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Averaging in GR using Cartan scalars

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Abstract. Averaging problem in GR and cosmology is of fundamental importance. It is still not clear how to unambiguously average Einstein equations and the metric tensor (despite some promising attempts). Here we will present a new approach to this problem using the theory of the Cartan scalars. After short review of the theory originally taken from the equivalence problem, averaging procedure is presented and some examples are given.

Keywords: inhomogeneous cosmology, Cartan scalars, averaging
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INTRODUCTION

In General relativity (GR) the evolution of the metric tensor is driven by the Einstein field equations. As emphasized in 80's by Ellis [3], averaging and evolution do not commute, i.e. $\langle E_{\mu\nu}(g_{\mu\nu})\rangle \neq E_{\mu\nu}(\langle g_{\mu\nu}\rangle)$. On the other hand, in cosmology one usually uses the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric and the smooth stress energy tensor of the perfect fluid. If we want to use a simple model and represent the dynamics of the universe by one single scale function $a(t)$ (not to use more general inhomogeneous cosmological model), we should put a new correlation term $C_{\mu\nu}$ into the equations .

$$E_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi\langle T_{\mu\nu}\rangle + C_{\mu\nu}, \tag{1}$$

which is defined by the construction

$$C_{\mu\nu} = E_{\mu\nu}(\langle g_{\mu\nu}\rangle) - \langle E_{\mu\nu}(g_{\mu\nu})\rangle . \tag{2}$$

It does not necessarily obey the usual energy condition and it can act as dark energy [2]. Averaging can be considered over some spacelike hypersurface, which depends on the selected slicing or over some spacetime interval, which can be covariantely defined. There are two main goals concerning averaging - the first is to construct averaged metric and the second is to obtain correlation term modifying Einstein equations.

There is a technical problem in a definition of an averaged tensor: Integrating a tensor field in curved spacetime does not result in a new tensor field (this is because of the addition of the tensors living in different spaces).

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CARTAN SCALARS

It can be shown that (pseudo)Riemannian geometry can be completely characterized by the Riemann tensor and the finite number of its covariant derivatives. We will start with the construction of the Cartan scalars (for the texts concerning equivalence problem see e.g. [4], [5]). Let $(\mathscr{M}, g)$ be n-dimensional differentiable manifold with a metric $\mathbf{g} = \eta_{ij}\omega^i \otimes \omega^j$, η_{ij} is constant symmetric matrix and ω^i, i=1,2...,n form the base of the cotangent space at the point x^μ. Now all the geometrical objects will be defined on the enlarged $\frac{1}{2}n(n+1)$ dimensional space $F(\mathscr{M})$ - the frame bundle of $\mathscr{M}$. Exterior derivative will be extended to $d = d_x + d_\xi$ and the Cartan equations has the form

$$d\omega^i = \omega^j \wedge \omega^i{}_j, \tag{3}$$

$$d\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l. \tag{4}$$

with a condition

$$\eta_{ik}\omega^k{}_j + \eta_{jk}\omega^k{}_i = 0. \tag{5}$$

Applying next exterior derivative we will obtain covariant derivatives of the curvature tensor.

$$\begin{aligned} \mathbf{d}R_{ijkl} &= R_{mjkl}\omega^m_i + R_{imkl}\omega^m_j + R_{ijml}\omega^m_k + R_{ijkm}\omega^m_l + R_{ijkl;m}\omega^m, \\ \mathbf{d}R_{ijkl;n} &= R_{mjkl;n}\omega^m_i + R_{imkl;n}\omega^m_j + \ldots + R_{ijkl;nm}\omega^m, \ldots \end{aligned} \tag{6}$$

Let R^p denotes the set $\{R_{ijkm}, R_{ijkm;n_1}, \ldots, R_{ijkm;n_1\ldots n_p}\}$, p is the lowest number such that R^{p+1} contains no element that is functionally independent (over $F(\mathscr{M})$) of the elements in R^p (two functions are functionally independent iff 1-forms $\mathbf{d}f$ and $\mathbf{d}g$ are linearly independent).

There exist quite elaborate algorithm [5] how to compute Cartan scalars. It uses the structure of isotropy group of R^q and in every step it restricts the frame requiring that R^q takes a standard form.

We have a set of the scalar functions R^{p+1} (Cartan scalars) that completely characterizes geometry. Now we can perform averaging. It can be shown that the Cartan scalars satisfy some algebraic and differential relations which are in general nonlinear [1]. It means that we have to restrict to the smaller set of the scalars, perform averaging and then construct a new set $\overline{R}^{p+1}$.

Using Cartan scalars we can average not only the geometry of spacetime but we can also obtain the correlation term in the averaged Einstein equations (left hand side consists of the finite sum of the Riemann tensor). As we will see in the next section, computation can be easily performed in the spinor formalism - we can read the form of the correction from the type of the averaged Ricci spinor (or Λ term).

FRW METRIC

Now we would like to compute zero order Cartan scalars of the FRW spacetimes. For simplicity we will show the result for the flat case.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \tag{7}$$

$$\Phi_{00} = \Phi_{22} = -\frac{1}{2}\frac{a(t)\left(\frac{d^2}{dt^2}a(t)\right) - \left(\frac{d}{dt}a(t)\right)^2}{a(t)^2}, \tag{8}$$

$$\Phi_{11} = -\frac{1}{4}\frac{a(t)\left(\frac{d^2}{dt^2}a(t)\right) - \left(\frac{d}{dt}a(t)\right)^2}{a(t)^2}, \tag{9}$$

$$\Lambda = \frac{1}{4}\frac{\left(\frac{d}{dt}a(t)\right)^2 + a(t)\left(\frac{d^2}{dt^2}a(t)\right)}{a(t)^2}. \tag{10}$$

Now if we have an inhomogeneous model, we can compare averaged Cartan scalars with the FRW case. By comparing two different sets of scalars, we can see under which conditions we can obtain effective FRW metric by averaging.

LTB METRIC

We will use the model of the inhomogeneous universe described by LTB metric. This is the exact spherically symmetric solution of the Einstein equations.

$$ds^2 = -dt^2 + \frac{R'^2(r,t)}{1+2E(r)}dr^2 + R^2(r,t)\,d\Omega^2. \tag{11}$$

Prime denotes partial derivative with respect to r. In the following we will set curvature function to zero $E(r) = 0$.

We will use WKB ansatz $R(r,t) = A(r,t)\exp\psi(r,t)$, where $\psi(r,t)$ is the highly fluctuating function. It can be shown that all zero-order Cartan scalar are in the first approximation equal to zero except

$$\Lambda = \frac{1}{2}\psi_{,t}^2. \tag{12}$$

CONCLUSION

We have presented a new method how to deal with the averaging in GR. It uses the theory of the Cartan scalars originally taken from the equivalence problem. In one example we have shown that specific type of the averaged LTB spacetime behaves like the effective FRW universe with a positive cosmological constant.

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