Recurrence of geodesics in a black-hole-disc field

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Recurrence of geodesics in a black-hole–disc field

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Abstract. Processes around astrophysical black holes are usually modeled with test (non-gravitating) matter and fields. However, the very fact that the holes are "visible" implies that *some* matter has to be present there. Even if very thin, it can influence the metric, mainly in higher derivatives, which should in turn affect its own behaviour, as well as motion of further (test) satellites. In particular, the geodesic dynamics, integrable in the fields of *isolated* stationary black holes, should become chaotic. Here we consider exact static, axially and reflection- symmetric space-times of a Schwarzschild black hole surrounded by a thin disc or ring, and examine how the presence of the additional source affects the dynamics of free test particles using recurence analysis.

Keywords: black holes, accretion discs, geodesic dynamics, chaos
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RECURRENCE ANALYSIS

Typical behaviour of a dynamical system is encoded in the pattern of its recurrences in the phase space. The pattern is different for regular, chaotic and random trajectories, so it can be used for classification of the systems and of their possible evolutions. Useful tool for visualisation of the recurrences are *recurrence plots* (RPs), developed in [1]. Recently this method was also used succesfully in general relativity, see e.g. [2] where recurrence analysis of the charged-particle motion around a magnetised rotating black hole was carried out. Here we draw RPs in order to study the geodesic dynamics in an exact static and axisymmetric space-time of a black hole surrounded by a massive thin disc, more specifically by the inverted first disc of the Morgan-Morgan family. (See [3] for details, including astrophysical motivation and analysis of the system by other methods.)

The drawing of RP requires knowledge of the test-particle phase trajectory with constant step of proper time (details are given in [4]). We find the orbits by solving numerically the corresponding geodesic equation, thus both position and velocity time series are known (in the case of just one known variable, the phase space can still be reconstructed from a sequence of its time-delayed series). The basic object of the recurrence analysis is the recurrence matrix defined by

$$\mathbf{R}_{i,j}(\varepsilon) = \Theta(\varepsilon - \parallel \vec{x}_i - \vec{x}_j \parallel), \qquad i,j = 1,...,N, \tag{1}$$

where $\vec{x}_i = \vec{x}(t_i)$ are (N) points of the phase trajectory, ε denotes a chosen threshold and Θ is the Heaviside step function. The matrix thus contains only 1's and 0's and can be easily visualised by plotting a black dot at the coordinates i, j whenever $\mathbf{R}_{i,j}(\varepsilon) = 1$. For regular orbits, the points are arranged in long diagonal lines, whereas for random behaviours they are scattered without order. Chaotic orbits yield more "artistic" plots:

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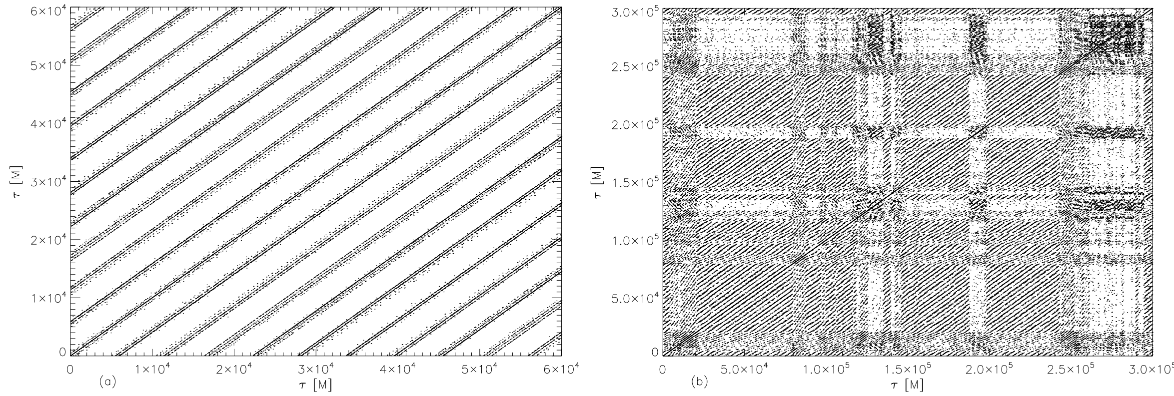

FIGURE 1. Recurrence plots for (a) regular and (b) chaotic trajectory. Note the scale change.

they contain blocks of almost-diagonal patterns as well as irregular ones, apparently placed one over another within horizontal and vertical structures. The almost-regular blocks correspond to time intervals when the trajectory sticks to some unstable periodic orbit. The more unstable this orbit is, the earlier the trajectory deviates from it and the smaller is the block in the RP. See Fig. 1 for an example of regular and chaotic RP.

Judging the degree of diagonal patterns within the RP by pure observation is of course subjective, so several quantifiers of the recurrence-matrix properties have been proposed. The simplest of them is the ratio of the recurrence points (black ones) within all points of the matrix, $RR(\varepsilon) = \frac{1}{N^2}\sum_{i,j=1}^{N} R_{i,j}(\varepsilon)$, called recurrence rate. Another, most important quantifier is the histogram of diagonal lines of a certain prescribed length l,

$$P(\varepsilon, l) = \sum_{i,j=1}^{N} (1 - R_{i-1,j-1}(\varepsilon))(1 - R_{i+l,j+l}(\varepsilon)) \prod_{k=0}^{l-1} R_{i+k,j+k}(\varepsilon). \tag{2}$$

Several further quantities can in turn be computed from this histogram. The one called DET is given by the ratio between all recurrence points and those of them which form a diagonal line longer than $l_{\min}$, thus $DET = \sum_{l=l_{min}}^{N} lP(l) / \sum_{l=1}^{N} lP(l)$. The length of the longest diagonal $L_{\max} = \max(\{l_i\}_{i=1}^{N})$ and its inverse $DIV = 1/L_{\max}$ are also of interest, since they are related to the rate of divergence of nearby orbits and serve as a rough estimate of the largest Lyapunov exponent. It is even possible to make this estimate more precise, because the cumulative histogram of diagonals turned out to yield the second-order Rényi's entropy K_2 which stands for a lower estimate of the sum of positive Lyapunov exponents (see [4]). Similarly, the histogram of vertical lines can be introduced,

$$P(\varepsilon, v) = \sum_{i,j=1}^{N} (1 - R_{i,j}(\varepsilon))(1 - R_{i,j+v}(\varepsilon)) \prod_{k=0}^{v-1} R_{i,j+k}(\varepsilon), \tag{3}$$

and also the respective measure of vertical structures $LAM = \sum_{v=v_{\min}}^{N} vP(v) / \sum_{v=1}^{N} vP(v)$. Finally, from the probabilities that some chosen diagonal/vertical line has length l,

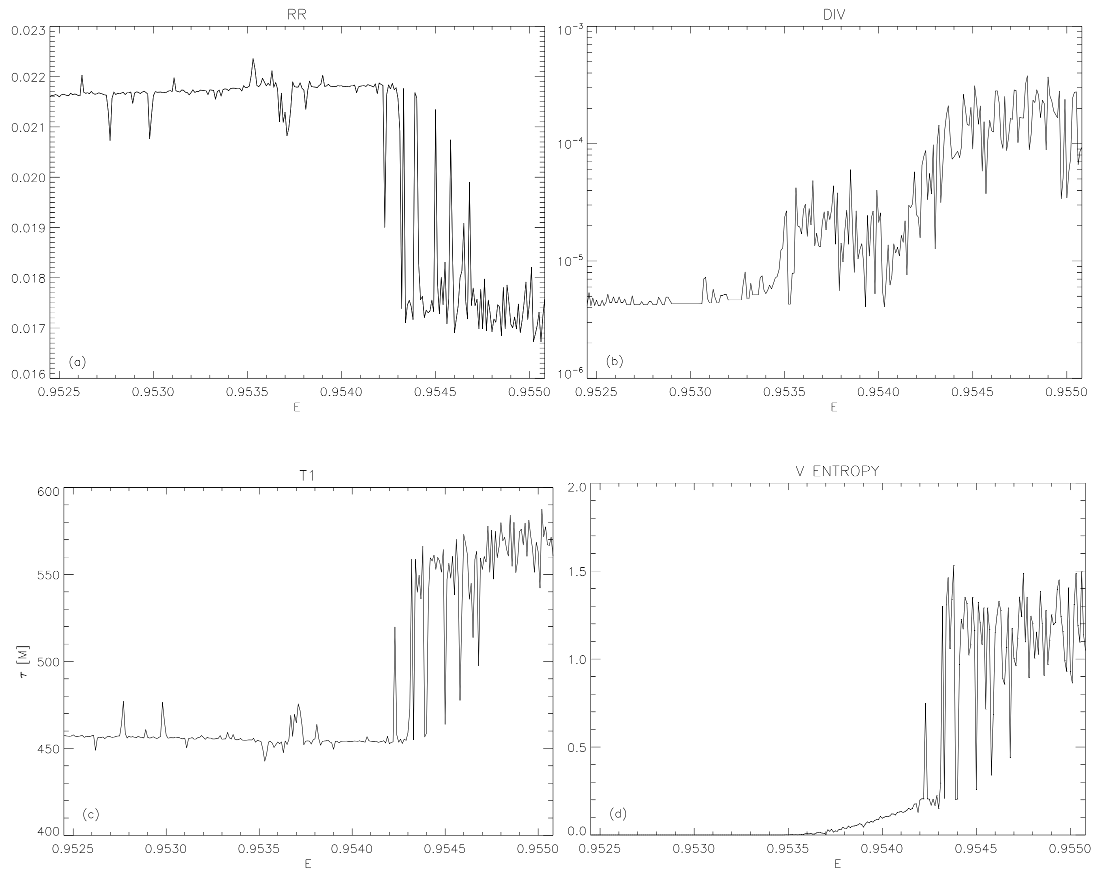

FIGURE 2. RQA measures as functions of energy: (a) the recurrence rate (RR), (b) longest-diagonal-line inverse (DIV), (c) first-type recurrence time (T1) and (d) entropy connected with the probability to find a vertical of length v (V ENTROPY).

$p(l) = P(l)/N_l$ and $p(v) = P(v)/N_v$, where N_l and N_v are the total numbers of diagonal/vertical lines, one can compute the Shannon entropies

$$ENTROPY = -\sum_{l=l_{min}}^{N} p(l)\ln p(l); \qquad VENTROPY = -\sum_{v=v_{min}}^{N} p(v)\ln p(v). \quad (4)$$

Another relevant indicator is the size of the "white gaps" between vertical lines. Namely, it is related to the recurrence times. For example, let us choose a point $\vec{x}_i$ on some trajectory and collect all the points which fall in its ε-neighbourhood, $\{\vec{x}_{j_1}, \vec{x}_{j_2}, \ldots\}$. Compute the recurrence times given by differences between serial numbers of the consecutive recurrence points $\vec{x}_{j_{k+1}}$, $\vec{x}_{j_k}$ multiplied by the respective proper-time steps, $\{T_k = (j_{k+1} - j_k)\Delta\tau\}$. The mean of T_k is called the recurrence time of the first type, $T1$.

RESULTS

We studied the behaviour of geodesics in our black-hole–disc system in dependence on their energy. In the above plots, the disc has mass $M/2$ and inner edge on $r = 20\,[M]$. The test particle is launched from $r = 6.25\,[M]$ with specific angular momentum $L = u_\phi = 3.75\,[M]$ and $u^\rho = 0$ and with specific energy $E = -u_t$ increasing from 0.951 by 0.00001 up to 0.95508 (M is the black-hole mass and u^μ is the particle's four-velocity). In each run we compute the trajectory numerically up to the time $t_{\max} = 250000\,[M]$, thus the minimal achievable DIV is 4×10^{-6}. For the recurrence analysis we use three spatial coordinates and velocities recorded with the proper-time step $\Delta\tau = 10\,[M]$. In order to be able to use the same threshold ε for different trajectories and to compare those of different extent in different directions, we normalize the series of each coordinate separately to zero mean and unit deviation. To exclude the sojourn points (the recurrence points resulting from tangential motion with ε large enough to capture more successive points of the trajectory) from the statistics, we set $l_{\min} = v_{\min} = 90\,[M]$.

For each trajectory we compute the RP and the corresponding RQA measures – see Fig. 2. The trajectory is regular initially. When the energy reaches about 0.95345, the motion becomes chaotic in a thin phase-space layer around the original regular orbit. The chaotic layer widens with increasing energy and around $E = 0.95423$ it spreads over a much larger region near the boundary of the accessible phase region. However, the process is not monotonic: for some higher energies the layer gets quite thin again, with its shape changing suddenly between these two possibilities. The computed RQA measures indicate the above behaviour in a slightly different way. Mainly, all of them clearly respond to the occurrence of a large chaotic region, but only DIV seems to be also sensitive to the thin-layer chaos. Namely, in a thin layer $DIV \simeq 3 \times 10^{-5}$ which is about ten times more than for regular motion while ten times less than in a large chaotic region. (The value 3×10^{-4} in a large chaotic region corresponds to a divergence time about several thousands $[M]$, which represents some ten cycles about the black hole.)

In the chaotic regime, the recurrence rate is lower and the vertical measures $T1$ and $VENTROPY$ grow significantly, which means that the system takes more time to get back to the chosen ε-neighbourhood.

For getting plausible results the parameters for RP (ε, $\Delta\tau$, $l_{\min}$, $v_{\min}$) have to be carefully set. More details of our results considering also this issue will be given elsewhere.

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