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Linear perturbations of a Schwarzschild black hole by thin disc - convergence

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Abstract. In order to find the perturbation of a Schwarzschild space-time due to a rotating thin disc, we try to adjust the method used by [4] in the case of perturbation by a one-dimensional ring. This involves solution of stationary axisymmetric Einstein's equations in terms of spherical-harmonic expansions whose convergence however turned out questionable in numerical examples. Here we show, analytically, that the series are almost everywhere convergent, but in some regions the convergence is not absolute.

Keywords: general relativity, black holes, perturbation techniques, accretion discs
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LINEAR PERTURBATION OF BLACK HOLE

The orthogonally transitive, stationary and axisymmetric space-time can be described by the metric [1]

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + r^2B^2\mathrm{e}^{-2\nu}\sin^2\theta(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 + \mathrm{e}^{2\zeta-2\nu}\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right), \qquad (1)$$

wwhere t, ϕ are Killing coordinates and r, θ are isotropic coordinates covering the meridional planes; B, ν, ω and ζ denote functions of r and θ which are determined by Einstein's equations. In the thin-disc case, the energy-momentum tensor reads

To calculate them we must specify the energy-momentum tensor. In the case of thin dust disc it can be expressed as

$$T^\alpha_\beta = \sigma\mathrm{e}^{2\zeta-2\nu}u^\alpha u_\beta\frac{1}{r}\delta(\cos\theta), \qquad (2)$$

where σ is a surface density and u^α is the disc-matter four-velocity which can be expressed as

$$u^\alpha = \frac{\mathrm{e}^{-\nu}}{1-v^2}(1,0,0,\Omega) \qquad (3)$$

in terms of linear velocity with respect to the zero-angular-momentum observer $v = r\sin\theta B\mathrm{e}^{-2\nu}(\Omega-\omega)$ ($\Omega = \mathrm{d}\phi/\mathrm{d}t$ is the corresponding angular velocity at infinity).

The field equation for B reads

$$\nabla\cdot(r\sin\theta\nabla B) = 0, \qquad (4)$$

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where ∇ and $\nabla\cdot$ denote gradient and divergence in a Euclidean 3D space (represented in spherical coordinates r, θ, ϕ). Otherwise B can be chosen arbitrarily[1].

The most important Einstein equations are those for the dragging and gravitational potentials, ν and ω,

$$
\begin{aligned}
\nabla\cdot(B\nabla\nu)-\frac{1}{2}r^2\sin^2\theta\mathrm{e}^{-4\nu}\nabla\omega\cdot\nabla\omega &= 4\pi B\sigma\frac{1+v^2}{1-v^2}\frac{1}{r}\delta(\cos\theta), &(5)\\
\nabla\cdot\left(r^2\sin^2\theta B^3\mathrm{e}^{-4\nu}\nabla\omega\right) &= -16\pi B^2\sigma\mathrm{e}^{-2\nu}\frac{v}{1-v^2}\delta(\cos\theta). &(6)
\end{aligned}
$$

Knowing B, ν and ω, the last function ζ can be obtained by line integration of the remaining two relevant field equations.

We are interested in a perturbation of the Schwarzschild metric which in isotropic coordinates reads

$$
\mathrm{d}s^2=-\left(\frac{2r-M}{2r+M}\right)^2\mathrm{d}t^2+\left(1+\frac{M}{2r}\right)^4\left(\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\phi^2\right) \tag{7}
$$

also it corresponds to $\nu=\ln\frac{2r-M}{2r+M}$ and $\omega=0$. We start from choosing $B=1-M^2/(4r^2)$. Then one can rewrite the equations (5), (6) using the perturbation expansions

$$
\nu=\ln\left(\frac{2r-M}{2r+M}\right)+\sum_{n=0}^{\infty}\delta\nu_n(x)P_n(\cos\theta)+O(\varepsilon^2), \tag{8}
$$

$$
\omega=\sum_{n=0}^{\infty}\delta\omega_n(x)C_n^{3/2}(\cos\theta)+O(\varepsilon^2), \tag{9}
$$

(with the O-remainders omitted), where $\varepsilon=\text{disc mass}/M$ is mass expansion parameter, $x\equiv\frac{r}{M}\left(1+\frac{M^2}{4r^2}\right)$ new "radial" coordinate and P_n and $C_n^{3/2}$ are Legendre and Gegenbauer polynomials, respectively, while decomposing the source terms on the r.h. sides in the same manner. Demanding equality in each n-order, the equations (5), (6) leads to

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\delta\nu_n\right]-n(n+1)\delta\nu_n=2(2n+1)\pi P_n(0)r\sigma(r)\frac{1+v^2}{1-v^2} \tag{10}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)^4\frac{\mathrm{d}}{\mathrm{d}x}\delta\omega_n\right]-\frac{(x+1)^3}{x-1}n(n+3)\delta\omega_n=-\frac{\pi C_n^{3/2}(0)(2n+3)(2r+M)^3}{8M^2(n+1)(n+2)(2r-M)}\frac{\sigma(r)v}{1-v^2}. \tag{11}
$$

This set can be solved to obtain expansions of the linear perturbation of functions ν and ω which are induced by the chosen source (thin disc) and which can be made regular both at the horizon and at radial infinity[2]; see e.g. [2].

[1] With the exception of the $B=1/(r\sin\theta)$ case which leads to plane-wave space-times.
[2] One can choose freely one constant which represents angular velocity of the horizon. We will omit it since it is not important for convergence. Besides that the system is determined.

TABLE 1. Coefficients of Green functions of the problem

	α	β	γ	δ	$N_n(x,x')$	$L_n(x,x')$	$\Sigma(r)$
$\mathscr{G}_n^{\nu}$	0	0	1/2	0	1	$-2\pi(2n+1)C_n^{\gamma}(0)$	$r\sigma(r)\frac{1+\nu^2}{1-\nu^2}$
$\mathscr{G}_n^{\omega}$	1	3	3/2	1	$(x-1)(x'-1)\frac{n+3}{n}$	$\frac{\pi(2n+3)(n+3)C_n^{\gamma}(0)(x-1)\sqrt{x'-1}(x'+1)^{3/2}}{2Mn(n+1)(n+2)}$	$r\sigma(r)\frac{\nu}{1-\nu^2}$

However, numerical illustrations of the results revealed unsatisfactory behaviour in the region close to the axis, mainly of the ω function. Suspecting bad convergence of the employed perturbation series, we have tried to check this issue analytically. As briefly summarized below, we found that the ω-expansion does not converge absolutely in certain regions and that its convergence may become problematic at certain parts of the axis.

CONVERGENCE OF THE SERIES

The general disc solution can be obtained convolving rings of matter. This ring solution (see [4]) corresponds (up to the numerical factor) to the Green function of equation,

$$\mathscr{G}_n(x,x') = N_n(x,x') \begin{cases} P_{n-\delta}^{(\alpha,\beta)}(x)Q_{n-\delta}^{(\alpha,\beta)}(x') & x < x' \\ P_{n-\delta}^{(\alpha,\beta)}(x')Q_{n-\delta}^{(\alpha,\beta)}(x) & x' < x \end{cases}, \tag{12}$$

where $P_n^{(\alpha,\beta)}$, $Q_n^{(\alpha,\beta)}$ are Jacobi functions of the first and second kind and the coefficients α, β, δ and the function $N(x,x')$ are written explicitly in the table 1.

The whole disc solution takes form

$$f(r,\theta) = \sum_{n=0}^{\infty} \int_{\text{disc}} C_n^{\gamma}(\cos\theta) L_n(x,x') \Sigma(x') P_{n-\delta}^{(\alpha,\beta)}(\min(x,x')) Q_{n-\delta}^{(\alpha,\beta)}(\max(x,x'))\mathrm{d}x', \tag{13}$$

where $f(r,\theta)$ is ν or ω and functions and constants are chosen in accordance to the table 1. It should be noted $C_n^{\gamma}(0)=0$ when n is odd so in the rest of this paper we will consider only "even" contributions to the Green functions.

To analyze asymptotic behaviour (with respect to $n\to\infty$) it is convenient to express Jacobi function in terms of Legendre functions:

$$P_n^{(1,3)}(x)Q_n^{(1,3)}(x') = H(x,x') \sum_{k=0}^{4}\sum_{l=0}^{4} X_{k\,l} \left[1+O\left(\frac{1}{n}\right)\right] P_{n+k}(x)Q_{n+l}(x'), \tag{14}$$

where $H(x,x')$ is rational function of x and x' and $X_{k\,l}$ are constants.

Using relations for modified Bessel functions I_0 and K_0 (see [3]) we can write

$$\begin{aligned} P_{n+k}(\cosh\zeta)Q_n(\cosh\xi) &= \sqrt{\frac{\xi\zeta}{\sinh\xi\sinh\zeta}} I_0\left(\tfrac{2n+2k+1}{2}\zeta\right) K_0\left(\tfrac{2n+1}{2}\xi\right)\left[1+O\left(\tfrac{1}{n}\right)\right] = \\ &= \frac{1}{2n\sqrt{\sinh\xi\sinh\zeta}} \mathrm{e}^{\zeta k}\mathrm{e}^{(\zeta-\xi)\left(n+\frac{1}{2}\right)}\left[1+O\left(\tfrac{1}{n}\right)\right]. \end{aligned} \tag{15}$$

We will also need to know asymptotic behaviour of the "spherical harmonics"

$$C_n^\gamma(\cos\theta) = \begin{cases} \frac{2n^{\gamma-1}\left[\cos\left((n+\gamma)\theta - \frac{\pi}{2}\gamma\right) + O\left(\frac{1}{n}\right)\right]}{\Gamma(\gamma)} & \text{when} \quad 0 < \theta < \pi \\ \frac{n^{2\gamma-1}}{\Gamma(2\gamma)}\left[1 + O\left(\frac{1}{n}\right)\right] & \text{when} \quad \cos\theta = 1 \end{cases}. \tag{16}$$

This expression can be also used to find properties of $L_n(x,x')$ when $n \to \infty$.

Taking all together we can conclude that

- At radii, where $\sigma(r) = 0$ (i.e. without matter) the exponential in (15) will dominate remaining terms and so there is exponential convergence.
- At the radii with $\sigma(r) \neq 0$ the behaviour depends on the actual position.
  - Between axis and equatorial plane there will be conditional convergence (as n^{-1}) in the case of ring and absolute convergence (as n^{-2}) in the case of disc.
  - At the equatorial plane there will be logarithmic divergence in the case of ring and absolute convergence (as n^{-2}) in the case of disc.
  - On the axis the situation is much more complicated. When the source is ring, there is conditional convergence (as $n^{-1/2}$) when considering gravitational potential and divergence (as $n^{1/2}$) for dragging. In the disc case convergence is one order faster, i.e. absolute for ν (as $n^{-3/2}$) and conditional for ω (as $n^{-1/2}$).

CONCLUSIONS

The method used by Will [4] when considering perturbation of a Schwarzschild black hole by a light slowly rotating ring can be extended to also enable disc perturbation. On the other hand, it involves expansions which do not behave well numerically. We have shown here that the series used in the first perturbation order are convergent almost everywhere, but the convergence is indeed slow at radii where the source is present (in the equatorial plane).

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