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The behaviour of geodesics in constant-curvature spacetimes with expanding impulsive gravitational waves

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Abstract. Motion of free test particles in spacetimes of constant curvature with an expanding impulsive gravitational wave is completely described. Explicit formulae which identify the particle positions and velocities "in front of" and "behind" the impulse are derived for a general impulse of this type. As a particular example, the effect of spherical impulsive wave generated by a snapped cosmic string is analyzed and visualized.

1. Introduction

In the classic work [1], Penrose described the construction of expanding spherical impulsive gravitational waves using his "cut and paste" method. Continuous metric form which represents this class of spacetimes was found later by Hogan [2]-[4] and, in the general case, by Podolský and Griffiths [5]. Motion of free test particles in flat background space with these waves was studied by Podolský and Steinbauer [6]. Recently we generalised these results to an arbitrary cosmological constant Λ in [7], where we described the behaviour of general C^1 geodesics in detail.

2. Spacetimes with expanding impulsive waves

Expanding impulsive gravitational waves of a spherical shape can be constructed using the Penrose "cut and paste" method, i.e., by "cutting" a constant-curvature spacetime along a null cone and "glueing it together" with a suitable warp. Continuous metric form which describes such a family of spacetimes can be written as

$$\mathrm{d}s^2 = \frac{2\left|\frac{V}{p}\mathrm{d}Z + U\Theta(U)p\bar{H}\mathrm{d}\bar{Z}\right|^2 + 2\mathrm{d}U\mathrm{d}V - 2\epsilon\mathrm{d}U^2}{\left[1+\frac{1}{6}\Lambda U(V-\epsilon U)\right]^2}\,, \tag{1}$$

where $p = 1+\epsilon Z\bar{Z}$, $\epsilon = -1, 0, +1$, $H = \frac{1}{2}\left[h'''/h' - \frac{3}{2}(h''/h')^2\right]$, and $h(Z)$ is an arbitrary complex function of the complex spatial coordinate Z (see [2]-[5] for more details). The impulse is located on the null hypersurface $U = 0$ which is a sphere expanding with the speed of light. Relations of this continuous line element to the conformally flat metric form of the background

$$\mathrm{d}s^{\pm 2}=\frac{-\mathrm{d}t^{\pm 2}+\mathrm{d}x^{\pm 2}+\mathrm{d}y^{\pm 2}+\mathrm{d}z^{\pm 2}}{\left[1+\frac{\Lambda}{12}(-t^{\pm 2}+x^{\pm 2}+y^{\pm 2}+z^{\pm 2})\right]^2}\,,\quad \text{i.e.,}\quad \mathrm{d}s^{\pm 2}=\frac{2\mathrm{d}\eta^{\pm}\mathrm{d}\bar\eta^{\pm}-2\mathrm{d}\mathcal{U}^{\pm}\mathrm{d}\mathcal{V}^{\pm}}{[1+\frac{1}{6}\Lambda(\eta^{\pm}\bar\eta^{\pm}-\mathcal{U}^{\pm}\mathcal{V}^{\pm})]^2} \tag{2}$$

with $\eta^{\pm}\equiv\frac{1}{\sqrt{2}}(x^{\pm}+\mathrm{i}y)$, $\mathcal{U}^{\pm}\equiv\frac{1}{\sqrt{2}}(t^{\pm}-z^{\pm})$, $\mathcal{V}^{\pm}\equiv\frac{1}{\sqrt{2}}(t^{\pm}+z^{\pm})$ [which is Minkowski ($\Lambda=0$), de Sitter ($\Lambda>0$) or anti-de Sitter ($\Lambda<0$) spacetime] are given by two different transformations, namely

$$\mathcal{V}^{+}=AV-DU\,,\qquad \mathcal{U}^{+}=BV-EU\,,\qquad \eta^{+}=CV-FU\,, \tag{3}$$

"in front of" the impulse ($U>0$), and

$$\mathcal{V}^{-}=\frac{V}{p}-\epsilon U\,,\qquad \mathcal{U}^{-}=\frac{Z\bar Z}{p}V-U\,,\qquad \eta^{-}=\frac{Z}{p}V\,, \tag{4}$$

"behind" the impulse ($U<0$), where the functions in (3) are given by

$$\begin{aligned}
A&=\frac{1}{p|h'|}\,,\qquad B=\frac{|h|^2}{p|h'|}\,,\qquad C=\frac{h}{p|h'|}\,,\\
D&=\frac{1}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'}\right|^2+\epsilon\left[1+\frac{Z}{2}\frac{h''}{h'}+\frac{\bar Z}{2}\frac{\bar h''}{\bar h'}\right]\right\}\,,\\
E&=\frac{|h|^2}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'}-2\frac{h'}{h}\right|^2+\epsilon\left[1+\frac{Z}{2}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)+\frac{\bar Z}{2}\left(\frac{\bar h''}{\bar h'}-2\frac{\bar h'}{\bar h}\right)\right]\right\}\,,\\
F&=\frac{h}{|h'|}\left\{\frac{p}{4}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)\frac{\bar h''}{\bar h'}+\epsilon\left[1+\frac{Z}{2}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)+\frac{\bar Z}{2}\frac{\bar h''}{\bar h'}\right]\right\}\,.
\end{aligned} \tag{5}$$

3. Refraction formulae

Suppose C^1 geodesics $Z=Z(\tau)$, $U=U(\tau)$ and $V=V(\tau)$ in the continuous coordinates of (1) and denote the positions and velocities at the interaction time τ_i when $U=0$ as $Z_i\equiv Z(\tau_i)$, $V_i\equiv V(\tau_i)$, $\dot U_i\equiv\dot U(\tau_i)$, $\dot V_i\equiv\dot V(\tau_i)$, $\dot Z_i\equiv\dot Z(\tau_i)$. We may apply the above two different transformations (4) and (3) in the half-spaces "behind" and "in front of" the impulse to obtain

$$\begin{aligned}
\mathcal{V}_i^{-}&=\frac{V_i}{p}\,, &\dot{\mathcal{V}}_i^{-}&=-\frac{\epsilon V_i}{p^2}(Z_i\dot{\bar Z}_i+\bar Z_i\dot Z_i)+\frac{\dot V_i}{p}-\epsilon\dot U_i\,,\\
\mathcal{U}_i^{-}&=\frac{|Z_i|^2}{p}V_i\,, &\dot{\mathcal{U}}_i^{-}&=\frac{V_i}{p^2}(Z_i\dot{\bar Z}_i+\bar Z_i\dot Z_i)+\frac{|Z_i|^2}{p}\dot V_i-\dot U_i\,,\\
\eta_i^{-}&=\frac{Z_i}{p}V_i\,, &\dot\eta_i^{-}&=\frac{V_i}{p^2}(\dot Z_i-\epsilon Z_iZ_i\dot{\bar Z}_i)+\frac{Z_i}{p}\dot V_i\,,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
h(Z_i)&=\frac{\eta_i^{+}}{\mathcal{V}_i^{+}}\,, &\dot Z_i&=\frac{p^2}{V_i}\left(\bar C_{,\bar Z}\,\dot\eta_i^{+}+C_{,\bar Z}\,\dot{\bar\eta}_i^{+}-A_{,\bar Z}\,\dot{\mathcal{U}}_i^{+}-B_{,\bar Z}\,\dot{\mathcal{V}}_i^{+}\right),\\
V_i=\frac{\mathcal{U}_i^{+}}{B}&=\frac{\mathcal{V}_i^{+}}{A}=\frac{\eta_i^{+}}{C}\,, &\dot V_i&=D\,\dot{\mathcal{U}}_i^{+}+E\,\dot{\mathcal{V}}_i^{+}-\bar F\,\dot\eta_i^{+}-F\,\dot{\bar\eta}_i^{+}+2\epsilon\,\dot U_i\,,\\
& &\dot U_i&=\frac{1}{V_i}\left(\bar\eta_i^{+}\dot\eta_i^{+}+\eta_i^{+}\dot{\bar\eta}_i^{+}-\mathcal{V}_i^{+}\dot{\mathcal{U}}_i^{+}-\mathcal{U}_i^{+}\dot{\mathcal{V}}_i^{+}\right)\,,
\end{aligned} \tag{7}$$

respectively. Combining (6) with (7), it is now straightforward to express the interaction parameters behind the impulse as specific functions of the parameters in front of the impulse. In the conformally flat coordinates of (2), for the positions of test particles we obtain

$$
\begin{aligned}
x_i^- &= |h_i'|\frac{Z_i+\bar{Z}_i}{h_i+\bar{h}_i}x_i^+ \,, \qquad & y_i^- &= |h_i'|\frac{Z_i-\bar{Z}_i}{h_i-\bar{h}_i}y_i^+ \,, \\
z_i^- &= |h_i'|\frac{|Z_i|^2-1}{|h_i|^2-1}z_i^+ \,, \qquad & t_i^- &= |h_i'|\frac{|Z_i|^2+1}{|h_i|^2+1}t_i^+ \,,
\end{aligned} \tag{8}
$$

and for their velocities we get

$$
\begin{aligned}
\dot{x}_i^- &= a_x\dot{x}_i^+ + b_x\dot{y}_i^+ + c_x\dot{z}_i^+ + d_x\dot{t}_i^+ \,, \qquad & \dot{y}_i^- &= a_y\dot{x}_i^+ + b_y\dot{y}_i^+ + c_y\dot{z}_i^+ + d_y\dot{t}_i^+ \,, \\
\dot{z}_i^- &= a_z\dot{x}_i^+ + b_z\dot{y}_i^+ + c_z\dot{z}_i^+ + d_z\dot{t}_i^+ \,, \qquad & \dot{t}_i^- &= a_t\dot{x}_i^+ + b_t\dot{y}_i^+ + c_t\dot{z}_i^+ + d_t\dot{t}_i^+ \,,
\end{aligned} \tag{9}
$$

where the coefficients a_p, b_p, c_p, d_p are (somewhat complicated) functions of Z_i and $h_i \equiv h(Z_i)$, see [7]. If we define natural angles which characterize the positions and velocity directions in the (x,z) and (y,z) planes as

$$
\tan\alpha^\pm \equiv \frac{x_i^\pm}{z_i^\pm} \,, \qquad \tan\beta^\pm \equiv \frac{\dot{x}_i^\pm}{\dot{z}_i^\pm} \,, \qquad \tan\gamma^\pm \equiv \frac{y_i^\pm}{z_i^\pm} \,, \qquad \tan\delta^\pm \equiv \frac{\dot{y}_i^\pm}{\dot{z}_i^\pm} \,, \tag{10}
$$

the expressions (8) identifying the positions can be written as

$$
\tan\alpha^- = \frac{(|h_i|^2-1)\,\mathrm{Re}\,Z_i}{(|Z_i|^2-1)\mathrm{Re}\,h_i}\tan\alpha^+ \,, \qquad \tan\gamma^- = \frac{(|h_i|^2-1)\,\mathrm{Im}\,Z_i}{(|Z_i|^2-1)\mathrm{Im}\,h_i}\tan\gamma^+ \,, \tag{11}
$$

while, using (9), the inclinations of the velocity vector on both sides of the impulse are related by

$$
\begin{aligned}
\tan\beta^- &= \frac{v_z^+(a_x\tan\beta^+ + b_x\tan\delta^+ + c_x) + d_x}{v_z^+(a_z\tan\beta^+ + b_z\tan\delta^+ + c_z) + d_z} \,, \\
\tan\delta^- &= \frac{v_z^+(a_y\tan\beta^+ + b_y\tan\delta^+ + c_y) + d_y}{v_z^+(a_z\tan\beta^+ + b_z\tan\delta^+ + c_z) + d_z} \,,
\end{aligned} \tag{12}
$$

where $v_z^+ \equiv \dot{z}_i^+/\dot{t}_i^+$. These are explicit and general refraction formulae for motion of free test particles influenced by any expanding impulsive gravitational wave.

4. Impulses generated by snapped cosmic strings

We may apply these general results to an important particular family of spacetimes in which the expanding spherical impulse is generated by a snapped cosmic string (identified by a deficit angle in the region $U > 0$ in front of the impulse). This coresponds to the choice of the function $h(Z) = Z^{1-\delta}$, where $\delta \in [0,1)$ is a real constant which characterizes the deficit angle $2\pi\delta$.

For simplicity, we consider here the ring of static test particles in the (x^+, z^+) plane, i.e., $y_i^+ = 0$, $\dot{x}^+ = \dot{y}^+ = \dot{z}^+ = 0$. The resulting motion remains in the (x^-, z^-) plane and general refraction formulae (11) and (12) reduce to

$$
\tan\alpha^- = \frac{\sinh((1-\delta)r)}{\sinh r}\tan\alpha^+ \,, \qquad \tan\beta^- = \frac{\delta(1-\frac{\delta}{2})\cosh((1-\delta)r)}{(1-\frac{\delta}{2})^2\sinh(\delta r) + \frac{\delta^2}{4}\sinh((2-\delta)r)} \,, \tag{13}
$$

where $r = \frac{1}{1-\delta}\log\left(\tan\frac{\alpha^+}{2}\right)$. These functions are drawn in Fig. 1. Using the above relations we can also study the deformation of such (initially static) ring of test particles, as shown in Fig. 2. We observe that the test particles are "dragged" along the z-direction by the impulse generated by moving "endpoints" of the snapped string (for more details see [7]).

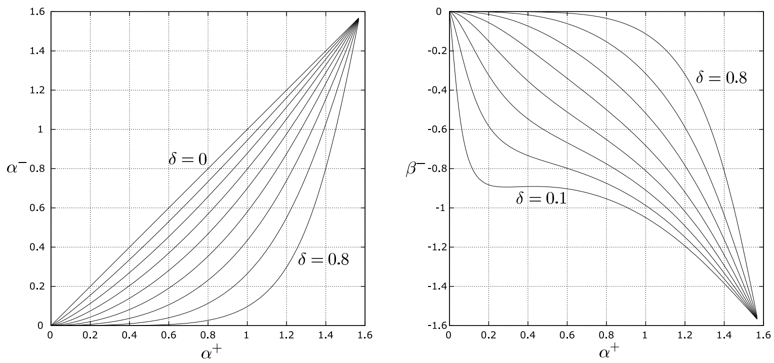

Figure 1. The function $\alpha^-(\alpha^+)$ which determines the displacement of the position of an initially static particle when it crosses the impulse generated by a snapped cosmic string (left). The function $\beta^-(\alpha^+)$ determines the dependence of the velocity vector inclination behind the impulse on the particle's position in front of the impulse (right). Here $\delta = 0.1, 0.2, \ldots, 0.8$.

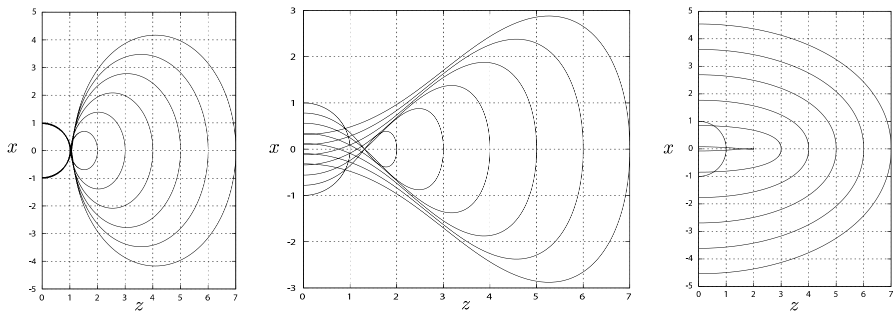

Figure 2. Time sequences which show the deformation of the ring of test particles (indicated here by initial semi-circles of unit radius for $\alpha^+ \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) caused by different impulses generated by snapped cosmic strings with $\delta = 0.005$ (left), $\delta = 0.2$ (middle) and $\delta = 0.8$ (right).

5. Conclusions

We presented the framework for a description of the influence of expanding spherical impulsive gravitational waves on free test particles in spacetimes of a constant curvature. This generalizes the previous results [6] obtained for Minkowski background to any cosmological constant Λ. We derived the general refraction formulae (11) and (12) characterizing the shift of positions and the change of velocity vectors of the particles. More details can be found in the work [7].

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