# SYSTEM OF GRAVITATIONALLY INTERACTING BODIES IN THE POSTMINKOWSKIAN HAMILTONIAN DESCRIPTION

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## ABSTRACT

The Hamiltonian description of a system of N bodies interacting by their gravitational field is given in the first-order post-Minkowskian approximation of the General relativity. The bodies are represented by their rest mass, canonical coordinates and momenta. Their velocity is not assumed to be small, as is the case in the post-Newtonian approximation, even particles with zero rest mass moving with the speed of light are allowed. The Hamiltonian given in [4] includes all terms linear in the gravitational constant. It has quite a simple form of a sum of kinetic energies of individual particles and binary interaction potentials. The dynamics of gravitational field is eliminated by solving inhomogeneous wave equations, applying transversetraceless projections, and using the Routh functional. To illustrate properties and possible applications of the post-Minkowskian Hamiltonian description of system of gravitationally interacting bodies several general-relativistic phenomena are discussed emphasizing the uniform treatment of gravitating and test bodies as well as test photons in this approach.

Key words: General relativity, post-Minkowskian approximation.

## 1. INTRODUCTION

While Einstein's theory of gravitation has a surprisingly simple formulation, the exact solution describing even a binary gravitationally interacting system is not available. In many situations, e.g. in the solar system, the gravitational field is weak enough so that approximations to the full general relativity can be used. Namely, the so-called post-Newtonian (PN) approximation based on expansion in the parameter $v^2/c^2 \ll 1$ can address this question, both for a binary and $N$-body system (see e.g. [6]). To describe situations, when radio waves or light are used to measure distances, due to the limitation $v^2/c^2 \ll 1$ these have to be treated in a different way, e.g., as test null particles in the gravitational field of massive bodies [3]. In this contribution, we present a different approach – an example of uniform description of the gravitational interaction of both massive and massless particles in the first post-Minkowskian (PM) approximation [4] given in terms of canonical Arnowitt-Deser-Misner (ADM) formulation of general relativity [2].

## 2. GRAVITATIONAL INTERACTION IN ADM CANONICAL FORMALISM

We assume that the source of the gravitational field is a set of $N$ point-like particles (without explicit multipolar structure, e.g., due to spin) described by action

$$I_M = \sum_a \int p_{a\mu} dx_a^\mu,$$

i.e. the energy-momentum tensor is proportional to $\delta$-function distributed along particles' trajectories $x_a^\mu(\lambda_a)$, $\mu = 0, 1, 2, 3$, for massive particles the affine parameter $\lambda_a$ is replaced with the proper time. In $3+1$ Hamiltonian form, particles numbered $a = 1, 2, .., N$ are thus represented by their invariant mass $m_a$, coordinates $\mathbf{x}_a$ and momenta $\mathbf{p}_a$. This is combined with the Einstein-Hilbert action

$$I_G = \frac{1}{16\pi G} \int R\sqrt{|g|}\, d^4x$$

for the gravitational field $g_{\mu\nu}(x^\nu)$ which is split into $3+1$ quantities (indices $i, j, .. = 1, 2, 3$) – lapse $\alpha$, shift $\beta^i$ and spatial metric

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}\,. \qquad (1)$$

Conjugated momenta $\pi^{ij}$ are linearly related to the extrinsic curvature tensor of a hypersurface $t = \mathrm{const.}$, i.e. to the time derivative of $g_{ij}$. If a suitable gauge ($3g_{ij,j} - g_{jj,i} = 0$ , $\pi^{ii} = 0$) is used, the degrees of freedom of the gravitational field are reduced to canonically conjugated $h_{ij}^{\mathrm{TT}}$ and $\pi^{\mathrm{TT}\,ij}$. Assuming that at the infinity the spacetime approaches asymptotically the Minkowski spacetime the dynamics of these fields and particles is then described by Hamiltonian [5]

$$H(h_{ij}^{TT}, \pi_{ij}^{TT}; \mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{16\pi G} \int \Delta\phi\; d^3x\,, \qquad (2)$$

where $\Delta\phi$ becomes a known quantity once constraints of Einstein equations in $3+1$ splitting are solved. Both constraints lead to elliptic partial differential equations which are in the PN approach solved iteratively assuming that all quantities can be expanded in parameter $1/c^2$, i.e. assuming that the velocity of particles $v/c \ll 1$.

## 3. FIRST-ORDER POST-MINKOWSKIAN APPROXIMATION

As an alternative, the expansion in gravitation interaction constant $G$ can also be used. Such expansion was given in [6] including terms $\sim G^2$. In the first-order PM approximation we neglect the terms $\sim G^2$. Then

$$H_{1\mathrm{PM}} = H_{\mathrm{Mink}} + H_{\mathrm{p}} + H_{\mathrm{pf}} + H_{\mathrm{f}} + O(G^2), \quad (3)$$

where (assuming $c = 1$) the Hamiltonian for free relativistic motion of particles is simply

$$H_{\mathrm{Mink}} = \sum_a \overline{m}_a \quad (4)$$

and the first-order terms read

$$\begin{aligned} H_{\mathrm{p}} =& -\frac{1}{2}G\sum_a\sum_{b\neq a}\frac{\overline{m}_a\overline{m}_b}{r_{ab}}\left(1+\frac{p_a^2}{\overline{m}_a^2}+\frac{p_b^2}{\overline{m}_b^2}\right) \quad (5)\\ &+\frac{1}{4}G\sum_a\sum_{b\neq a}\frac{1}{r_{ab}}\left(7\,\mathbf{p}_a.\mathbf{p}_b+\mathbf{p}_a.\mathbf{n}_{ab}\,\mathbf{p}_b.\mathbf{n}_{ab}\right),\\ H_{\mathrm{pf}} =& -\frac{1}{2}\sum_a\frac{p_{a\,i}p_{a\,j}}{\overline{m}_a}\,h_{ij}^{TT}(\mathbf{x}=\mathbf{x}_a), \quad (6)\\ H_{\mathrm{f}} =& \frac{1}{16\pi G}\int\left(\frac{1}{4}h_{ij,k}^{TT}\,h_{ij,k}^{TT}+\pi^{ij\,TT}\pi^{ij\,TT}\right)d^3x. \quad (7) \end{aligned}$$

We denote $\overline{m}_a = \sqrt{m_a^2+\mathbf{p}_a^2}, r_{ab} = |\mathbf{x}_a-\mathbf{x}_b|$, and $\mathbf{n}_{ab} = (\mathbf{x}_a-\mathbf{x}_b)/r_{ab}$. In this approximation, we do not need to distinguish covariant and contravariant indices, the contraction is performed using flat $\delta_{ij}$.

Equations of motion for particles are the usual Hamilton equations, but those for fields

$$\begin{aligned} \dot{\pi}^{ij\,TT} &= -\,16\pi G\,\delta_{kl}^{TT\,ij}\,\frac{\delta H}{\delta h_{kl}^{TT}} \quad (8)\\ \dot{h}_{ij}^{TT} &= 16\pi G\,\delta_{ij}^{TT\,kl}\,\frac{\delta H}{\delta\pi^{kl\,TT}} \quad (9) \end{aligned}$$

require application of TT-projection $\delta_{ij}^{TT\,kl}$ which among others involves a solution of a (fourth-order) elliptic equation $\Delta^2 f = g$.

The field equations up to the first order in interaction constant $G$ implied by (3) read

$$\Box h_{ij}^{TT} = -16\pi G\,\delta_{ij}^{TT\,kl}\sum_a\frac{p_{a\,k}p_{a\,l}}{\overline{m}_a}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a), \quad (10)$$

which means that $h_{ij}^{TT}$ is composed of waves generated by point-like particles with positions $\mathbf{x}_a(t)$. The shift and lapse functions are on the other hand given by the gauge choice and read (again, neglecting $O(G^2)$ terms)

$$\begin{aligned} \beta^i(\mathbf{x}) &= -\frac{G}{2}\sum_a p_k^a\left(7\frac{\delta^{ik}}{|\mathbf{x}-\mathbf{z}_a|}+\frac{x_a^i x_a^k}{|\mathbf{x}-\mathbf{z}_a|^3}\right), \quad (11)\\ \alpha(\mathbf{x}) &= 1-G\sum_a\frac{\overline{m}_a}{|\mathbf{x}-\mathbf{z}_a|}. \quad (12) \end{aligned}$$

When one writes down the particle's equation of motion

$$\dot{x}_a^i = \frac{\partial H}{\partial p_i^a},\ \ \dot{p}_i^a = -\frac{\partial H}{\partial x_a^i} \quad (13)$$

it contains two types of interaction terms – those which look like ‘action on a distance’ (such as $G\overline{m}_a\overline{m}_b\mathbf{n}_{ab}/r^2$) and those which can be interpreted as interaction of the particle with gravitational field $h_{ij}^{TT}$. Indeed, also the former terms represent the interaction with the gravitational field, namely the lapse and shift which have been determined through the choice of the gauge, which is of elliptic nature and thus it yields the terms which mimic an immediate force between the particles.

An important simplification is then applied in [4]. Since $h_{ij}^{TT} \sim G$ and also the acceleration of particles is proportional to $G$, the changes in the gravitational field due to the acceleration of the particles are $\sim G^2$ and can be neglected when we determine the particle's dynamics in desired approximation. Thus we only need to solve (10) for field $h_{ij}^{TT}$ surrounding an unaccelerated particle. It is well known that both retarded and advanced waves generated by the unaccelerated particle are given by the same boosted field of a static particle. This is complicated by the fact that in ADM gauge this gravitational field has nontrivial form since the TT-projection has to be applied to a simple solution of the wave equation. Fortunately, the solution of (10) can be found in an exact form in the given approximation:

$$\begin{aligned} h_{ij}^{TT}(\mathbf{x};\mathbf{x}_b,\mathbf{p}_b,\dot{\mathbf{x}}_b) \;=\; &\sum_b\frac{G}{|\mathbf{x}-\mathbf{x}_b|}\frac{1}{\overline{m}_b}\frac{1}{y_b(1+y_b)^2}\\ &\times\Big\{\left[y_b\mathbf{p}_b^2-(\mathbf{n}_b.\mathbf{p}_b)^2(3y_b+2)\right]\delta_{ij}\\ &+2\left[1-\dot{\mathbf{x}}_b^2(1-2\cos^2\theta_b)\right]p_{bi}p_{bj}\\ &+\left[(2+y_b)\,(\mathbf{n}_b.\mathbf{p}_b)^2-\left(2+3y_b-2\dot{\mathbf{x}}_b^2\right)\mathbf{p}_b^2\right]n_{bi}n_{bj}\\ &+2(\mathbf{n}_b.\mathbf{p}_b)\left(1-\dot{\mathbf{x}}_b^2+2y_b\right)\left(n_{bi}p_{bj}+p_{bi}n_{bj}\right)\Big\}, \quad (14) \end{aligned}$$

where $y_b = \sqrt{1-\dot{\mathbf{x}}_b^2\sin^2\theta_b}, \mathbf{n}_b = (\mathbf{x}-\mathbf{x}_b)/|\mathbf{x}-\mathbf{x}_b|$, and $\cos\theta_b = \dot{\mathbf{x}}_b.\mathbf{n}_b/|\mathbf{x}_b|$.

Finally, the Hamiltonian for particle and field has to be converted into another one which is function of particle variables only. Since variational derivative of Hamiltonian yields (generally) non-vanishing time derivative of the conjugated field, one cannot simply substitute the known solution of the field equations into the Hamiltonian. Rather the Legendre transformation to the so-called

Routh functional has to be performed

$$R(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{TT}, \dot{h}_{ij}^{TT}) = H - \frac{1}{16\pi G} \int d^3x \, \pi^{TT\,ij} \, \dot{h}_{ij}^{TT}.$$

It has vanishing variational derivatives (with respect to $h_{ij}^{TT}, \pi^{TT\,ij}$) if field equation are satisfied and thus it plays the role of particles-only Hamiltonian, after few more technical details are resolved [4]. Then the gravitational interaction of particles in 1PM is described by Hamiltonian

$$\begin{aligned}
H &= \sum_a \overline{m}_a - \frac{1}{2} G \sum_{a,b\neq a} \frac{\overline{m}_a \overline{m}_b}{r_{ab}} \left(1 + \frac{\mathbf{p}_a^2}{\overline{m}_a^2} + \frac{\mathbf{p}_b^2}{\overline{m}_b^2}\right) \\
&+ \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \left(7\, \mathbf{p}_a {\cdot} \mathbf{p}_b + (\mathbf{p}_a {\cdot} \mathbf{n}_{ab})(\mathbf{p}_b {\cdot} \mathbf{n}_{ab})\right) \\
&- \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \frac{(\overline{m}_a \overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}} \\
&\times \Bigg\{ \frac{2}{\overline{m}_b^2} \Big[ 2(\mathbf{p}_a{\cdot}\mathbf{p}_b)^2 (\mathbf{p}_b{\cdot}\mathbf{n}_{ba})^2 + (\mathbf{p}_a{\cdot}\mathbf{n}_{ba})^2 \mathbf{p}_b^4 \\
&\quad - 2(\mathbf{p}_a{\cdot}\mathbf{n}_{ba})(\mathbf{p}_b{\cdot}\mathbf{n}_{ba})(\mathbf{p}_a{\cdot}\mathbf{p}_b)\mathbf{p}_b^2 - (\mathbf{p}_a{\cdot}\mathbf{p}_b)^2 \mathbf{p}_b^2 \Big] \\
&\quad - 2\mathbf{p}_a^2 (\mathbf{p}_b{\cdot}\mathbf{n}_{ba})^2 + 4(\mathbf{p}_a{\cdot}\mathbf{n}_{ba})(\mathbf{p}_b{\cdot}\mathbf{n}_{ba})(\mathbf{p}_a{\cdot}\mathbf{p}_b) \\
&\quad + 2(\mathbf{p}_a{\cdot}\mathbf{n}_{ba})^2 \left((\mathbf{p}_b{\cdot}\mathbf{n}_{ba})^2 - \mathbf{p}_b^2\right) + 2(\mathbf{p}_a{\cdot}\mathbf{p}_b)^2 \\
&\quad + \Big[ (\mathbf{p}_a{\cdot}\mathbf{n}_{ba})^2 (\mathbf{p}_b{\cdot}\mathbf{n}_{ba})^2 - 3\mathbf{p}_a^2 (\mathbf{p}_b{\cdot}\mathbf{n}_{ba})^2 \\
&\qquad + 8(\mathbf{p}_a{\cdot}\mathbf{n}_{ba})(\mathbf{p}_b{\cdot}\mathbf{n}_{ba})(\mathbf{p}_a{\cdot}\mathbf{p}_b) + \mathbf{p}_a^2 \mathbf{p}_b^2 \\
&\qquad - 3(\mathbf{p}_a{\cdot}\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] y_{ba} \Bigg\}, \qquad (15)
\end{aligned}$$

where we use

$$y_{ba} = \sqrt{\frac{m_b^2 + (\mathbf{n}_{ba}{\cdot}\mathbf{p}_b)^2}{m_b^2 + \mathbf{p}_b^2}}. \qquad (16)$$

Let us mention that the particles' coordinates $\mathbf{x}_a$ and momenta $\mathbf{p}_a$ appearing in the Hamiltonian (15) are not identical to those which appear in (3), (12), or (11) since several canonical transformations were performed along the way: First, part of $H_f$ which has a form of a total time derivative was dropped which is equivalent to some canonical transformation. Also, since (14) contains particles' velocities $\dot{\mathbf{x}}_a$, after the Routh functional had been used to obtain particle-only Hamiltonian, it still depended on $\dot{\mathbf{x}}_a$. To convert this generalized Hamiltonian to the standard one depending on positions and momenta only, another canonical transformation had to be used.

## 4. DYNAMICS OF MASSLESS PARTICLES

To illustrate the properties of the obtained 1PM Hamiltonian for gravitationally interacting point particles, let us discuss its description of the dynamics of massless particles. The simplest example which takes advantage of PM approach is the Hamilton for a test photon ($m_2 = 0, \mathbf{x}_2 = \mathbf{x}, \mathbf{p}_2 = \mathbf{p}$) in the static ($\mathbf{p}_1 = 0, \mathbf{x}_1 = 0$) field of massive ($m_1 = M$) particle

$$H(\mathbf{x}, \mathbf{p}) = \left(1 + 2\frac{GM}{|\mathbf{x}|}\right) |\mathbf{p}|. \qquad (17)$$

The factor 2 comes from the ultra-relativistic limit of factor $1 + \mathbf{p}^2/\overline{m}^2$ unreachable by a simple PN expansion. This is identical to the truncated expansion of the Hamiltonian one would obtain for a test photon in isotropic Schwarzschild coordinates. Neglecting terms proportional to higher powers of $G$ it correctly describes such effects as *ligth deflection* and *Shapiro delay* in gravitational field. In this approximation there also is an unstable circular *photon orbit*. (The orbital frequency $\omega_0 = (8GM)^{-1}$ is about two thirds of the correct value, but the 1PM approximation can also describe a system, where the momentum of the orbiting null particle cannot be neglected with respect to the mass of the massive companion or even two massless particles in mutual orbits.)

When the gravitating object is not static ($\mathbf{p}_1 \neq 0$) the situation becomes much more complicated, but it is easy to check, that even though (16) reduces to $y_{ba} = |\mathbf{n}_{ba}{\cdot}\mathbf{p}_b|/|\mathbf{p}_b|$ for massless ($m_b = 0$) particle, Hamiltonian (15) does not contain terms proportional to $|\mathbf{n}_{ba}{\cdot}\mathbf{p}_b|^{-1}$ divergent at plane $\mathbf{n}_{ba}{\cdot}\mathbf{p}_b = 0$. Still, for a massless particle the non-smooth term $|\mathbf{n}_{ba}{\cdot}\mathbf{p}_b|$ will remain in Hamiltonian. It is related to the well-known fact that gravitational field accompanying the massless particle has form of an impulsive wave [1]. Thus the solutions of equations of motion implied by (15) are continuous functions but they are not generally smooth when massless particles are involved.

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