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Filipe Moura • Estelita Vaz
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Relative Motions of Free Test Particles in Robinson–Trautman Spacetimes of Any Dimension

Robert Švarc and Jiří Podolský

Abstract Using the invariant form of equation of geodesic deviation we analyze the relative deformations of a congruence of free test particles in general non-twisting, shearfree and expanding geometries. In four dimensions this class of exact solutions includes important classes of expanding gravitational waves. On the other hand, higher-dimensional Robinson–Trautman spacetimes can only be of algebraic type D. We emphasize the difference between the standard four-dimensional solutions and their arbitrary-dimensional extensions from the physical point of view of a geodesic observer.

1 Robinson–Trautman Geometries

The optical scalars A^2 (twist), σ^2 (shear) and Θ (expansion) characterizing affinely parameterized null geodesic congruence k^a are in arbitrary dimension D given by

$$A^2 = -k_{[a;b]}k^{a;b}\,, \quad \sigma^2 = k_{(a;b)}k^{a;b} - \frac{1}{D-2}(k^a{}_{;a})^2\,, \quad \Theta = \frac{1}{D-2}k^a{}_{;a}\,. \qquad (1)$$

The Robinson–Trautman class of spacetimes is defined as the geometries admitting nontwisting ($A = 0$), shearfree ($\sigma = 0$) and expanding ($\Theta \neq 0$) null geodesic congruence. The line element of a general nontwisting spacetime takes the form

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Abstract Using the invariant form of equation of geodesic deviation we analyze the relative deformations of a congruence of free test particles in general non-twisting, shearfree and expanding geometries. In four dimensions this class of exact solutions includes important classes of expanding gravitational waves. On the other hand, higher-dimensional Robinson–Trautman spacetimes can only be of algebraic type D. We emphasize the difference between the standard four-dimensional solutions and their arbitrary-dimensional extensions from the physical point of view of a geodesic observer.

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$$\mathrm{d}s^2 = g_{ij}(r,u,x)\,\mathrm{d}x^i\mathrm{d}x^j + 2g_{ui}(r,u,x)\,\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \quad (2)$$

where $i, j = 2, \ldots, D-1$, $u = const$ defines null hypersurfaces with normal $k^a = \partial_r$, r is an affine parameter along the geodesic congruence, and x^i represent $D-2$ spatial coordinates in a transverse Riemannian space. As shown in [1], the shearfree condition $\sigma = 0$ and the vacuum Einstein equations than imply $g_{ui} = 0$ in (2) and fully determine the r-dependence of the D-dimensional Robinson–Trautman metric as

$$\mathrm{d}s^2 = r^2 h_{ij}(u,x)\,\mathrm{d}x^i\mathrm{d}x^j - 2\,\mathrm{d}u\mathrm{d}r - 2H(r,u,x)\,\mathrm{d}u^2\,, \quad (3)$$

with the function $2H$ given by

$$2H = \frac{\mathscr{R}}{(D-2)(D-3)} + \frac{2(\ln\sqrt{h})_{,u}}{D-2}\,r - \frac{2\Lambda}{(D-2)(D-1)}\,r^2 - \frac{\mu}{r^{D-3}}\,, \quad (4)$$

where $\mathscr{R}(u,x)$ is the scalar curvature calculated with respect to the spatial metric h_{ij},

$$h_{ij}(u,x) = P^{-2}(u,x)\,\gamma_{ij}(x) \quad \text{and} \quad \det\gamma_{ij} = 1\,, \quad (5)$$

$h(u,x)$ is defined as $h \equiv \det h_{ij} = P^{2(2-D)}$, Λ is a cosmological constant, and $\mu(u,x)$ is an arbitrary function. For this general form of the vacuum Robinson–Trautman line element (3) the nonvanishing components of the Weyl tensor explicitly become

$$C_{ruru} = -(D-2)(D-3)\frac{\mu}{2r^{D-1}}\,, \qquad C_{rpuq} = -\frac{r^2 h_{pq}}{D-2}C_{ruru}\,,$$
$$C_{kplq} = r^2\mathscr{R}_{kplq} - \frac{2r^2 h_{k[l}h_{q]p}}{(D-2)(D-3)}\left(2r^2 C_{ruru} + \mathscr{R}\right)\,,$$
$$C_{upkq} = \frac{2r\mathscr{R}_{,[k}h_{q]p}}{(D-2)^2(D-3)}\,, \qquad C_{ruup} = \frac{\mathscr{R}_{,p}}{(D-2)^2 r}\,,$$
$$C_{upuq} = 2H C_{rpuq} + W_{pq} - \frac{h_{pq}}{D-2}\,h^{ij}\,W_{ij}\,, \quad (6)$$

where $\mathscr{R}_{kplq}$ is the Riemann tensor of the transverse space h_{ij}, and W_{pq} denotes

$$W_{pq} \equiv H_{,pq} - \frac{1}{2}H_{,k}\,h^{kl}\left(2h_{l(p,q)} - h_{pq,l}\right)\,. \quad (7)$$

Other restrictions on the transverse metric h_{ij} and the parameters contained in the metric function H (in general depending on u and x^i coordinates) follow from the remaining vacuum Einstein equations and significantly depend on the number of dimensions D, see the detailed discussion in [1]. For our purpose here notice that in any higher dimension $D > 4$ the coordinate dependence of these metric functions is

$$\mathscr{R} = \mathscr{R}(u)\,, \qquad \mu = \mu(u)\,, \qquad P(x)\,, \qquad P(u,x)\ \text{for}\ \mu = 0\,, \quad (8)$$

while in standard four-dimensional case we obtain

$$\mathscr{R} = \mathscr{R}(u,x)\,, \qquad \mu = \mu(u)\,, \qquad P(u,x)\,, \qquad h_{ij} = P^{-2}(u,x)\,\delta_{ij}\,. \quad (9)$$

2 Geodesic Deviation

In our work [2] we discussed specific influence of an arbitrary gravitational field in any dimension D on relative motion of geodesic particles. In the case of vacuum spacetimes the equation of geodesic deviation takes the invariant form

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)}\,Z^{(1)} + \Psi_{2S}\,Z^{(1)} + \frac{1}{\sqrt{2}}\left(\Psi_{1Tj} - \Psi_{3Tj}\right)Z^{(j)}\,,$$
$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)}\,Z^{(i)} - \Psi_{2T(ij)}\,Z^{(j)} + \frac{1}{\sqrt{2}}\left(\Psi_{1Ti} - \Psi_{3Ti}\right)Z^{(1)}$$
$$-\frac{1}{2}\left(\Psi_{0ij} + \Psi_{4ij}\right)Z^{(j)}\,, \quad (10)$$

with $i, j = 2, \ldots, D-1$. Here $Z^{(1)}, Z^{(2)}, \ldots, Z^{(D-1)}$ are spatial components of the separation vector $\mathbf{Z} = Z^a\,\mathbf{e}_a$ between the test particles in a natural interpretation orthonormal frame $\{\mathbf{e}_a\}$, i.e., $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}$, where $\mathbf{e}_{(0)} \equiv \mathbf{u} = \dot{r}\partial_r + \dot{u}\partial_u + \dot{x}^i\partial_i$ is the velocity vector of the fiducial test particle, $\ddot{Z}^{(1)}, \ddot{Z}^{(2)}, \ldots, \ddot{Z}^{(D-1)}$ are the corresponding relative accelerations, and the scalars $\Psi_{A\ldots}$ are defined as the components of the Weyl tensor in the null frame $\{\mathbf{k}, \mathbf{l}, \mathbf{m}_i\}$ adapted to observer's D-velocity $\mathbf{u}$, namely,

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{u} + \mathbf{e}_{(1)}) = \frac{1}{\sqrt{2}\dot{u}}\partial_r\,,$$
$$\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{e}_{(1)}) = \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\partial_r + \sqrt{2}\dot{u}\,\partial_u + \sqrt{2}\dot{x}^i\,\partial_i\,,$$
$$\mathbf{m}_i = \mathbf{e}_{(i)} = \frac{1}{\dot{u}}\,g_{kl}\,\dot{x}^k m_i^l\partial_r + m_i^j\,\partial_j\,, \quad (11)$$

and the projections of the Weyl tensor (grouped by their boost weight) are

$$\Psi_{0ij} = C_{abcd}\,k^a\,m_i^b\,k^c\,m_j^d\,,$$
$$\Psi_{1ijk} = C_{abcd}\,k^a\,m_i^b\,m_j^c\,m_k^d\,, \qquad \Psi_{1Ti} = C_{abcd}\,k^a\,l^b\,k^c\,m_i^d\,,$$

$$\begin{aligned}
\Psi_{2ijkl} &= C_{abcd}\, m_i^a\, m_j^b\, m_k^c\, m_l^d\,, & \Psi_{2S} &= C_{abcd}\, k^a\, l^b\, l^c\, k^d\,, \\
\Psi_{2ij} &= C_{abcd}\, k^a\, l^b\, m_i^c\, m_j^d\,, & \Psi_{2T^{ij}} &= C_{abcd}\, k^a\, m_i^b\, l^c\, m_j^d\,, \\
\Psi_{3ijk} &= C_{abcd}\, l^a\, m_i^b\, m_j^c\, m_k^d\,, & \Psi_{3T^i} &= C_{abcd}\, l^a\, k^b\, l^c\, m_i^d\,, \\
\Psi_{4ij} &= C_{abcd}\, l^a\, m_i^b\, l^c\, m_j^d\,, & &
\end{aligned} \tag{12}$$

where $i, j, k, l = 2, \ldots, D-1$.

However, for the vacuum Robinson–Trautman spacetimes using the explicit form of the Weyl tensor (6) we find that Ψ_{0ij}, Ψ_{1ijk} and Ψ_{1T^i} (which correspond to the highest boost weights $+2$ and $+1$) vanish identically. The only non-trivial Weyl scalars (12) with respect to the null frame (11) that are present in (10) take the form

$$\Psi_{2S} = -C_{ruru}\,, \qquad \Psi_{2T^{ij}} = m_i^p m_j^q\, C_{rpuq}\,, \tag{13}$$

$$\Psi_{3T^j} = \sqrt{2}\, m_j^p \big[\dot{x}^k (g_{kp} C_{ruru} - C_{rkup}) - \dot{u}\, C_{ruup}\big]\,, \tag{14}$$

$$\begin{aligned}
\Psi_{4ij} = 2\, m^p_{(i} m^q_{j)} \big\{ & \dot{x}^k \dot{x}^l \big[g_{kl} C_{rpuq} - g_{pk}(2C_{rluq} - g_{lq} C_{ruru}) + C_{kplq} \big] \\
& + 2\dot{x}^k \dot{u} (C_{upkq} - g_{kq} C_{ruup}) + \dot{u}^2 (C_{upuq} - 2H C_{rpuq}) \big\}\,,
\end{aligned} \tag{15}$$

where the Weyl tensor components are explicitly given by (6) and, due to (8) and (9), significantly depend on the number of dimensions D.

The Weyl scalars (13)–(15) represent specific combinations of observer's kinematics with the curvature of the spacetime. The overall relative motion measured by an arbitrary geodesic observer in any dimension D with velocity $\mathbf{u}$ described by equations (10) thus, in general, consists of the isotropic influence of the cosmological constant Λ, Newton-like deformation induced by the terms Ψ_{2S} and $\Psi_{2T^{(ij)}}$, the longitudinal effects encoded in Ψ_{3T^i}, and the transverse deformations corresponding to Ψ_{4ij}, see [2] for the physical interpretation of the $\Psi_{A\ldots}$ scalars.

However, the terms in (13)–(15) containing spatial components $\dot{x}^i$ of observer's velocity $\mathbf{u}$ can be (at least locally) removed by a suitable particular choice of the fiducial geodesic with $\dot{x}^i = 0$. These 'radial' observers thus measure 'pure' effects of the vacuum Robinson–Trautman gravitational field and are able to distinguish between four and a higher dimensional spacetime. To be more specific:

- The higher-dimensional constraints (8) imply than in the case $\dot{x}^i = 0$ the only nonvanishing Weyl scalars are Ψ_{2S} and $\Psi_{2T^{(ij)}}$ representing Newton-like tidal deformations governed by the 'mass' parameter μ, see (13) and (6).
- From (9) in four dimensions it follows that all Weyl scalars Ψ_{2S}, $\Psi_{2T^{(ij)}}$, Ψ_{3T^i} and Ψ_{4ij} are in general nonvanishing and the test particles in vacuum Robinson–Trautman spacetimes are thus affected by Newton-like tidal deformation (13), the longitudinal effects (14), and by the transverse gravitational waves (15).

This agrees with previous results of [1, 3] that vacuum Robinson–Trautman spacetimes in $D > 4$ are only of algebraic type D, while in $D = 4$ they are of type II, or more special.

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