^2\psi = 0. \quad (5.8)$$

For $W = \rho, \omega = 0$ we recover the Einstein–Rosen waves. In their case one can determine $\gamma(t, \rho)$ by a quadrature. Now the equations

$$\mathcal{R}_{ab} = 2\langle\partial_a\psi\partial_b\psi\rangle = 2\mathcal{S}_{ab}, \quad (5.9)$$

with \mathcal{R}_{ab} given by (5.6) and (5.7) should determine the functions γ, W and ω .

It is well known that the gravitational field in the three-dimensional spacetime has no dynamical degrees of freedom; these are all contained in the matter fields, in our case in the field ψ . The entire curvature (5.1) is determined by the local distribution of ψ . Outside matter the spacetime is flat. In the canonical gravity language, the two-dimensional metric and the conjugate momenta have six independent components, three of which can be annulled by the choice of a spacelike hypersurface (time) and two suitable coordinates on the hypersurface. In addition, there are three constrained field equations (see, e.g., [39] for an interesting exposition of Einstein’s theory in a three-dimensional spacetime).

We have chosen suitable coordinates in the metric (5.4) already. Hence, we have to look at the constraints. One is evident. The equation

$$\mathcal{R}_{02} = \frac{1}{2\rho}(\rho^3\omega')' = 2\langle\dot{\psi}\partial_\varphi\psi\rangle, \quad (5.10)$$

following from (5.6), gives with suitable boundary conditions the ‘angular velocity’ $\omega(t, \rho)$ in terms of the averaged ‘angular momentum’ component \mathcal{S}_{02} of rotating ψ -waves. This equation will be analyzed in depth in the accompanying paper [14]. In the explicit construction we shall demonstrate how rotating gravitational waves rotate local inertial frames in a ‘global sense’, like rotating ‘ordinary’ matter does. Alternatively, the integration function of time is fixed by the equation $\mathcal{R}_{12} = 2\mathcal{S}_{12}$, see (5.7). The equation for ω still leaves ω undetermined up to an arbitrary function of time $\omega_0(t)$. This arbitrariness, however, corresponds just to going over to rotating axes (cf the end of section 4). Another constraint equation, $\mathcal{R}_{01} = 2\mathcal{S}_{01}$, see (5.6), can be written as

$$\frac{1}{\rho}\mathcal{F} = 2\mathcal{S}_{01} \quad \text{where} \quad \mathcal{F} = \gamma - W'. \quad (5.11)$$

This becomes (2.8) for the Einstein–Rosen waves ($W = \rho$).

There is still a constraint equation, the one corresponding to the Hamiltonian constraint in canonical gravity. Its left-hand side is given by $\mathcal{R}_{00} - \frac{1}{2}g_{00}\mathcal{R}$ which, in our approximation, is equal to $\frac{1}{2}(\mathcal{R}_{00} + \mathcal{R}_{11} + \rho^{-2}\mathcal{R}_{22})$. Substituting for the Ricci tensor from (5.6) and (5.7), we obtain the constraint in the form

$$\frac{1}{\rho}\mathcal{F}' = \frac{1}{\rho}(\gamma - W')' = \mathcal{S}_{00} + \mathcal{S}_{11} + \frac{1}{\rho^2}\mathcal{S}_{22}. \quad (5.12)$$

Before we indicate how to solve the field equations, we investigate the consistency of (5.10) and (5.11). This should generalize the condition $(\dot{\gamma})' = (\gamma')'$ for the Einstein–Rosen

waves when $\dot{\gamma}$ is expressed from (2.8), whereas γ' from the sum of (2.6) and (2.7). There the consistency is guaranteed by the wave equation (2.9) for the source ψ . Now ψ satisfies the flat-space wave equation (5.8) in cylindrical coordinates with $\partial_\varphi \psi \neq 0$. As a consequence of this equation the effective energy–momentum tensor

$$T_{ab} = \partial_a \psi \partial_b \psi - \frac{1}{2} \bar{g}_{ab} \bar{g}^{cd} \partial_c \psi \partial_d \psi, \quad \bar{g}_{ab} = (1, -1, -\rho^2), \quad (5.13)$$

satisfies the conservation law

$$\nabla_b T_a^b = \frac{1}{\sqrt{-\bar{g}}} \partial_b (\sqrt{-\bar{g}} T_a^b) - \frac{1}{2} T^{cd} \partial_b \bar{g}_{cd} = 0. \quad (5.14)$$

Denoting by $s_{ab} = \partial_a \psi \partial_b \psi$ without the averaging so that $\langle s_{ab} \rangle = \mathcal{S}_{ab}$, we easily get the relations

$$T_{ab} = s_{ab} - \frac{1}{2} s g_{ab} \quad \Rightarrow \quad s_{ab} = T_{ab} - T g_{ab} \quad \text{where} \\ s = -2T = \dot{\psi}^2 - \psi'^2 - \frac{1}{\rho^2} (\partial_\varphi \psi)^2. \quad (5.15)$$

The conservation law (5.14) written in terms of s_{ab} reads explicitly as follows:

$$\dot{s}_{00} - \frac{1}{\rho} s_{01} - s'_{01} - \frac{1}{\rho^2} \partial_\varphi s_{02} - \frac{1}{2} \dot{s} = 0, \quad (5.16)$$

$$\dot{s}_{01} - \frac{1}{\rho} s_{11} - s'_{11} - \frac{1}{\rho^2} \partial_\varphi s_{12} + \frac{1}{\rho^3} s_{22} - \frac{1}{2} s' = 0, \quad (5.17)$$

$$\dot{s}_{02} - \frac{1}{\rho} s_{12} - s'_{12} - \frac{1}{\rho^2} \partial_\varphi s_{22} - \frac{1}{2} \partial_\varphi s = 0. \quad (5.18)$$

The averaging over φ makes the $\partial_\varphi s_{ab}$ -terms to vanish so that \mathcal{S}_{ab} satisfies the equations

$$\dot{\mathcal{S}}_{00} - \frac{1}{\rho} \mathcal{S}_{01} - \mathcal{S}'_{01} - \frac{1}{2} \dot{\mathcal{S}} = 0, \quad \dot{\mathcal{S}}_{01} - \frac{1}{\rho} \mathcal{S}_{11} - \mathcal{S}'_{11} + \frac{1}{\rho^3} \mathcal{S}_{22} - \frac{1}{2} \mathcal{S}' = 0, \quad (5.19)$$

$$\dot{\mathcal{S}}_{01} - \frac{1}{\rho} \mathcal{S}_{12} - \mathcal{S}'_{12} = 0 \quad \text{with} \quad \mathcal{S} = \mathcal{S}_{00} - \mathcal{S}_{11} - \frac{1}{\rho^2} \mathcal{S}_{22}. \quad (5.20)$$

The compatibility of (5.11) and (5.12) requires

$$(\dot{\mathcal{F}})' = 2(\rho \mathcal{S}_{01})' = (\mathcal{F}') \cdot = \rho \left(\dot{\mathcal{S}}_{00} + \dot{\mathcal{S}}_{11} + \frac{1}{\rho^2} \dot{\mathcal{S}}_{22} \right). \quad (5.21)$$

This is indeed satisfied as a consequence of (5.19).

What boundary conditions do we impose? As emphasized above, all the dynamics is governed by the field $\psi(t, \rho, \varphi)$ satisfying the wave equation (5.8). We consider decaying solutions at spatial infinity which guarantee that the three-dimensional spacetime is asymptotically flat although the metric at spatial infinity has a ‘conical singularity’ which is the measure of the total energy of the scalar field ψ computed using the Minkowski metric. This will generalize naturally the case of the Einstein–Rosen waves (see [13, 15] for the details). Hence we assume the solutions of (5.8) to have the expansion on each Cauchy surface $t = \text{const}$ of the form

$$\psi(t, \rho, \varphi) = \frac{1}{\rho^{1/2}} \left[f_0(t, \varphi) + \sum_{k=1}^{\infty} \frac{f_k(t, \varphi)}{\rho^k} \right]. \quad (5.22)$$

The solutions of the wave equation in 2+1 dimensions well behaved at infinity involve the Bessel functions which decay as $\rho^{-1/2}$. Our explicit solutions constructed in the accompanying paper [14] will all admit the expansion of the form (5.22).

Since the spacetime should be asymptotically flat at infinity the function W must behave as follows:

$$W(t, \rho) = \rho + w_0(t) + \sum_{k=1}^{\infty} \frac{w_k(t)}{\rho^k}. \quad (5.23)$$

However, regarding the field equation (5.7),

$$\mathcal{R}_{22} = \rho(\ddot{W} - W'') = 2\mathcal{S}_{22}, \quad (5.24)$$

we see that $w_0(t) = w_1(t) = 0$ since $\mathcal{S}_{22}/\rho \sim \rho^{-2}$ as a consequence of (5.22). Hence, we may write

$$W(t, \rho) = \rho + \sum_{k=2}^{\infty} \frac{w_k(t)}{\rho^k}. \quad (5.25)$$

A similar argument based on the field equation $\mathcal{R}_{02} = 2\mathcal{S}_{02}$ implies the following behavior of the ‘dragging factor’ ω :

$$\omega(t, \rho) = \frac{\varpi_{\infty}(t)}{\rho^2} + \sum_{k=1}^{\infty} \frac{\pi_k(t)}{\rho^{k+2}}. \quad (5.26)$$

Since $\mathcal{S}_{12} \sim \rho^{-2}$, the equation $\mathcal{R}_{12} = 2\mathcal{S}_{12}$ with \mathcal{R}_{12} given in (5.7) implies

$$\varpi_{\infty}(t) = \text{const.} \quad (5.27)$$

This constant is directly related to the total angular momentum in the waves (see equation (4.9)).

Now in order to guarantee a regular axis, without a conical singularity present, we require the following conditions at $\rho = 0$:

$$W(t, 0) = \rho, \quad W'(t, 0) = 1, \quad \omega(t, 0) = \omega_0(t), \quad \omega'(t, 0) = 0, \quad \gamma(t, 0) = 0. \quad (5.28)$$

The only missing boundary condition is that on γ at infinity. To find it, we return back to the ‘Hamiltonian’ constraint, multiply by ρ and integrate to obtain

$$\gamma(t, \rho) - \gamma(t, 0) - W'(t, \rho) + W'(t, 0) = \int_0^{\rho} \bar{\rho} \left(\mathcal{S}_{00} + \mathcal{S}_{11} + \frac{1}{\bar{\rho}^2} \mathcal{S}_{22} \right) d\bar{\rho} + \lambda(t), \quad (5.29)$$

where $\lambda(t)$ is an arbitrary function of time, and $\gamma(t, 0) = 0$, $W'(t, 0) = 1$ by the boundary conditions (5.28). Regarding the behavior of $W(t, \rho)$ in accordance with (5.25) and realizing that $\mathcal{S}_{01} = \langle \dot{\psi} \psi' \rangle \sim \rho^{-2}$, we can deduce from the constraint (5.11) that $\lambda(t)$ must be a constant. Since, in addition, we have to satisfy the boundary condition $\gamma(t, 0) = 0$, this constant has to vanish and the integrated constraint equation (5.29) becomes

$$\gamma(t, \rho) - W'(t, \rho) + 1 = \int_0^{\rho} \bar{\rho} \left(\mathcal{S}_{00} + \mathcal{S}_{11} + \frac{1}{\bar{\rho}^2} \mathcal{S}_{22} \right) d\bar{\rho}. \quad (5.30)$$

At $\rho = 0$, $W'(t, 0) = 1$ and, indeed, $\gamma(t, 0) = 0$. Now for $\rho \rightarrow \infty$, $W'(t, \infty) = 1 + \mathcal{O}(\rho^{-3})$, (cf (5.25)), so putting $\gamma(t, \infty) \equiv \gamma_{\infty}$, we get

$$\gamma_{\infty} = \int_0^{\infty} \left(\mathcal{S}_{00} + \mathcal{S}_{11} + \frac{1}{\rho^2} \mathcal{S}_{22} \right) \rho d\rho = \int_0^{\infty} \left(\langle \dot{\psi}^2 \rangle + \langle \psi'^2 \rangle + \frac{\langle (\partial_{\varphi} \psi)^2 \rangle}{\rho^2} \right) \rho d\rho. \quad (5.31)$$

Therefore the function γ at spatial infinity approaches a constant which is equal to the total energy of the scalar field ψ computed using the Minkowski metric. In the case of the Einstein–Rosen waves, the term $\rho^{-2} \langle (\partial_{\varphi} \psi)^2 \rangle$ vanishes, and we recover the result (2.35) in

[13]. Although γ_∞ is energy for weak fields, the physical Hamiltonian turns out to be a non-polynomial function of γ_∞ for in general strong fields [15, 40].

In order to find the complete solution of the averaged Einstein equations up to order $\mathcal{O}(\epsilon^2)$ we have to solve equation (5.24) with (5.28), and with the right-hand side given in terms of ψ . The function γ can then be determined from (5.30). Alternatively, we can take the combination $\frac{1}{2}(-\mathcal{R}_{00} + \mathcal{R}_{11} - \rho^{-1}\mathcal{R}_{22})$ and obtain an equation for γ , similar to (5.24) for W , in the form

$$\ddot{\gamma} - \gamma'' = \mathcal{S}_{11} - \mathcal{S}_{00} + \frac{1}{\rho^2}\mathcal{S}_{22}. \quad (5.32)$$

Both equations (5.24) and (5.32) are one-dimensional wave equations with given right-hand side and boundary conditions $\gamma = \gamma_\infty$, $W = 1$ at $\rho \rightarrow \infty$ and $\gamma = 0$, $W = 1$ at $\rho = 0$. Both equations have thus the same form as the equation for the motion of an elastic string with fixed end points under the influence of an external force. There are standard techniques for solving such equations. For example, one can find the desired result by expanding the solution in time t in terms of eigenfunctions $\sin n\bar{\rho}$ and expanding the ‘external force’ similarly (see [41]).

To determine the ‘dragging potential’ $\omega(t, \rho)$ we employ simpler equations $\mathcal{R}_{02} = 2\mathcal{S}_{02}$ and $\mathcal{R}_{12} = 2\mathcal{S}_{12}$, with \mathcal{R}_{02} and \mathcal{R}_{12} given by (5.6) and (5.7). The explicit solution of these equations with the source field ψ representing specific ingoing and outgoing pulses of the rotating gravitational waves demonstrates nicely the dragging effects of such waves. We treat this problem in-depth in the following paper [14].

6. Dragging of inertial frames in axisymmetric spacetimes with cylindrical waves and spinning string

The vacuum metrics (4.7) considered by Mashhoon *et al* [31] as ‘rotating gravitational waves’ cannot have a regular axis since this requires the Killing orbits to admit orthogonal surfaces. However, the axis can represent a rotating cosmic string. Let us assume the ‘rotation parameter’ ω in (4.7) is small so that the terms in $\mathcal{O}(\omega^2)$ can be neglected. The inspection of vacuum field equations following from the ansatz (4.7) then reveals that one can choose $W = \rho$ and equations for ψ and γ are the same as (2.9) and (2.10), see the appendix. The rotational perturbation ω is determined by the equation

$$R_{12} = -\frac{1}{2\rho}(\rho^3 e^{-2\gamma} \omega')' = 0, \quad (6.1)$$

and the constraint equation

$$R_{02} = -\frac{1}{2\rho}(\rho^3 e^{-2\gamma} \omega')' = 0. \quad (6.2)$$

These two equations are directly related to the properties of the angular momentum of cylindrical systems derived from the Komar expressions (4.1) or (4.3) and (4.6). In (4.6) we indicated that one can evaluate the integral at $\rho \rightarrow \infty$ since its value in vacuum does not depend on the location of the cylinder \mathcal{C} . For the metrics of the form (4.7) with small ω we obtained the result (4.9) which again one can evaluate at $\rho \rightarrow \infty$. However, it is evident from the constraint equation (6.2) that the same value for J results at any $\rho \neq 0$ since $\rho^3 e^{-2\gamma} \omega'$ is independent of ρ and by (6.1) independent of time.

Integration of (6.1) and (6.2) yields

$$\omega' = \frac{e^{2\gamma(t,\rho)} \kappa}{\rho^3 \pi} J, \quad (6.3)$$

where the integration constant was chosen in accordance with the expression (4.9) for the angular momentum. The angular momentum is a small constant parameter with $\frac{\kappa J}{\pi}$ having the dimensions of length. Hence,

$$\omega(t, \rho) = -\frac{\kappa}{\pi} J \int_{\rho}^{\infty} \frac{e^{2\gamma(t,\rho)}}{\rho^3} d\rho + \omega_0(t); \quad (6.4)$$

we put $\omega_0 = 0$ because it can be transformed away by going to rotating axes, without a change of J (cf section 4).

When no wave is present, (6.4) gives $\omega \equiv \omega_S = -\frac{\kappa}{2\pi} \frac{J}{\rho^2}$; the metric (4.7) with $W = \rho$ becomes precisely the metric (4.11) of the massless spinning string with angular momentum J . The ‘string perturbation’ ω_S causes rotation of local inertial frames at some finite ρ as compared with frames at infinity. However, this relative rotation is time independent as, e.g., in the Kerr metric.

Now consider ψ and γ to represent an exact cylindrical gravitational wave, e.g., the Bonnor/Weber–Wheeler wave given in (2.12) and (2.13). Since the rotational effects are given by a linear perturbation of the metric we can write the total perturbation (6.3) as $\omega' = \omega'_S + \omega'_W$, the time independent ω'_S is attributed to the spinning string, whereas the time-dependent part

$$\omega'_W = \frac{\kappa J}{\pi \rho^3} (e^{2\gamma(t,\rho)} - 1) \quad (6.5)$$

is associated with the presence of a cylindrical gravitational wave interacting with the string.

Now at $t^2 \gg a^2 + \rho^2$, (2.13) implies $\gamma \approx 0$, so when the pulse is far away from the axis, the dragging near the axis is dominated by the string. As the wave proceeds towards the axis, the dragging due to its presence increases. The expansion of γ at small $\tilde{\rho}$ (see (2.11)) reads

$$\gamma = 8b^2 \tilde{\rho}^2 \frac{\tilde{t}^2}{(1 + \tilde{t}^2)^4} + \mathcal{O}(\tilde{\rho}^4). \quad (6.6)$$

Hence, at small $\tilde{\rho}$ equation (6.5) gives

$$\omega'_W \simeq \frac{16b^2 \kappa J}{\pi a^3} \frac{1}{\tilde{\rho}} \frac{\tilde{t}^2}{(1 + \tilde{t}^2)^4}, \quad (6.7)$$

so that

$$\omega_W \simeq \frac{16b^2 \kappa J}{\pi a^2} \ln \tilde{\rho} \frac{\tilde{t}^2}{(1 + \tilde{t}^2)^4}. \quad (6.8)$$

Since we assumed ω —or, correspondingly, J —small, our approximation will break down¹¹ at $\rho = 0$, but it should be reliable at all ρ 's for which $\kappa J/\pi\rho \ll 1$; this can be assured at any $\rho \neq 0$ by choosing J sufficiently small.

At spatial infinity, $t = \text{const}$, $\rho \rightarrow \infty$, the function γ approaches a constant

$$\gamma = \frac{1}{2} b^2 \left[1 - \frac{2}{\tilde{\rho}^2} + \mathcal{O}(\tilde{\rho}^{-4}) \right]; \quad (6.9)$$

¹¹ A somewhat analogous situation arises in the problem of slowly rotating collapsing spherical shells when the shells approach the horizon [8].

the time-dependent terms appear only in $\mathcal{O}(\tilde{\rho}^{-4})$. Substituting this expansion into (6.5) and integrating, we find the angular velocity of the rotation of inertial frames at large $\tilde{\rho}$,

$$\omega_W = \frac{\kappa J}{\pi \rho^2} (e^{b^2} - 1) + \mathcal{O}(\tilde{\rho}^{-4}). \quad (6.10)$$

The perturbation ω_S corresponding to the string also decays at infinity, so local inertial frames do not rotate with respect to the ‘fixed stars’ at infinity, i.e., with respect to the ‘lines’ $\varphi = \text{const}$. However, they do rotate with respect to these lines close to the axis, at small $\tilde{\rho}$, with the time-dependent angular velocity ω_W given by (6.8). This vanishes at times $|\tilde{t}| \gg 1$ when the incoming or outgoing pulse is far away from the axis. It becomes maximal ($\dot{\omega}_W = 0$) at $|\tilde{t}| = 1/\sqrt{3}$. Then the pulse is close to the axis.

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Appendix. The Ricci tensor for the averaged rotating gravitational waves in 2+1 dimensional spacetime

Assume the metric is of the form (5.4). The Ricci tensor components read as follows:

$$\begin{aligned} \mathcal{R}_{00} &= -\ddot{\gamma} + \gamma'' - \frac{\ddot{W}}{W} + \frac{W'}{W} \gamma' + \frac{\dot{W}}{W} \dot{\gamma} + e^{-2\gamma} W \left(2\gamma' W \omega \omega' + \ddot{W} \omega^2 - W'' \omega^2 - 3W' \omega \omega' \right. \\ &\quad \left. - \frac{1}{2} W \omega'^2 - W \omega \omega'' - \frac{1}{2} W^3 \omega^2 \omega'^2 e^{-2\gamma} \right), \\ \mathcal{R}_{01} &= \frac{W'}{W} \dot{\gamma} - \frac{\dot{W}'}{W} + \frac{\dot{W}}{W} \gamma' + e^{-2\gamma} W \left(W \dot{\gamma} \omega \omega' - \frac{1}{2} W \omega \dot{\omega}' - \frac{3}{2} \dot{W} \omega \omega' \right), \\ \mathcal{R}_{02} &= e^{-2\gamma} W \left(\frac{3}{2} W' \omega' + \frac{1}{2} W \omega'' + W'' \omega - W \gamma' \omega' - \dot{W} \omega + \frac{1}{2} W^3 \omega \omega'^2 e^{-2\gamma} \right), \\ \mathcal{R}_{11} &= \ddot{\gamma} - \gamma'' + \frac{W'}{W} \gamma' - \frac{W''}{W} + \frac{\dot{W}}{W} \dot{\gamma} + \frac{1}{2} W^2 \omega'^2 e^{-2\gamma}, \\ \mathcal{R}_{12} &= \frac{1}{2} e^{-2\gamma} W (W \dot{\omega}' + 3 \dot{W} \omega' - 2W \dot{\gamma} \omega'), \\ \mathcal{R}_{22} &= e^{-2\gamma} W \left(\ddot{W} - W'' - \frac{1}{2} W^3 \omega'^2 e^{-2\gamma} \right). \end{aligned}$$

References

- [1] Bini D and de Felice F 2000 Gyroscopes and gravitational waves *Class. Quantum Grav.* **17** 4627
- [2] Sorge F, Bini D and de Felice F 2001 Gravitational waves, gyroscopes and frame dragging *Class. Quantum Grav.* **18** 2945
- [3] Herrera L and Hernández P 2000 On the influence of gravitational radiation on a gyroscope *Class. Quantum Grav.* **17** 3617 (Preprint gr-qc/0006080)

- [4] Valiente-Kroon J A 2001 Can one detect a non-smooth null infinity? *Class. Quantum Grav.* **18** 4311 (Preprint [gr-qc/0108049v1](#))
- [5] Mohseni M and Sepangi HR 2000 Gravitational waves and spinning test particles *Class. Quantum Grav.* **17** 4615 (Preprint [gr-qc/0009070v1](#))
- [6] Mohseni M, Tucker R W and Wang C 2001 On the motion of spinning test particles in plane gravitational waves *Class. Quantum Grav.* **18** 3007 (Preprint [gr-qc/0308042v1](#))
- [7] Lindblom L and Brill D R 1974 Inertial effects in the gravitational collapse of a rotating shell *Phys. Rev. D* **10** 3151
- [8] Katz J, Lynden-Bell D and Bičák J 1998 Instantaneous inertial frames but retarded electromagnetism in rotating relativistic collapse *Class. Quantum Grav.* **15** 3177 (Preprint [gr-qc/9805071](#))
- [9] Bičák J, Lynden-Bell D and Katz J 2004 Do rotations beyond the cosmological horizon affect the local inertial frame? *Phys. Rev. D* **69** 064011 (Preprint [gr-qc/0309126v1](#))
Bičák J, Lynden-Bell D and Katz J 2003 Toroidal perturbations of Friedmann–Robertson–Walker universes *Phys. Rev. D* **69** 064012 (Preprint [gr-qc/0309127v1](#))
- [10] Bičák J, Katz J and Lynden-Bell D 2007 Cosmological perturbation theory, instantaneous gauges, and local inertial frames *Phys. Rev. D* **76** 063501 (Preprint [arXiv:0803.4106v1](#))
- [11] Corvino J 2000 Scalar curvature deformation and a gluing construction for the Einstein constraint equations *Commun. Math. Phys.* **214** 137
- [12] Corvino J and Schoen R M 2006 On the asymptotics for the vacuum Einstein constraint equations *J. Diff. Geom.* **73** 185
- [13] Ashtekar A, Bičák J and Schmidt B G 1997 Asymptotic structure of symmetry reduced general relativity *Phys. Rev. D* **55** 669 (Preprint [gr-qc/9608042](#))
- [14] Lynden-Bell D, Bičák J and Katz J 2008 Inertial frame rotation induced by rotating gravitational waves *Class. Quantum Grav.* **25** 165018
- [15] Ashtekar A, Bičák J and Schmidt B G 1997 Behavior of Einstein–Rosen waves at null infinity *Phys. Rev. D* **55** 687 (Preprint [gr-qc/9608041](#))
- [16] Einstein A and Rosen N 1937 On gravitational waves *J. Franklin Inst.* **223** 43
- [17] Bičák J and Schmidt B G 1989 Asymptotically flat spacetimes with boost-rotation symmetry: the general structure *Phys. Rev. D* **40** 1827
- [18] Bonnor W B 1957 Non-singular fields in general relativity *J. Math. Mech.* **6** 203
- [19] Weber J and Wheeler J A 1957 Reality of the cylindrical waves of Einstein–Rosen *Rev. Mod. Phys.* **29** 509
- [20] Weber J 1961 *General Relativity and Gravitational Waves* (New York: Interscience)
- [21] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press) p 276
- [22] Anderson M R 2003 *The Mathematical Theory of Cosmic Strings (Cosmic Strings in the Wire Approximation)* (Bristol: Institute of Physics Publishing)
- [23] Berger B K, Chruściel P T and Moncrief V 1995 On 'asymptotically flat' space–times with G_2 -invariant Cauchy surfaces *Ann. Phys., NY* **237** 322 (Preprint [gr-qc/9404005](#))
- [24] Carot J, Senovilla J M M and Vera R 1999 On the definition of cylindrical symmetry *Class. Quantum Grav.* **16** 3025 (Preprint [gr-qc/9905059](#))
- [25] Mena Marugán G A 2000 Gauge fixing and the Hamiltonian for cylindrical spacetimes *Phys. Rev. D* **63** 024005 (Preprint [gr-qc/0011068](#))
- [26] Bonnor W B 1980 The rigidly rotating relativistic cylinder *J. Phys. A: Math. Gen.* **13** 2121
- [27] Bondi H 1994 The angular momentum of cylindrical systems in general relativity *Proc. R. Soc. Lond. A* **446** 57
- [28] Komar A 1959 Covariant conservation laws in general relativity *Phys. Rev.* **113** 934
- [29] Winicour J 1980 Angular momentum in general relativity *General Relativity and Gravitation* vol 2, ed H Held (New York: Plenum)
- [30] Wald R M 1984 *General Relativity* (Chicago, IL: Chicago University Press)
- [31] Mashhoon B, McClune J C and Quevedo H 2000 On a class of rotating gravitational waves *Class. Quantum Grav.* **17** 533
- [32] Deser S, Jackiw R and 't Hooft G 1984 Three-dimensional Einstein gravity: dynamics of flat space *Ann. Phys., NY* **152** 220
- [33] Gal'tsov D V and Letelier P S 1993 Spinning strings and cosmic dislocations *Phys. Rev. D* **47** 4273
- [34] Xanthopoulos B C 1986 Cylindrical waves and cosmic strings of Petrov type D *Phys. Rev. D* **34** 3608
- [35] Xanthopoulos B C 1987 Cosmic strings coupled with gravitational and electromagnetic waves *Phys. Rev. D* **35** 3713
- [36] Economu A and Tsoubelis D 1988 Rotating cosmic strings and gravitational soliton waves *Phys. Rev. D* **38** 498

- [37] Dagotto A D, Gleiser R J and Nicasio C O 1990 New exact solution describing the interaction of Einstein–Maxwell radiation and a rotating cosmic string *Phys. Rev. D* **42** 424
- [38] Manojlović N and Mena Marugán G A 2001 Asymptotic behaviour of cylindrical waves interacting with spinning strings *Class. Quantum Grav.* **18** 2065 (*Preprint gr-qc/0011080*)
- [39] Giddings S, Abbott J and Kuchař K 1984 Einstein theory in a 3-dimensional spacetime *Gen. Rel. Grav.* **16** 751
- [40] Ashtekar A and Varadarajan M 1994 Striking property of the gravitational Hamiltonian *Phys. Rev. D* **50** 494 (*Preprint gr-qc/9406040*)
- [41] Courant R and Hilbert D 1966 *Methods of Mathematical Physics* vol I (New York: Interscience) chapter V