Gen Relativ Gravit (2014) 46:1648
DOI 10.1007/s10714-013-1648-0

RESEARCH ARTICLE

The regularity of geodesics in impulsive *pp*-waves

Alexander Lecke · Roland Steinbauer · Robert Švarc

Received: 4 October 2013 / Accepted: 22 November 2013

Abstract We consider the geodesic equation in impulsive *pp*-wave space-times in Rosen form, where the metric is of Lipschitz regularity. We prove that the geodesics (in the sense of Carathéodory) are actually continuously differentiable, thereby rigorously justifying the $\mathcal{C}^1$-matching procedure which has been used in the literature to explicitly derive the geodesics in space-times of this form.

Keywords Impulsive *pp*-waves · Geodesics · Carathéodory-solutions

Mathematics Subject Classification (2010) 83C15 · 34A36 · 83C35

1 Introduction

Impulsive *pp*-waves [16] have become text-book examples of exact solutions modeling gravitational wave pulses, see [9, Ch. 20] and [17] for an overview. They can be described by the line element in Brinkmann form

$$\mathrm{d}s^2 = 2H(\zeta, \bar{\zeta})\delta(\mathcal{U})\mathrm{d}\mathcal{U}^2 - 2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} + 2\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}, \tag{1}$$

A. Lecke · R. Steinbauer (✉)
Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
e-mail: roland.steinbauer@univie.ac.at

A. Lecke
e-mail: alexander.lecke@univie.ac.at

R. Švarc
Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, 180 00 Praha 8, Czech Republic
e-mail: robert.svarc@mff.cuni.cz

Published online: 17 December 2013

where for convenience we have used complex coordinates

$$\zeta = \frac{1}{\sqrt{2}}\big(x + \mathrm{i}y\big), \quad \bar{\zeta} = \frac{1}{\sqrt{2}}\big(x - \mathrm{i}y\big),$$

and $(\mathcal{U}, \mathcal{V}, x, y) \in \mathbb{R}^4$. Here H is a real-valued function of the spatial variables which we assume to be smooth (except for possible singularities which we then remove from the space-time) and δ denotes the Dirac-function. In these coordinates the metric takes manifestly Minkowskian form in front and behind the wave impulse which is located on the null hypersurface $\{\mathcal{U} = 0\}$. This, however, comes at the expense of introducing a *distributional* coefficient into the metric. Alternatively the space-time is described in Rosen form [4,16,24]

$$\mathrm{d}s^2 = 2\left|\mathrm{d}Z + U_+\big(H_{,Z\bar{Z}}\mathrm{d}Z + H_{,\bar{Z}\bar{Z}}\mathrm{d}\bar{Z}\big)\right|^2 - 2\mathrm{d}U\mathrm{d}V, \tag{2}$$

where again we have used complex coordinates in the transverse space

$$Z = \frac{1}{\sqrt{2}}\big(X + \mathrm{i}Y\big) \quad \bar{Z} = \frac{1}{\sqrt{2}}\big(X - \mathrm{i}Y\big),$$

and $(U, V, X, Y) = (U, V, X^2, X^3) \in \mathbb{R}^4$. Moreover,

$$U_+(U) = \begin{cases} 0 & \text{if } U \leq 0, \\ U & \text{if } U \geq 0 \end{cases}$$

denotes the kink-function and hence the metric (2) is *Lipschitz continuous*.

The geodesics of (1), which actually are broken and refracted straight lines with a jump in the $\mathcal{V}$-coordinate, have been derived in [7], while in [2,26] the geodesic equations (which are non-linear ODEs with distributional right hand sides, hence mathematically delicate) have been treated rigorously. Finally in [14] the geodesic equations of (1) have been proven to possess unique global solutions in a suitable space of (non-linear) generalized functions [3,8]. This result in turn enabled a mathematically sensible treatment [13,5] of the discontinuous "coordinate transform" introduced by Penrose [16] which relates (1) and (2) [(see also (9), below].

On the other hand the geodesics in space-times similar to (2) (impulsive *pp*-waves [25,27], non-expanding (Kundt) impulsive waves with a cosmological constant [19, 21], and expanding impulsive waves [22,23]) have been derived by pasting together the geodesics of the background in a $\mathcal{C}^1$-manner. More precisely, in our case assuming the geodesics to be $\mathcal{C}^1$-curves in the continuous metric (2) one may match the straight line solutions given in manifestly Minkowskian coordinates on either side of the wave to obtain explicit global geodesics (see also section 3, below). While this "$\mathcal{C}^1$-matching procedure" basically gives the correct answer [27, Sec. 4] the key assumption that allows for the matching at all has remained unproven. In fact, the Christoffel symbols of the Lipschitz continuous metric (2) and hence the right hand side of the geodesic equations are only locally bounded but discontinuous, and at first sight the $\mathcal{C}^1$-property as well as uniqueness of the geodesics seems to be too much to hope for.

In this short note, we *prove* that the geodesic equation of (2) actually possesses unique $\mathcal{C}^1$-solutions. To this end we employ the most natural solution-concept available for ODEs with discontinuous right hand sides of this form, which is due to Carathéodory (see e.g. [6, Ch. 1]). It is a minimal extension of the classical solution concept and provides an existence and uniqueness theorem for systems of the form

$$\dot{x}(t) = f(t, x(t))$$

basically assuming f to be Lipschitz continuous only with respect to x and merely measurable w.r.t. t. Moreover the solutions are guaranteed to be absolutely continuous. For the convenience of the reader we have collected the basic facts on Carathéodory solutions in an appendix, for all details we refer to the literature.

We prove the $\mathcal{C}^1$-property of the geodesics in Sect. 2 and using the (now justified) matching procedure derive an explicit description of the geodesics in generic impulsive *pp*-waves in Sect. 3.

2 The regularity of geodesics for impulsive *pp*-waves

In this section we explicitly calculate the geodesic equations for impulsive *pp*-waves in the continuous form of the metric (2) and demonstrate that the coefficients obey the assumptions of Carathéodory's existence and uniqueness theorem (Theorem 2). In this way we prove that the (Carathéodory) solutions of the geodesic equations are continuously differentiable.

We start by rewriting metric (2) in real form

$$\mathrm{d}s^2 = g_{ij}\,(U, X^k)\,\mathrm{d}X^i \mathrm{d}X^j - 2\,\mathrm{d}U\mathrm{d}V, \tag{3}$$

where the spatial metric is given by

$$g_{ij} = \delta_{ij} + 2U_+ H_{,ij} + (U_+)^2 \delta^{kl} H_{,ik} H_{,jl} \tag{4}$$

for $i, j, k, l = 2, 3$. So the g_{ij} and hence the entire metric is smooth w.r.t. X^i but merely Lipschitz continuous w.r.t. U. Recall that by Rademacher's theorem (locally) Lipschitz continuous functions are differentiable almost everywhere with derivative belonging (locally) to L^∞. Taking derivatives of the metric coefficients will always be understood in this sense. Therefore we deliberately do not denote the kink function by $U\Theta(U)$ to avoid any confusion which might arise from using multiplication rules on $(U\Theta(U))^2$ (possibly obscuring the fact that $U_+^2 \in \mathcal{C}^1$).

The non-vanishing Christoffel symbols are

$$\Gamma^V_{jk} = \frac{1}{2} g_{jk,U}, \quad \Gamma^i_{Uk} = \frac{1}{2} g^{ij} g_{jk,U}, \quad \Gamma^i_{kl}, \tag{5}$$

where Γ^i_{kl} $(i, k, l = 2, 3)$ are the Christoffel symbols of the spatial metric (4). Since all the Christoffel symbols of the form $\Gamma^U_{\mu\nu}$ vanish we may use U as an affine parameter for the geodesics. Setting $\dot{U} = 1$ the geodesic equations take the form

$$\ddot{V} + \frac{1}{2} g_{ij,U} \dot{X}^i \dot{X}^j = 0, \quad \ddot{X}^i + \Gamma^i_{kl} \dot{X}^k \dot{X}^l + g^{ij} g_{jk,U} \dot{X}^k = 0 \tag{6}$$

and after some calculations we explicitly obtain

$$\begin{aligned} \ddot{V} &= -\big[(U_+)_{,U}\, H_{,ij} + \tfrac{1}{2}(U_+^2)_{,U}\, \delta^{mn} H_{,im} H_{,jn}\big]\dot{X}^i \dot{X}^j\,, \\ \ddot{X}^i &= -g^{ij}\big[U_+\, H_{,jkl} + U_+^2\, \delta^{mn} H_{,jm} H_{,kln}\big]\dot{X}^k \dot{X}^l \\ &\quad -2g^{ij}\big[(U_+)_{,U}\, H_{,jk} + \tfrac{1}{2}(U_+^2)_{,U}\, \delta^{mn} H_{,jm} H_{,kn}\big]\dot{X}^k\,, \end{aligned} \tag{7}$$

where the coefficients of the inverse spatial metric are of course given by

$$g^{ij} = D^{-1} g_{pq} (\delta^{ij}\delta^{pq} - \delta^{ip}\delta^{jq}), \;\; D \equiv \det g_{ij} = \frac{1}{2}\big(\delta^{ij}\delta^{pq} - \delta^{ip}\delta^{jq}\big) g_{ij} g_{pq}.$$

We now interpret system (7) as a first order system in the dependent variables $\mathbf{X} := (V, \tilde{V} = \dot{V}, X = X^2, \tilde{X} = \dot{X}^2, Y = X^3, \tilde{Y} = \dot{X}^3)$ and the independent variable U and check that the conditions of Theorem 2 (see appendix) are satisfied. Since these are obviously fulfilled for the trivial equations $\dot{V} = \tilde{V}$, $\dot{X} = \tilde{X}$ and $\dot{Y} = \tilde{Y}$ we are left with the task of verifying conditions (A)–(C) of Theorem 2 for the right hand sides of (7), locally around every point p lying on the shock surface $\{U = 0\}$.

Clearly every such point p has some neighborhood $\mathcal{W}$ where D is bounded away from 0, hence the inverse spatial metric on $\mathcal{W}$ is smooth w.r.t. (X, Y) and Lipschitz continuous w.r.t. U. Consequently the r.h.s. of (7) is smooth w.r.t. $\mathbf{X}$ and (due to the terms $(U_+)_{,U} = \Theta(U)$ merely) L^∞ w.r.t. U on $\mathcal{W}$. This, however, gives (A) (with the exceptional value $U = 0$), (B), and (C) (with $\mathfrak{m}$ actually in L^∞) on $\mathcal{W}$.

Now given arbitrary data at p, Theorem 2 provides us with a unique solution $\mathbf{X}$ locally around the shock hypersurface. In addition $\mathbf{X}$ is absolutely continuous which implies that the velocities $(\dot{V}, \dot{X}, \dot{Y})$ are continuous. Hence the geodesics are $\mathcal{C}^1$. Since off the shock hypersurface the space-time is just Minkowski space we may match the solutions obtained above to the geodesics of the background to obtain global solutions.

Hence we have shown that impulsive *pp*-waves in the continuous form possess unique global $\mathcal{C}^1$-geodesics. More precisely we may state the following theorem.

Theorem 1 *The geodesic equations for the impulsive* pp-*wave metric* (2) *are uniquely globally solvable in the sense of Carathéodory and the solutions are continuously differentiable. (In fact they possess absolutely continuous velocities.)*

Finally we remark that our method crucially depends on the fact that the coordinate U can be used as an affine parameter for the geodesics. This is, however, not the case for more general classes of impulsive gravitational waves such as non-expanding impulsive waves on (anti) de-Sitter background [19,21] as well as expanding impulsive waves in all constant curvature backgrounds [1,10–12,15,16,18,20]. In this situation the geodesic equations have to be treated as an autonomous system of ODEs and in this case Carathéodory's theorem provides no advantage over the classical theory, i.e., it also needs the right hand side to be Lipschitz continuous; but of course the Christoffel symbols will again only be bounded. A thorough investigation of this case is subject to current research.

3 $\mathcal{C}^1$-matching

Using Theorem 1 we now apply the $\mathcal{C}^1$-matching procedure outlined in the introduction to derive the explicit form of the geodesics for impulsive *pp*-waves. We start with Minkowski space-time in the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} \tag{8}$$

and consider the (formal) transformation [9, eq. (20.4)]

$$\begin{aligned}
\mathcal{U} &= U,\\
\mathcal{V} &= V + \Theta(U)H + \tfrac{1}{2}U_+\big((H_{,X})^2 + (H_{,Y})^2\big),\\
x &= X + U_+ H_{,X},\\
y &= Y + U_+ H_{,Y},
\end{aligned} \tag{9}$$

which is discontinuous in $\mathcal{V}$ at $U = 0$ and exactly gives Penrose's junction conditions used in his "scissors and paste" approach [16]. If we employ the transformation separately in the regions $U < 0$ and $U > 0$ we obtain the continuous line element (3). Observe that if one formally applies (9) for all U one obtains the distributional form (1) of the metric (a procedure which has been made mathematically precise in [13,5]).

We now stay with the continuous form (2) of the metric and apply Theorem 1 to obtain global $\mathcal{C}^1$-geodesics which we denote by

$$V = V(U), \quad X = X(U), \quad Y = Y(U), \tag{10}$$

again using U as an affine parameter. We now employ transformation (9) separately for $U < 0$ and $U > 0$ and consider the geodesics (10) in the manifestly Minkowskian "halves" on either side of the impulse. We denote their limits and the limits of their velocities as we approach the impulse from the region $U < 0$ by

$$\mathcal{V}_\mathrm{i}^-,\ \dot{\mathcal{V}}_\mathrm{i}^-,\ x_\mathrm{i}^-,\ \dot{x}_\mathrm{i}^-,\ y_\mathrm{i}^-,\ \dot{y}_\mathrm{i}^-$$

and the limits as we approach the impulse from the region $U > 0$ by

$$\mathcal{V}_\mathrm{i}^+,\ \dot{\mathcal{V}}_\mathrm{i}^+,\ x_\mathrm{i}^+,\ \dot{x}_\mathrm{i}^+,\ y_\mathrm{i}^+,\ \dot{y}_\mathrm{i}^+.$$

Here the subscript i stands for "(time of) interaction". Now the $\mathcal{C}^1$-property of the geodesics (10) allows us to relate these sets of "interaction parameters" to one another. From (9) we explicitly obtain

$$
\begin{aligned}
\mathcal{V}_{\mathrm{i}}^{-} &= \mathcal{V}_{\mathrm{i}}^{+} - H_{\mathrm{i}}, \\
\dot{\mathcal{V}}_{\mathrm{i}}^{-} &= \dot{\mathcal{V}}_{\mathrm{i}}^{+} - H_{\mathrm{i},X}\dot{x}_{\mathrm{i}}^{+} - H_{\mathrm{i},Y}\dot{y}_{\mathrm{i}}^{+} + \tfrac{1}{2}\big((H_{\mathrm{i},X})^2 + (H_{\mathrm{i},Y})^2\big), \\
x_{\mathrm{i}}^{-} &= x_{\mathrm{i}}^{+}, \\
\dot{x}_{\mathrm{i}}^{-} &= \dot{x}_{\mathrm{i}}^{+} - H_{\mathrm{i},X}, \\
y_{\mathrm{i}}^{-} &= y_{\mathrm{i}}^{+}, \\
\dot{y}_{\mathrm{i}}^{-} &= \dot{y}_{\mathrm{i}}^{+} - H_{\mathrm{i},Y},
\end{aligned}
$$

where H_{i}, $H_{\mathrm{i},X}$ and $H_{\mathrm{i},Y}$ denote the value of H respectively of its derivatives on the respective geodesic (10) at interaction time $U = 0$. So the geodesics, *as seen in the Minkowskian "halves" in front and behind the wave*, suffer a jump in the $\mathcal{V}$-component and are refracted in the $\mathcal{V}$-direction as well as in both spatial directions.

These formulae coincide with the (distributional limits of the) geodesics derived in the distributional form (1) of the metric in [14, Thm. 3] and we have thus given a second rigorous way of explicitly deriving the geodesics for impulsive *pp*-waves.

Acknowledgments We thank Jiří Podolský for kindly sharing his expertise and Clemens Sämann, and Milena Stojković for helpful discussions. This work was supported by FWF-grant P25326 and OeAD WTZ-project CZ15/2013 resp. 7AMB13AT003.

4 Appendix: Carathéodory solutions

In this appendix we briefly summarize the basic facts of Carathéodory's extension of the classical existence theory for ODEs and explicitly state the theorem used to prove our main result in Sect. 2. For all details we refer to [6, Ch. 1] and [28, Ch. 3 §10, Suppl. 2].

We consider the initial value problem for a non-autonomous system of first order ODEs

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{11}$$

Here $f : I \times D \to \mathbb{R}^d$, I is an open interval containing t_0 and $D \subseteq \mathbb{R}^d$ is an open and connected set which contains x_0.

A *Carathéodory solution* of (11) on an interval J with $t_0 \in J \subseteq I$ is an absolutely continuous function $x\colon J \to D$ which solves Eq. (11) almost everywhere (in the sense of the Lebesgue measure) and $x(t_0) = x_0$. We recall that *absolute continuity* is a strengthening of continuity, which is weaker than Lipschitz continuity. More precisely, a function $x\colon J \to \mathbb{R}^d$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite sequence of pairwise disjoint sub-intervals (a_k, b_k) of J with total length $\sum_k |b_k - a_k| < \delta$ we have $\sum_k |x(b_k) - x(a_k)| < \varepsilon$. Equivalently the derivative $\dot{x}$ of x exists almost everywhere and we have

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)\,\mathrm{d}\tau.$$

Hence x is a Carathéodory solution of (11) iff it solves the equivalent integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, (x(s)) \, \mathrm{d}s.$$

Of course every classical solution of (11) is a Carathéodory solution but the existence of the latter is guaranteed even for certain discontinuous right hand sides f. More precisely, we have the following basic existence and uniqueness theorem (cf. e.g. [28, §10, XVIII]).

Theorem 2 *Let the function* $f : I \times D \to \mathbb{R}^d$ *satisfy the conditions*

(A) $f(t, x)$ *is continuous in* x *for almost all* t,
(B) $f(t, x)$ *is measurable in* t *for all* x,
(C) There exists $\mathfrak{m} \in L^1(I)$ *with* $|f(t, 0)| \leq \mathfrak{m}(t)$ *and*

$$|f(t, x) - f(t, y)| \leq \mathfrak{m}(t) \, |x - y|.$$

Then there exists a unique (absolutely continuous) Carathéodory solution x *of* (11) *on some interval* J *with* $t_0 \in J \subseteq I$.

References

1. Aliev, A.N., Nutku, Y.: Impulsive spherical gravitational waves. Class. Quant. Grav. **18**(5), 891–906 (2001)
2. Balasin, H.: Geodesics for impulsive gravitational waves and the multiplication of distributions. Class. Quant. Grav. **14**(2), 455–462 (1997)
3. Colombeau, J.F.: Elementary Introduction to New Generalized Functions. North Holland, Amsterdam (1985)
4. D'Eath, P.D.: High-speed black-hole encounters and gravitational radiation. Phys. Rev. D (3) **18**(4), 990–1019 (1978)
5. Erlacher, E., Grosser, M.: Inversion of a 'discontinuous coordinate transformation' in general relativity. Appl. Anal. **90**(11), 1707–1728 (2011)
6. Filippov, A.F.: Differential Equations with Discontinuous Righthand Sides. Kluwer Academic Publishers, Dordrecht (1988)
7. Ferrari, V., Pendenza, P., Veneziano, G.: Beam like gravitational waves and their geodesics. J. Gen. Rel. Grav. **20**(11), 1185–1191 (1988)
8. Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.: Geometric Theory of Generalized Functions. Volume 537 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht (2001)
9. Griffiths, J., Podolský, J.: Exact Space-Times in Einstein's General Relativity. Cambridge University Press, Cambridge (2009)
10. Hogan, P.A.: A spherical gravitational wave in the de Sitter universe. Phys. Lett. A **171**(1–2), 21–22 (1992)
11. Hogan, P.A.: A spherical impulse gravity wave. Phys. Rev. Lett. **70**(2), 117–118 (1993)
12. Hortacsu, M.: Quantum fluctuations in the field of an impulsive spherical gravitational wave. Class. Quant. Grav. **7**(8), L165–L169, (1990). Erratum: Class. Quant. Grav. **9**(3), 799 (1992)
13. Kunzinger, M., Steinbauer, R.: A note on the Penrose junction conditions. Class. Quant. Grav. **16**(4), 1255–1264 (1999)
14. Kunzinger, M., Steinbauer, R.: A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves. J. Math. Phys. **40**(3), 1479–1489 (1999)
15. Nutku, Y., Penrose, R.: On impulsive gravitational waves. Twist. Newslett. **34**, 9–12 (1992)

16. Penrose, R.: The geometry of impulsive gravitational waves. In: O'Raifeartaigh, L. (ed.) General Relativity, pp. 101–115. Clarendon Press, Oxford (1972)
17. Podolský, J.: Exact impulsive gravitational waves in spacetimes of constant curvature. In: Semerák, O., Podolský, J., Žofka, M. (eds.) Gravitation: Following the Prague Inspiration. A Volume in Celebration of the 60th Birthday of Jiří Bičák., pp. 205–246. World Scientific, Singapore (2002)
18. Podolský, J., Griffiths, J.B.: Expanding impulsive gravitational waves. Class. Quant. Grav. **16**(9), 2937–2946 (1999)
19. Podolský, J., Griffiths, J.B.: Nonexpanding impulsive gravitational waves with an arbitrary cosmological constant. Phys. Lett. A **261**(1–2), 1–4 (1999)
20. Podolský, J., Griffiths, J.B.: The collision and snapping of cosmic strings generating spherical impulsive gravitational waves. Class. Quant. Grav. **17**(6), 1401–1413 (2000)
21. Podolský, J., Ortaggio, M.: Symmetries and geodesics in (anti-) de Sitter spacetimes with non-expanding impulsive waves. Class. Quant. Grav. **18**(14), 2689–2706 (2001)
22. Podolský, J., Steinbauer, R.: Geodesics in spacetimes with expanding impulsive gravitational waves. Phys. Rev. D (3), **67**(6), 064013, 13 (2003)
23. Podolský, J., Švarc, R.: Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant. Phys. Rev. D (3), **81**(12), 124035, 19 (2010)
24. Podolský, J., Veselý, K.: Continuous coordinates for all impulsive pp-waves. Phys. Lett. A **241**(3), 145–147 (1998)
25. Podolský, J., Veselý, K.: Smearing of chaos in sandwich pp-waves. Class. Quant. Grav. **16**(11), 3599–3618 (1999)
26. Steinbauer, R.: Geodesics and geodesic deviation for impulsive gravitational waves. J. Math. Phys. **39**(4), 2201–2212 (1998)
27. Steinbauer, R.: On the geometry of impulsive gravitational waves. arXiv:gr-qc/9809054v2
28. Walter, W.: Ordinary Differential Equations, volume 182 of Graduate Texts in Mathematics. Springer, New York, 1998. Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics