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Critical Curves of Triple Gravitational Lenses

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Abstract. Gravitational microlensing is a useful tool for finding exoplanets orbiting distant stars by detecting their gravitational influence on the passage of light from a more distant star. Knowledge of the lens caustic (formed by positions of point sources with infinite amplification) is necessary for the analysis and interpretation of observed light curves. It is well known that the binary lens can form caustics of three topologies, which correspond to three topologies of the critical curve (primary image of the caustic). Here we focus on the case of the triple lens. While the binary lens is characterised by two parameters, three more parameters are needed to describe the triple lens. We analyse a two-dimensional cut through the five-dimensional parameter space, identifying the boundaries of regions with different critical-curve topology. For each region we present corresponding critical curves and caustics. We also include sample results for a three-dimensional cut of a general triple lens with equal masses.

Introduction

Gravitational lensing provides a means of detecting matter through its gravitational influence on the passage of light from a more distant source. Gravitational microlensing is a special regime of gravitational lensing in which the lensing body is of stellar or sub-stellar mass and the deflection angle is too small for individual images to be resolved. The main measurable quantity is the time-dependent amplification of flux from the source, i.e., the light curve of a microlensing event.

The single lens with a mass distribution approximated by a point mass was discussed in detail by *Refsdal* (1964). The first theoretical study of two-point-mass lenses was carried out by *Schneider* & *Weiss* (1984). The single-lens light curve has a simple symmetric peak and diverges only if the point source, the observer and the point-mass lens are exactly aligned. In the case of binary or more complex lenses the amplification of a point source diverges when the source, the observer and some point on the lens caustic are aligned. During such alignment a pair of infinitely amplified images appears or disappears at some point on the critical curve (for sources entering or exiting the caustic, respectively). Both the lens caustic and the critical curve consist of the same number of closed curves. A complete analysis of caustic topologies of the binary lens was performed by *Erdl* & *Schneider* (1993).

Several papers discussed the possibility of detecting triple lenses (e.g., star with an exoplanet with a moon by *Wambsganss* & *Liebig* (2010), *Han* (2008); planet in a binary star system by *Lee et al.* (2008), *Chung* & *Park* (2010)). In 2006 the first convincing triple-lens microlensing event, OGLE-2006-BLG-109, was found (*Gaudi et al.* 2008), with a Sun + Jupiter + Saturn analogue acting as the lens. At the same time, gravitational microlensing by a system of three bodies (3 stars, 2 stars + planet, star + 2 planets, star + planet + moon) has not yet been satisfactorily analysed theoretically. Inspired by the *Erdl* & *Schneider* (1993) analysis of the parameter dependence of binary lensing topologies, we extend their approach to special cases of the triple lens, focusing on the classification of critical-curve topologies.

Basics of gravitational microlensing

For an $n-$point-mass lens system, the mapping from the lens plane to the source plane is expressed as

$$\zeta = z - \sum_{i=1}^{n} \frac{\mu_i}{\bar{z} - \bar{z}_i}, \tag{1}$$

where ζ is the source position, z is the image position, z_i are the point-mass positions (barred variables $\bar{z}$, $\bar{z}_i$ are complex conjugates of z, z_i), and μ_i are their relative masses with unit total mass ($\sum_{i=1}^{n} \mu_i = 1$). This complex notation was introduced by *Witt* (1990). All angular positions are normalized to the total-mass Einstein radius. The amplification of a given image is obtained as the reciprocal value of the determinant of the Jacobi matrix,

$$\det J = \left|\frac{\partial\zeta}{\partial z}\right|^2 - \frac{\partial\zeta}{\partial\bar{z}}\frac{\partial\bar{\zeta}}{\partial z} = 1 - \frac{\partial\zeta}{\partial\bar{z}}\frac{\partial\bar{\zeta}}{\partial z}. \tag{2}$$

The set of image positions z with $\det J = 0$ (i.e., infinite amplification) is called the critical curve. Using (1) in (2) the critical curve equation is

$$\sum_{i=1}^{n} \frac{\mu_i}{(z-z_i)^2} = e^{-2\mathrm{i}\phi}\,, \tag{3}$$

where the real phase ϕ varies along the curve. In this paper we focus on changes of critical curve topology. Because the critical curve is the $\det J = 0$ contour in the lens plane, it borders regions of positive and negative $\det J$. Hence, the point on a critical curve at which loops of the critical curve merge is also a saddle point of $\det J$. The additional saddle-point condition for such a merging point is

$$\sum_{i=1}^{n} \frac{\mu_i}{(z-z_i)^3} = 0. \tag{4}$$

Binary-lens topologies

The Jacobian of the binary lens has two poles at the lens positions, three saddle points and two maxima. The binary lens can be parametrised by the separation of lenses d and relative mass of one lens μ, so that $z_2 = -z_1 = d/2$, $\mu_1 = 1-\mu_2 = \mu$. It was shown by *Erdl* & *Schneider* (1993) that there are only three possible critical-curve topologies for the binary lens, usually referred to as 'wide', 'resonant' and 'close'. The critical curve of 'wide' topology forms two loops, which in the limit $d \to \infty$ become Einstein rings of two single lenses. The caustic of 'wide' topology forms two four-cusped *Chang* & *Refsdal* (1984) caustics that shrink to points at the point mass positions for $d \to \infty$. The critical curve of 'resonant' topology forms a single loop and the caustic forms one curve with six cusps. The critical curve of 'close' topology forms one big loop with two smaller loops inside. In the $d \to 0$ limit the smaller loops shrink and disappear at the origin, while the big loop becomes a single-lens Einstein ring. The caustic of 'close' topology consists of two parts with three cusps each and one four-cusped part. In the $d \to 0$ limit the four-cusped caustic shrinks to a point at the origin and the two triangular caustics go to infinity.

In order to divide the whole parameter space into different topological regions *Erdl* & *Schneider* (1993) had to find μ and d values for which critical-curve loops merge. They computed the determinant of the Sylvester matrix of two polynomials in z that emerged from (3) and (4) multiplied by their denominators. The resulting determinant can be expressed as a polynomial in $e^{\mathrm{i}\phi}$, then conjugated and multiplied by $e^{\mathrm{i}n\phi}$ to get another polynomial in $e^{\mathrm{i}\phi}$. For the second pair of polynomials they again computed the Sylvester matrix determinant, factorised it and got the merging conditions as polynomials in d with coefficients as functions of μ.

Triple lens: two-parameter example

As mentioned in the Introduction, an analysis of possible critical-curve topologies based on solving (3) and (4) has not been published yet for the triple lens. Articles concerning triple lenses (e.g., *Wambsganss* & *Liebig*, 2010) primarily computed amplification maps obtained by inverse ray shooting. This method is useful for taking into account extended sources. However, by solving (3) and (4) we can get analytic conditions for merging points, and thus map the different topologies in parameter space.

The triple lens is described by three more parameters than the binary lens. Namely, we have two relative masses and three position parameters. We were not able to find merging conditions as equations combining all five parameters because of computational time and complexity (we used the Maple 9.5 software package). However, we got such a solution for a three-parameter cut through parameter space, for the lens configuration $\mu_1 = \mu$, $\mu_2 = \mu_3 = \frac{(1-\mu)}{2}$, $z_1 = 0$, $z_2 = \bar{z_3} = de^{i\theta}$ with μ, d and θ as variables, as well as for several two-parameter cuts.

To give a two-parameter example, we present here the results for a symmetric collinear triple-lens system with variable middle mass and lens separation. Masses and lens-point positions in this model are $\mu_1 = \mu$, $\mu_2 = \mu_3 = \frac{(1-\mu)}{2}$, $z_1 = 0$, $z_2 = -z_3 = d$. Using the Sylvester matrix method of *Erdl* & *Schneider* (1993) we got three simple independent conditions for critical-curve merger:

$$8d^6 + (9\mu + 15)d^4 + 6(18\mu^2 - 15\mu + 1)d^2 + 9\mu - 1 = 0, \tag{5}$$

$$8d^6 - (9\mu + 15)d^4 + 6(18\mu^2 - 15\mu + 1)d^2 - 9\mu + 1 = 0, \tag{6}$$

$$64d^{12} - 48(18\mu^2 - 15\mu + 1)d^8 + 3(3\mu + 5)(9\mu - 1)d^4 - (9\mu - 1)^2 = 0. \tag{7}$$

By solving (5), (6) or (7) we get parameter combinations at which two or more loops of the critical curve merge. These solutions form boundaries that divide the parameter space into several regions. However, it is necessary to check for every solution of (5), (6) or (7) whether it also fulfils (3) and (4).

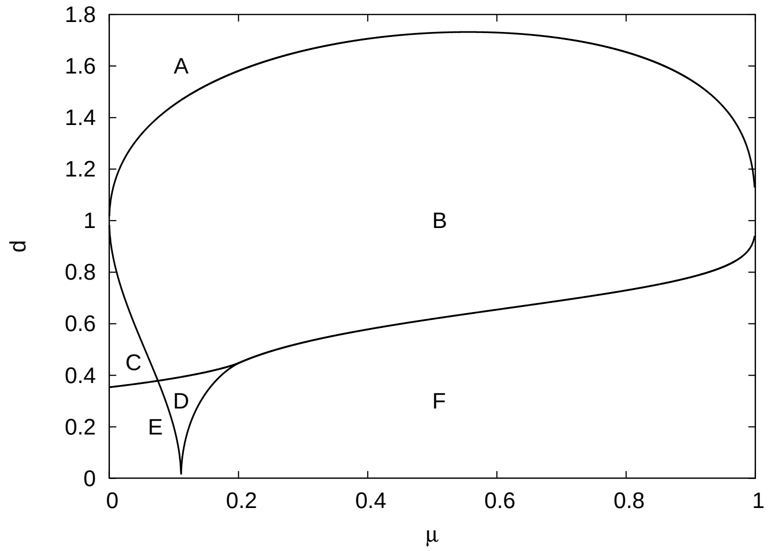

Figure 1. Parameter space of symmetric collinear triple lens configuration. Boundary between B and D, D and F, C and E: equation (5); boundary between A and B, B and C, D and E: equation (6); boundary between B and F: equation (7).

In Fig. 1 we plotted the (μ, d) parameter space of the symmetric collinear triple lens with curves given by (5), (6) and (7). These curves divide the parameter space into six regions: A, B, C, D, E, F.

In Fig. 2 we plotted samples of critical curves and caustics of all six regions. There are four different topologies of the critical curve. The topologies in regions E and F are the same, so are those in C and D. The critical curve of region A is analogous to the 'wide' type binary lens but with three loops instead of two, and in the limit $d \to \infty$ it becomes a set of Einstein rings of three single lenses. The caustic of A consists of three parts, each with four cusps. The critical curve of region B is similar to the 'resonant' binary lens topology but its caustic has six extra cusps and forms self-crossing features called butterflies. The critical curve of region C forms one big loop with two small loops inside. The caustic of region C is similar to the 'resonant' caustic of the binary lens but has two additional disconnected parts, each with three cusps. The critical curve and caustic of region D are similar to those of 'close' binary lenses except that the upper and lower parts of the caustic have four cusps. Critical curves and caustics of regions E and F are analogous to those of 'close' binary lenses but the critical curve has two extra inner loops and the caustic has two extra pairs of upper and lower caustics (second pair in E not discernible in Figure).

Two other two-dimensional cuts can be found in *Daněk* (2010).

Triple lens: three-parameter example

Using the Sylvester matrix method for three-parameter triple-lens configurations usually leads to multi-page expressions for merging conditions. For most parametrisations we could not factorise the final result into several polynomials in terms of model parameters. For this reason it is necessary to try some different method. We noticed a nice property of (3) and (4) that can be used to obtain parameters of critical-curve mergers. Multiplying both sides of (3) by a real positive number α and performing the transformation $z' = \alpha^{-1/2}z$, $z'_i = \alpha^{-1/2}z_i$ we obtain the equation of a Jacobian contour line with

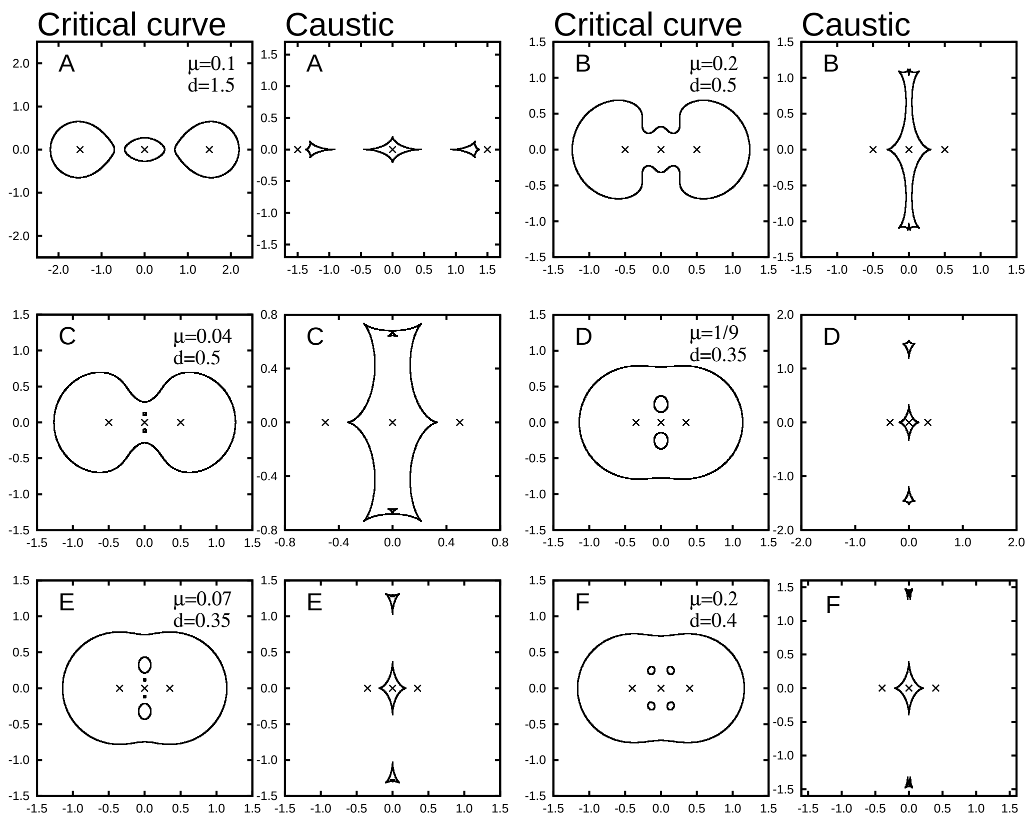

Figure 2. Critical curves and caustics of symmetric collinear triple-lens configuration. First and third column: critical curves; second and fourth column: caustics; crosses: point-mass positions.

$\det J' = 1 - \alpha^2$ in z'. The equation of saddle points (4) holds for both transformed and untransformed values. This enables us to find values of $\det J'$ in a saddle point z' for some chosen lens positions ${z'}_i$ and a contour line going through that saddle point. By inverting the transformation we can find lens parameters z_i with a critical curve corresponding to the contour line. In the triple-lens case, we obtain six values of $\det J'$ that tell us how to transform lens positions to obtain parameters of critical-curve mergers.

As an example of a three-parameter cut we have chosen a triple-lens system with equal masses and arbitrary lens positions. The system is parametrised by the circumference o and three lengths of sides as fractions of the circumference a, b, c, constrained by $a + b + c = 1$. Taking advantage of the symmetry of this problem we draw $o = const.$ cuts as ternary plots. In Fig. 3 we plotted a cut of the three-dimensional parametrisation for $o = 1.68$. This part of the parameter space is divided into thirteen regions but because of the symmetry of the problem there are only four distinct types of regions. Therefore, we sorted the regions into groups A, B, C, D.

In Fig. 4 we plotted the critical curves and caustics corresponding to these groups. Critical curves from region B have the same topology as critical curves from region C. The critical curve and caustic of region A has the same properties as those from region C of the symmetric collinear triple-lens configuration, just extremely deformed. Critical curves of regions B and C have a topology with one outer and three inner loops that does not occur in the case of the symmetric collinear triple lens. The caustic of region B consists of four parts, three with three cusps and one with five cusps. The caustic of region C has also four parts, two with four cusps and two with three cusps. The critical curve of region D has the same topology as those of regions E and F of the symmetric collinear triple lens. The caustic of region D has five parts, four with three cusps and one with six cusps. The fourth part with three cusps is inside the central one with six cusps, which has the shape of a self-intersecting concave triangle.

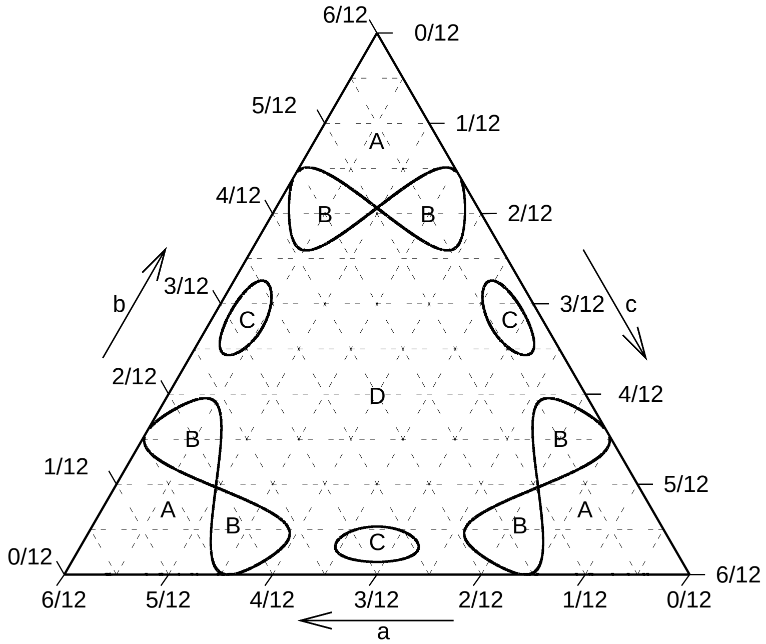

Figure 3. Parameter space of triple lens with equal masses and fixed circumference $o = 1.68$. Lengths of sides a, b, c are in units of the circumference.

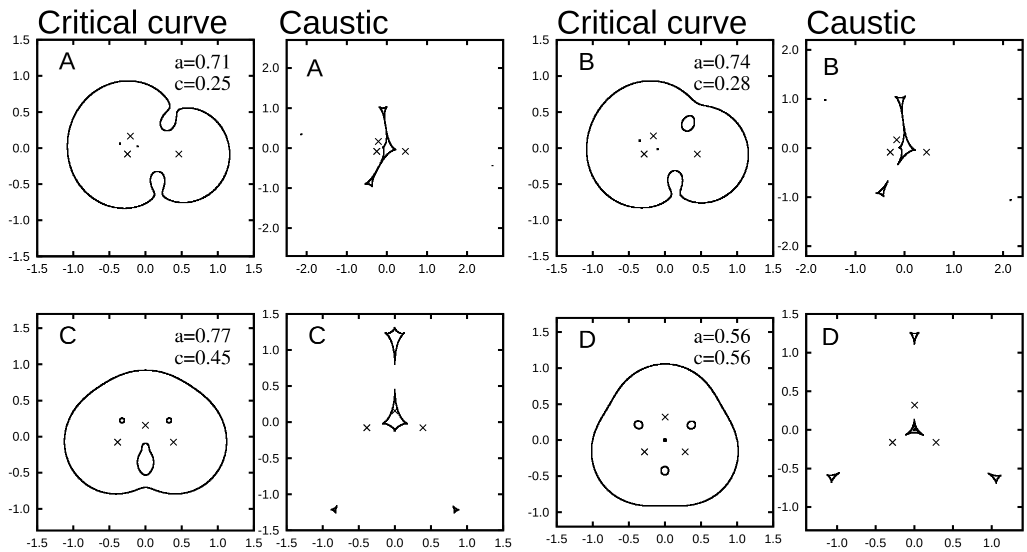

Figure 4. Critical curves and caustics of triple lens with equal masses and fixed circumference $o = 1.68$. First and third column: critical curves; second and fourth column: caustics; crosses: point-mass positions.

Conclusion

The topological analysis of critical curves and caustics of binary lenses provides a priceless theoretical background for analysing observed light curves. A similar analysis for systems of more than two lenses is still missing. We have shown that it is possible to find algebraic conditions for the merger of critical-curve loops for special two-dimensional cuts through parameter space. For general multiple lenses we can obtain merger conditions by an algorithm employing the equivalence of contour lines of the Jacobian going through its saddle point and the critical curve of a differently scaled lens configuration.

Still, there is much work to be done. It is unclear how many critical-curve topologies are possible for triple lenses. The next step closer to actual light curves is the analysis of caustics. The knowledge of critical curve topology gives us some insight into caustics, because mergers of critical-curve loops and mergers of caustic parts coincide. However, the caustic analysis is much more complex because the number of cusps can change without mergers. For a complete caustic-topology analysis, also self-intersections and overlaps should be taken into account.

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