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# The global existence, uniqueness and $C^1$ -regularity of geodesics in nonexpanding impulsive gravitational waves

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## Abstract

We study geodesics in the complete family of nonexpanding impulsive gravitational waves propagating in spaces of constant curvature, that is Minkowski, de Sitter and anti-de Sitter universes. Employing the continuous form of the metric we prove the existence and uniqueness of continuously differentiable geodesics (in the sense of Filippov) and use a  $C^1$ -matching procedure to explicitly derive their form.

Keywords: impulsive gravitational waves, geodesics, low regularity

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## 1. Introduction

Impulsive gravitational waves describe short but intense bursts of gravitational radiation and have become physically interesting models of exact radiative spacetimes in Einstein's theory. At the same time they are spacetimes of low regularity: while one prominent form of the metric is only continuous (actually locally Lipschitz continuous—a fact that will turn out to be essential in our analysis) another one is even distributional. Hence these geometries are also mathematically interesting in the context of non-smooth Lorentzian geometry, where they can serve as relevant test models.

In this work we treat the entire class of nonexpanding impulsive waves propagating on spaces of constant curvature—Minkowski space, de Sitter and anti-de Sitter universes (with vanishing, positive and negative cosmological constant  $\Lambda$ , respectively). We focus on particle motion using the continuous form of the metric. In particular, we prove that the geodesics are

continuously differentiable curves and then we apply a  $C^1$ -matching procedure to compute them explicitly.

Specifically, we begin in section 2 with a review of the class of nonexpanding impulsive gravitational waves in spaces of constant curvature, including various methods of their construction. In section 3 we focus on the main topic of this work: the geodesic equation in the continuous form of the metric. After reviewing in sections 3.1 and 3.2 what has been done so far we derive in section 3.3 our key mathematical result, a general existence and uniqueness theorem for the geodesic equation in a class of locally Lipschitz continuous spacetimes. For this purpose we use *Filippov's solution concept* for differential equations with discontinuous right-hand side whose basics are collected in the appendix. After explicitly deriving the geodesic equations in section 3.4, in sections 3.5 and 3.6 we apply these mathematical findings to establish our main result (for  $\Lambda = 0$  and  $\Lambda \neq 0$ , respectively), which says that the entire class of nonexpanding impulsive waves has unique, continuously differentiable global geodesics. In section 4 we explicitly derive the geodesics for this class of spacetimes using a  $C^1$ -matching procedure.

## 2. Impulsive waves and methods of their construction

From the physical point of view, an impulsive gravitational wave can most naturally be understood as a limit of a suitable family of gravitational waves with sandwich profiles of ever 'shorter duration'  $\varepsilon$  which simultaneously become 'stronger' as  $\varepsilon^{-1}$ . Mathematically, this amounts to a distributional limit in which a sequence of sandwich profiles converges to the profile  $\delta(u)$ , the Dirac function. An impulsive gravitational wave is thus localized on a *single wave-front*  $u = 0$ , which is a null hypersurface. Across  $u = 0$  the first derivative of the metric with respect to  $u$  is discontinuous, introducing a Dirac delta in the curvature tensor representing the gravitational impulse.

Interestingly, there exist several alternative methods of construction of such exact non-expanding solutions to Einstein's vacuum field equations. They will now be summarized and compared, together with the appropriate references to original works.

### 2.1. The 'cut and paste' method

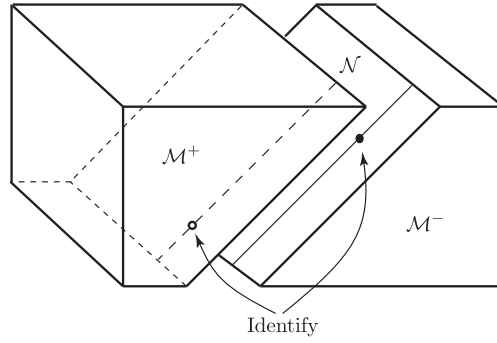
Let us start with an elegant geometrical method for constructing impulsive plane gravitational waves in a Minkowski background presented by Penrose in now classic works [1–3]. His 'cut and paste' approach<sup>3</sup> is based on the removal of the null hypersurface  $\mathcal{N}$  given by  $\mathcal{U} = 0$  from the spacetime

$$ds_0^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V}}{\left[ 1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right]^2}, \quad (2.1)$$

(with  $\Lambda = 0$ ), and re-attaching the 'halves'  $\mathcal{M}^-(\mathcal{U} < 0)$  and  $\mathcal{M}^+(\mathcal{U} > 0)$  by making an identification with a 'warp' in the coordinate  $\mathcal{V}$  such that (see figure 1)

$$\left[ \eta, \bar{\eta}, \mathcal{V}, \mathcal{U} = 0_- \right]_{\mathcal{M}^-} \equiv \left[ \eta, \bar{\eta}, \mathcal{V} - H(\eta, \bar{\eta}), \mathcal{U} = 0_+ \right]_{\mathcal{M}^+}, \quad (2.2)$$

<sup>3</sup> Penrose's approach is in some respects similar to that of Israel [4, 5] and closely related to the Dray and 't Hooft method of 'shift function' [6].



**Figure 1.** Minkowski space is cut into two parts  $\mathcal{M}^-$  and  $\mathcal{M}^+$  along the null hyperplane  $\mathcal{N}$ . The parts are then re-attached with a ‘warp’ in which points are shunted along the null generators of the cut and then identified. The same construction in de Sitter or anti-de Sitter space generates impulsive gravitational waves in these backgrounds.

where  $H(\eta, \bar{\eta})$  is an *arbitrary* real-valued function of  $\eta$  and  $\bar{\eta}$ . It was shown in [3] that impulsive components are introduced into the curvature tensor proportional to  $\delta(\mathcal{U})$  representing gravitational (plus possibly null-matter) impulsive waves.

In [3] Penrose considered only a Minkowski background, in which case the impulsive surface  $\mathcal{U} = 0$  is obviously a plane, and impulsive  $pp$ -waves are obtained. In [7] it was demonstrated that exactly the same junction conditions (2.2) applied to a general background spacetime (2.1) of constant curvature also introduce impulsive waves in the de Sitter ( $\Lambda > 0$ ) or anti-de Sitter ( $\Lambda < 0$ ) universes. However, the geometries of these impulses are different since the null hypersurface  $\mathcal{U} = 0$ , along which the spacetime is cut and pasted, is described by the 2-metric  $d\sigma^2 = 2(1 + \frac{1}{6}\Lambda\eta\bar{\eta})^{-2} d\eta d\bar{\eta}$ . This is a two-dimensional space of constant Gaussian curvature  $K = \frac{1}{3}\Lambda$ , which is a plane for  $\Lambda = 0$ . In the  $\Lambda \neq 0$  cases it is either a *sphere* ( $\Lambda > 0$ ) or a *hyperboloid* ( $\Lambda < 0$ ). These geometries have been described in detail in [8] using various coordinate representations, and it was explicitly demonstrated that the wave surfaces are nonexpanding.

## 2.2. The continuous form of the metric

While the ‘cut and paste’ method describes the identification of points on both sides of the impulse it does not provide explicit metric forms of the complete spacetimes. Thus the next step is to find a suitable coordinate system in which the metric is continuous as a function of  $\mathcal{U}$ .

Starting again with the metric (2.1) we perform the transformation

$$\mathcal{U} = U, \quad \mathcal{V} = V + H + UH_{,Z}H_{,\bar{Z}}, \quad \eta = Z + UH_{,\bar{Z}}, \quad (2.3)$$

where  $H(Z, \bar{Z})$  is an arbitrary real-valued function. This yields the metric

$$ds_0^2 = \frac{2 \left| dZ + U(H_{,ZZ}dZ + H_{,\bar{Z}\bar{Z}}d\bar{Z}) \right|^2 - 2 dUdV}{\left[ 1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - UG) \right]^2}, \quad (2.4)$$

where  $G(Z, \bar{Z}) \equiv H - ZH_{,Z} - \bar{Z}H_{,\bar{Z}}$ . We now consider (2.4) for  $U > 0$  and combine this with the line element (2.1) in which we set  $\mathcal{U} = U$ ,  $\mathcal{V} = V$ ,  $\eta = Z$  for  $U < 0$ . The resulting metric can be written as

$$ds^2 = \frac{2 \left| dZ + U_+ (H_{,ZZ} dZ + H_{,ZZ} d\bar{Z}) \right|^2 - 2 dU dV}{\left[ 1 + \frac{1}{6} \Lambda (Z\bar{Z} - UV - U_+ G) \right]^2}, \quad (2.5)$$

where

$$U_+ \equiv U_+(U) = \begin{cases} 0 & \text{if } U \leq 0, \\ U & \text{if } U \geq 0 \end{cases} \quad (2.6)$$

is the *kink-function*. Since the kink function is Lipschitz continuous the metric (2.5) is locally Lipschitz in the variable  $U$ . Thus, apart from possible singularities of the function  $H$  and its derivatives (which indeed occur in physically realistic models, see e.g. section 2.4, below), the spacetime is locally Lipschitz. Observe that any locally Lipschitz metric  $g$  (denoted by  $g \in C^{0,1}$ ) possesses a locally bounded connection and so the curvature is a distribution. Also we are well within the ‘maximal’ distributional curvature framework as identified by Geroch and Traschen [9]. For locally Lipschitz metrics there is no bound on the curvature (in  $L^\infty$ ). Indeed, the discontinuity in the derivatives of the metric introduces impulsive components in the Weyl and curvature tensors [7], namely  $\Psi_4 = (1 + \frac{1}{6} \Lambda Z\bar{Z})^2 H_{,ZZ} \delta(U)$  and  $\Phi_{22} = [(1 + \frac{1}{6} \Lambda Z\bar{Z})^3 ((1 + \frac{1}{6} \Lambda Z\bar{Z})^{-1} H)_{,ZZ} + \frac{1}{3} \Lambda H] \delta(U)$ , in a natural tetrad. The metric (2.5) thus explicitly describes impulsive waves in de Sitter, anti-de Sitter or Minkowski backgrounds. For  $\Lambda = 0$ , the conformal factor is one, and the line element (2.5) reduces to the Rosen form of impulsive  $pp$ -waves [3, 10, 11].

The transformations relating (2.1) and (2.5) separately for  $U < 0$  and  $U > 0$  can be written in a combined way using the *Heaviside function*  $\Theta = \Theta(U)$  as

$$\mathcal{U} = U, \quad \mathcal{V} = V + \Theta H + U_+ H_{,Z} H_{,\bar{Z}}, \quad \eta = Z + U_+ H_{,\bar{Z}}, \quad (2.7)$$

which is discontinuous in the coordinate  $\mathcal{V}$  on  $\{\mathcal{U} = 0\}$ . From (2.7) we, in particular, obtain the Penrose junction condition (2.2) for reattaching  $\mathcal{M}^-$  and  $\mathcal{M}^+$  with a warp. Thus, the above procedure is indeed an *explicit* Penrose’s ‘cut and paste’ construction of all nonexpanding impulsive gravitational waves.

### 2.3. The distributional form of the metric

The most intuitive way of constructing impulsive waves is the *distributional limit* of suitable families of gravitational waves with smooth sandwich profiles alluded to above. For vacuum  $pp$ -waves, such a procedure was considered in [1, 2, 12] and later elsewhere (e.g. [13]). In this simplest case one obtains the metric

$$ds^2 = 2 d\xi d\bar{\xi} - 2 du dv + \mathcal{H}(\xi, \bar{\xi}) \delta(u) du^2, \quad (2.8)$$

which is the well-known Brinkmann form of impulsive  $pp$ -waves in a Minkowski background [1–3].

More general impulsive waves within the Kundt class (see [14], [15, chapter 31], [16, chapter 18]) of nonexpanding spacetimes with  $\Lambda \neq 0$  can be obtained similarly. It was demonstrated in [17] that *all* nonexpanding impulses in Minkowski or (anti-)de Sitter universes can be constructed from the general class of type N Kundt solutions [18, 19], upon considering the distributional limit  $\mathcal{H}(\xi, \bar{\xi}) \delta(u)$  of the structural function, i.e.,

$$ds^2 = \frac{2}{P^2} d\xi d\bar{\xi} - 2 \frac{Q^2}{P^2} du dv + \left[ 2k \frac{Q^2}{P^2} v^2 - \frac{(Q^2)_{,u}}{P^2} v + \frac{Q}{P} \mathcal{H}(\xi, \bar{\xi}) \delta(u) \right] du^2, \quad (2.9)$$

where  $P = 1 + \frac{1}{6}\Lambda\xi\bar{\xi}$ ,  $Q = (1 - \frac{1}{6}\Lambda\xi\bar{\xi})a + \bar{b}\xi + b\bar{\xi}$ , and  $k = \frac{1}{6}\Lambda a^2 + b\bar{b}$ . The metric (2.9) contains a single  $\delta(u)$  and obviously generalizes the Brinkmann form of impulsive  $pp$ -waves (2.8) to which it reduces when  $P = 1 = Q$  for  $\Lambda = 0$ .

Of course, a distributional term in the metric leads us out of the Geroch–Traschen class [9] of metrics, for which the metric is of regularity  $W_{\text{loc}}^{2,1} \cap L_{\text{loc}}^\infty$ , which guarantees the curvature to exist in distributions. However, due to its simple geometrical structure the metric (2.9) nevertheless allows one to calculate the curvature as a distribution.

For a *generic* function  $\mathcal{H}(\xi, \bar{\xi}, u)$  there exist distinct canonical subclasses of type N Kundt solutions characterized by specific choices of the parameters  $a$  and  $b$ , see [16, 18, 19]. Surprisingly, it was proven in [17] that *impulsive* limits  $\mathcal{H}(\xi, \bar{\xi})\delta(u)$  of these subclasses for a given  $\Lambda$  become (locally) *equivalent*. For example, in the case of  $\Lambda = 0$  there are two subclasses, namely the  $pp$ -waves and the Kundt waves KN. However, the transformation

$$\mathcal{U} = (\xi + \bar{\xi})(1 + uv)u, \quad \mathcal{V} = (\xi + \bar{\xi})v - 1, \quad \eta = \xi + (\xi + \bar{\xi})uv, \quad (2.10)$$

converts the impulsive KN metric (2.9) with  $a = 0$ ,  $b = 1$  to the impulsive  $pp$ -wave metric with  $a = 1$ ,  $b = 0$ . The only non-trivial impulsive gravitational waves of the form (2.9) in Minkowski space are thus the impulsive  $pp$ -waves (2.8). Similar results hold also for  $\Lambda > 0$  and  $\Lambda < 0$ . Generically, there are three distinct subclasses of nonexpanding waves, namely  $\text{KN}(\Lambda)\text{I}$  given by  $a = 0$ ,  $b = 1$ ,  $\text{KN}(\Lambda^-)\text{II}$  given by  $a = 1$ ,  $b = 0$ ,  $\Lambda < 0$ , and generalized Siklos waves [20, 21]  $\text{KN}(\Lambda^-)\text{III}$  for which  $k = 0$  and  $\Lambda < 0$ , see [16]. In all these cases it was shown in [17] that although these canonical subclasses are different for extended profiles, they are equivalent for impulsive profiles. In this way, by considering the above distributional limit (2.9) of the class we obtain an explicit form of *all* solutions representing nonexpanding impulses.

Interestingly, there exists yet another metric form of representing this complete family of impulsive solutions. It is obtained from the continuous form of the impulsive wave metric (2.5) by applying the transformation (2.7) not separately for  $U < 0$  and  $U > 0$  but (formally) for all values of  $U$  including on the impulse. Explicitly, if we keep the terms arising from the derivatives of  $\Theta$ , this transformation relates (2.5) to

$$ds^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V} + 2H(\eta, \bar{\eta})\delta(\mathcal{U})d\mathcal{U}^2}{\left[ 1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right]^2}. \quad (2.11)$$

Observing also from (2.7) that  $\eta = Z$  if  $U = 0 = \mathcal{U}$ , the function  $H(Z, \bar{Z})$  of (2.5) agrees with  $H(\eta, \bar{\eta})$  of (2.11) on the wave surface. Again, in a Minkowski background this is just the Brinkmann form of a general impulsive  $pp$ -wave (2.8)<sup>4</sup>.

<sup>4</sup> In fact, the metric (2.11) is conformal to (2.8). Recall in this context that Siklos [20] proved that Einstein spaces conformal to  $pp$ -waves only occur when  $\Lambda < 0$ . The *impulsive* case shows that this result does not hold in low regularity.

The explicit transformation relating (2.9) to (2.11) for the KN( $\Lambda$ )I subclass is

$$\begin{aligned} \mathcal{U} &= \frac{(\xi + \bar{\xi})(1 + uv)u}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u}, \\ \mathcal{V} &= \frac{(\xi + \bar{\xi})v + \frac{1}{6}\Lambda\xi\bar{\xi}}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u} - 1, \\ \eta &= \frac{\xi + (\xi + \bar{\xi})uv}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u}, \end{aligned}$$

which reduces to (2.10) in the case  $\Lambda = 0$ . Similar transformations exist also for the subclasses KN( $\Lambda^-$ )II and KN( $\Lambda^-$ )III, see [17]. Therefore, the full family of impulsive limits (2.9) of nonexpanding sandwich waves of the Kundt class is indeed equivalent to the distributional form of the solutions (2.11), and consequently to the continuous metric (2.5) obtained by the ‘cut and paste’ method.

Of course, the discontinuity in the complete transformation (2.7) is mathematically delicate. However, in the special case of impulsive  $pp$ -waves it was rigorously analyzed using a general regularization procedure in [11]. Indeed it was shown within the geometric theory of nonlinear generalized functions [22] (based on Colombeau algebras [23]) that (2.7) is a (generalized) coordinate transformation, a result which puts the formal (‘physical’) equivalence of both forms of impulsive spacetimes on a solid ground<sup>5</sup>.

#### 2.4. Boosting static sources

As demonstrated in 1971 by Aichelburg and Sexl in a classic paper [26], a specific impulsive gravitational  $pp$ -wave solution (in distributional form) can be obtained by boosting the Schwarzschild black hole to the speed of light, while its mass is scaled to zero in an appropriate way. Such a solution represents an axially symmetric impulsive gravitational wave in Minkowski space generated by a single *null monopole* particle moving along the axis. Note that the continuous coordinate system for the Aichelburg–Sexl solution was found by D’Eath [27] and used for investigation of ultrarelativistic black-hole encounters.

Using a similar approach, numbers of other specific impulsive waves in flat space have been obtained by boosting more general black hole spacetimes [28–32], multipole sources [33] or black rings [34, 35]. This method has been generalized to the  $\Lambda \neq 0$ -cases by Hotta and Tanaka [36], who boosted the Schwarzschild–de Sitter solution to obtain a nonexpanding spherical impulsive gravitational wave generated by a pair of null monopole particles in the de Sitter background. They also described an analogous solution in the anti-de Sitter universe. Their main ‘trick’ was to consider the boost in the five-dimensional representation of the (anti-)de Sitter spacetime (see also section 2.5, below), where the boost can explicitly (and consistently) be performed.

Details on boosting monopole particles to the speed of light in the (anti-)de Sitter universe, the geometry of the nonexpanding wave surfaces, and discussion of various useful coordinates can be found in [8]. It was also shown that although the impulsive wave surface is nonexpanding, for  $\Lambda > 0$  this coincides with the horizon of the closed de Sitter universe. The background space contracts to a minimum size and then re-expands in such a way that the

<sup>5</sup> Interestingly, these studies have triggered a corresponding line of research in generalized functions [24, 25].

impulse in fact propagates with the speed of light from the ‘north pole’ of the universe across the equator to its ‘south pole’.

There are also particular impulsive waves generated by *null multipole* particles obtained by boosting static multipole sources [33]. Such solutions with  $\Lambda = 0$  can be written in the form (2.8) with

$$\mathcal{H} = -b_0 \log \rho + \sum_{m=1}^{\infty} b_m \rho^{-m} \cos \left[ m(\phi - \phi_m) \right], \quad (2.12)$$

where  $\xi = \frac{1}{\sqrt{2}} \rho e^{i\phi}$  and  $b_m, \phi_m$  are constants. The term given by  $b_0 \log \rho$  represents the Aichelburg–Sexl solution [26] for a single null monopole particle. The terms with  $m \geq 1$  correspond to the multipole components of an impulsive  $pp$ -wave generated by a source of an arbitrary multipole structure [37]. Indeed, the field equations relate them to a source localized at  $\rho = 0$  on the impulsive wavefront  $u = 0$ , which is described by  $T_{uu} = J(\rho, \phi) \delta(u)$  with

$$J(\rho, \phi) = \frac{1}{4} b_0 \delta(\rho) + \sum_{m=1}^{\infty} \frac{1}{4} b_m \frac{(-1)^m}{(m-1)!} \delta^{(m)}(\rho) \cos \left[ m(\phi - \phi_m) \right]. \quad (2.13)$$

Observe that any function  $\mathcal{H}$  of the form (2.12) is singular on the axis  $\rho = 0$  leading to curvature singularities in the spacetime at  $\rho = 0, u = 0$ .

Interestingly, as demonstrated in [38], there are analogous impulsive solutions also in the case  $\Lambda \neq 0$ . For their description it is, however, more convenient to use the formalism based on embedding the (anti-)de Sitter universe into the five-dimensional Minkowski space, as we will detail next.

### 2.5. Embedding to five dimensions

The full class of nonexpanding impulsive waves in spaces of constant curvature with  $\Lambda \neq 0$  can be obtained in a five-dimensional formalism as metrics

$$ds^2 = dZ_2^2 + dZ_3^2 + \epsilon dZ_4^2 - 2d\tilde{U}d\tilde{V} + \mathcal{H}(Z_2, Z_3, Z_4) \delta(\tilde{U}) d\tilde{U}^2, \quad (2.14)$$

with the constraint  $Z_2^2 + Z_3^2 + \epsilon Z_4^2 - 2\tilde{U}\tilde{V} = \epsilon a^2$ , where  $\tilde{U} = \frac{1}{\sqrt{2}}(Z_0 + Z_1)$ ,  $\tilde{V} = \frac{1}{\sqrt{2}}(Z_0 - Z_1)$ ,  $a = \sqrt{3/|\Lambda|}$  and  $\epsilon = \text{sign } \Lambda$ . As shown in [38] this metric represents impulsive waves propagating in the (anti-)de Sitter universe with the impulse located on the null hypersurface  $\tilde{U} = 0$ , i.e.,

$$Z_2^2 + Z_3^2 + \epsilon Z_4^2 = \epsilon a^2, \quad (2.15)$$

which is a *nonexpanding* two-sphere in the de Sitter universe and a hyperboloidal two-surface in the anti-de Sitter universe, respectively. Various four-dimensional coordinate parametrizations of (2.14) can be considered. For example,

$$\tilde{U} = \frac{\mathcal{U}}{\Omega}, \quad \tilde{V} = \frac{\mathcal{V}}{\Omega}, \quad Z_2 + iZ_3 = \frac{x}{\Omega} + i\frac{y}{\Omega} = \frac{\sqrt{2}\eta}{\Omega}, \quad Z_4 = a \left( \frac{2}{\Omega} - 1 \right), \quad (2.16)$$

where  $\Omega = 1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) = 1 + \frac{1}{12}\Lambda(x^2 + y^2 - 2\mathcal{U}\mathcal{V})$ , brings the metric to the previous form (2.11) with the function

$$\mathcal{H} = 2H / \left( 1 + \frac{1}{6}\Lambda\eta\bar{\eta} \right). \quad (2.17)$$

Other natural coordinates which parametrize (2.14) have been discussed in [8].

The metric (2.14) may describe impulsive gravitational waves and/or impulses of null matter. Purely gravitational waves occur when the vacuum field equation



$$\left(\Delta + \frac{2}{3}\Lambda\right)\mathcal{H} = 0 \tag{2.18}$$

is satisfied [7, 39, 40], where  $\Delta \equiv \frac{1}{3}\Lambda \{ \partial_z [(1 - z^2)\partial_z] + (1 - z^2)^{-1} \partial_\phi \partial_\phi \}$  is the Laplacian on the impulsive surface (2.15), parametrized by  $Z_2 = a\sqrt{\epsilon(1 - z^2)} \cos \phi$ ,  $Z_3 = a\sqrt{\epsilon(1 - z^2)} \sin \phi$ , and  $Z_4 = az$ . It was demonstrated in [38] that non-trivial solutions of (2.18) can be written as

$$\mathcal{H}(z, \phi) = b_0 Q_1(z) + \sum_{m=1}^{\infty} b_m Q_1^m(z) \cos [m(\phi - \phi_m)], \tag{2.19}$$

where  $Q_1^m(z)$  are associated Legendre functions of the second kind generated by the relation  $Q_1^m(z) = (-\epsilon)^m |1 - z^2|^{m/2} \frac{d^m}{dz^m} Q_1(z)$ . The first term for  $m = 0$ , i.e.,  $Q_1(z) = \frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1$ , represents the simplest axisymmetric Hotta–Tanaka solution [36]. The components with  $m \geq 1$  describe nonexpanding impulsive gravitational waves in the (anti-)de Sitter universe generated by null point sources with an  $m$ -pole structure, localized on the wave-front at the singularities  $z = \pm 1$ .

### 2.6. Summary of the construction methods

To end this review we collect the various methods of constructing nonexpanding impulsive waves in spaces of constant curvature and the corresponding references in the following table. For more details see, e.g., [41, 42] and [16, chapter 20].

Method of construction	$\Lambda = 0$	$\Lambda \neq 0$
‘Cut and paste’	[1], [2], [3], [6]	[7], [40]
Continuous coordinates	[3], [27], [10], [11]	[7], [17]
Limits of sandwich waves	[1], [2], [12], [13], [46], [47], [17]	[17], [38]
Boosts	[26], [28]–[35], [37]	[36], [8], [33]
Embedding	—	[36], [8], [38], [39], [7]

## 3. Geodesics in nonexpanding impulsive waves

In this section we shift our focus on to the main theme of this work, i.e., the analysis of the geodesic equation in spacetimes with nonexpanding impulsive waves. While we will mainly be concerned with the continuous form of the metric (2.5) we start by reviewing some results obtained using the distributional form of the metric (2.11).

### 3.1. Distributional form

Geodesics in Minkowski space with impulsive  $pp$ -waves were discussed in many works, e.g. in [6, 40, 43–45], all deriving that they are refracted straight lines with a jump in the  $\mathcal{V}$ -direction. However, the corresponding geodesic (and also the geodesic deviation) equations in standard coordinates (2.8) cannot be formulated consistently in distributions, since they contain ill-defined products. These equations have been rigorously analyzed in [46, 47] using the geometric theory of nonlinear generalized functions [22]. In particular, existence and uniqueness results have been obtained in a space of generalized functions and the geodesics have been shown again to be broken straight lines. The benefit of the rigorous approach is the

following: there it is *proven* that the geodesics really cross the impulsive wave rather than being reflected or trapped, a possibility which is ruled out *a priori* by the approaches using multiplication rules and other tricks from the gray area of (linear) distribution theory. As a consequence, in the rigorous approach *geodesic completeness* of impulsive *pp*-waves is proven—a result which has been recently generalized to models allowing for a non-flat wave surface [48, 49].

In [50] the geodesics in nonexpanding impulsive waves in all constant curvature backgrounds with any  $\Lambda$  have been studied. They have been derived using the embedding of (anti-) de Sitter spacetime into five-dimensional Minkowski space as detailed in section 2.5. The advantage of this approach lies in the fact that it yields a system of differential equations which is distributionally accessible at all, if not rigorously. In particular, there is no square of  $\delta$ , contrary to other ‘direct’ approaches, such as those of [40] which used the coordinates introduced in [6]. The general results of [50] in the special case  $\Lambda = 0$  reduce to those *rigorously* derived in [47]. Nevertheless, a desirable nonlinear distributional analysis of the geodesic equation in the  $\Lambda \neq 0$ -cases is still subject to ongoing research.

### 3.2. Continuous form

Recently the continuous form of the metric (2.5) has been employed in [51] to derive the geodesics in impulsive *pp*-waves. Here the geodesic equation, which has a discontinuous right-hand side, has been *uniquely solved in the sense of Carathéodory*. In this way it has, in particular, been shown that the spacetimes are geodesically complete with the geodesics being *continuously differentiable* curves. This justifies the  $C^1$ -matching of the geodesics of the background to obtain the geodesics of the entire spacetime. In fact such an approach has been used in [45, 52], as well as in [7, 50] for nonvanishing  $\Lambda$ , and in [53, 54] for expanding impulsive waves.

The analysis of [51] is based on the fact that for *pp*-waves the coordinate  $U$  can be used as a parameter along the geodesics. However, this is no longer possible if  $\Lambda \neq 0$ , where the geodesic equations take the form of an autonomous system of ODEs with discontinuous right-hand side. In this case Carathéodory’s concept provides no advantage over the classical theory and is not applicable to the equations at hand.

On the other hand it was also recently shown in [55] that the geodesic equations for any locally Lipschitz continuous semi-Riemannian metric possess solutions *in the sense of Filippov* and that these geodesics in addition are continuously differentiable. Filippov’s solution concept [56] is a general and nowadays widely applied approach (e.g. in non-smooth mechanics, see [57]) and, for the convenience of the reader, we have collected its basics used in this work in the [appendix](#). Since we will make use of this result in the following, we give its precise formulation:

**Theorem 3.1.** (*[55, theorem 2]*) *Let  $(M, g)$  be a smooth manifold with a  $C^{0,1}$ -semi-Riemannian metric  $g$ . Then there exist Filippov solutions of the geodesic equations which are  $C^1$ -curves.*

However, in general we cannot expect the geodesic equation in locally Lipschitz spacetimes to be *uniquely* solvable. The threshold for unique solvability of the geodesic equations is the regularity class  $C^{1,1}$ , i.e., the first derivatives of the metric being locally Lipschitz. In fact, in this class classical ODE-theory provides unique solvability of the geodesic equation with the geodesics being  $C^2$ . Moreover, as has been recently shown [58, 59], the exponential map retains maximal regularity. On the other hand there exist

metrics in any Hölder class  $C^{1,\alpha}$ , with  $\alpha < 1$ , for which the initial value problem for the geodesic equation fails to be uniquely solvable. Here we recall the following classical example due to [60] for a  $C^1$  (hence, in particular, locally Lipschitz) Riemannian metric. Consider the line element  $ds^2 = h(y)(dx^2 + dy^2)$  on  $\mathbb{R}^2$  with the function  $h(y) \equiv 1 + y^{\frac{4}{3}}$ . The corresponding geodesic equations then reduce to

$$2 \frac{d^2 y}{dx^2} = \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) \frac{d}{dy} \log(h(y)).$$

Thus  $y(x) = 0$  and  $y(x) = x^3$  are two distinct solutions starting at 0 with initial velocity 0.

For our purpose it is worth while to observe that these geodesics are classical solutions ( $C^2$  and they satisfy the differential equation everywhere) so they are also Filippov solutions. Consequently in this case we have non-unique Filippov solutions.

However, in the case we are interested in, that is the continuous form of the metric for nonexpanding impulsive waves in all constant curvature backgrounds (2.5), the metric *in addition* to being locally Lipschitz is also *smooth off a null hypersurface*. In particular, it is piecewise smooth and in such a case uniqueness of the geodesics can indeed be established (as we will see next), thereby justifying the  $C^1$ -matching procedure.

### 3.3. Unique $C^1$ -geodesics for a class of locally Lipschitz metrics

To formulate the result announced above we need some preparations. We consider a spacetime  $(M, g)$  with *locally Lipschitz continuous* metric  $g$  and global null coordinates  $U, V$  such that  $M$  is separated into two parts by a *totally geodesic* null hypersurface  $N = \{U = 0\}$ . By  $N$  being totally geodesic we mean that every geodesic (in the sense of Filippov) starting in  $N$  and being initially tangential to  $N$  stays (initially) in  $N$ .

**Remark 3.2.** Observe that the notion of a totally geodesic (null) hypersurface  $N$  of a spacetime  $(M, g)$  with  $g \in C^{0,1}$  is somewhat subtle. Indeed, if geodesics are not unique, then from the classically equivalent characterizations

- (1) the second fundamental form of  $N$  vanishes (in  $L_{loc}^\infty$ ),
  - (2) every geodesic in  $N$  is a geodesic in  $M$ ,
  - (3) every geodesic starting in  $N$  tangential to  $N$  stays (initially) in  $N$ ,
- condition (3) (i.e., our definition) becomes stronger while it implies the still equivalent conditions (1) and (2). Note, however, that in the spacetimes we are dealing with we prove uniqueness of geodesics and so (1)–(3) become again equivalent.

We want to rewrite the geodesic equation in *first order form*. To this end we denote by  $X, Y$  some local spatial coordinates and introduce

$$\mathbf{W} \equiv (U, \hat{U}, V, \hat{V}, X, \hat{X}, Y, \hat{Y}) \equiv (U, \dot{U}, V, \dot{V}, X, \dot{X}, Y, \dot{Y}). \quad (3.1)$$

We assume that the geodesic equation for  $(M, g)$ , as a first order system, has the form

$$\dot{W}^a = \frac{E^a(\mathbf{W})}{P^a(U, V, X, Y)} \Theta(U) + \frac{F^a(\mathbf{W})}{Q^a(U, V, X, Y)} \equiv T^a(\mathbf{W}), \quad (3.2)$$

for  $a = 1, \dots, 8$  labeling the components  $W^a$  of  $\mathbf{W}$  in (3.1), and analogously for  $\mathbf{E}, \mathbf{F}, \mathbf{P}, \mathbf{Q}, \mathbf{T}$ . Here we further assume that  $\mathbf{E}, \mathbf{F}$  are *smooth*, and that  $\mathbf{P}, \mathbf{Q}$  are *smooth* in  $V, X, Y$  but are *polynomial* in  $U_+$  (of degree at most five), locally bounded away from zero, hence  $\mathbf{T} \in L_{loc}^\infty$ .

Note that  $U_+$  is Lipschitz continuous and  $\Theta$  is discontinuous at  $U = 0$ , so by restricting ourselves to the form (3.2) of the geodesic equation we allow only for non-smoothness in  $U$  at  $U = 0$  given by a jump ( $\Theta$ ) and possibly by (higher order) kinks ( $U_+$ ,  $U_+^2$ , etc). Now we may state and prove the following:

**Theorem 3.3.** *Let  $(M, g)$  be a spacetime with a  $C^{0,1}$ -metric  $g$  as above and assume the geodesic equation to be of the form (3.2). Then given initial data with  $U \neq 0$  the geodesic equation possesses unique  $C^1$ -solutions in the sense of Filippov.*

**Proof.** Existence follows from theorem 3.1, since the metric is locally Lipschitz continuous by assumption. Therefore it only remains to prove *uniqueness* and we aim at applying corollary A.5 of the appendix.

Rewriting the geodesic equations as first order system as above and setting  $D^- \equiv \{\mathbf{W}: U < 0\}$ ,  $D^+ \equiv \{\mathbf{W}: U > 0\}$ ,  $N \equiv \partial D^- = \partial D^+ = \{\mathbf{W}: U = 0\}$ , we obviously get that  $\mathbf{T}$  is smooth except on  $N$ , where it is discontinuous, hence it is piecewise continuous. Moreover  $\mathbf{T}$  satisfies

$$(T^a)^- = \left( \frac{F^a}{Q^a} \right) \Big|_{\bar{D}^-} \in C^\infty(\bar{D}^-), \quad (3.3)$$

$$(T^a)^+ = \left( \frac{E^a}{P^a} + \frac{F^a}{Q^a} \right) \Big|_{\bar{D}^+} \in C^\infty(\bar{D}^+), \quad (3.4)$$

where  $\mathbf{T}^+$ ,  $\mathbf{T}^-$  denote the extensions of  $\mathbf{T}$  to the boundary from  $D^+$  and  $D^-$ , respectively (see the text above theorem A.4). A normal to  $N$ , pointing from  $D^-$  to  $D^+$  is  $n \equiv e_1$ , the first standard unit vector. Hence the projection of  $\mathbf{T}^+$  onto  $n$ , denoted by  $\mathbf{T}_n^+$ , is just its first component and the same holds true for  $\mathbf{T}_n^-$ . To apply corollary A.5 we have to show that  $\mathbf{T}_n^+(\mathbf{W}) > 0$  and  $\mathbf{T}_n^-(\mathbf{W}) > 0$  (for  $\mathbf{W} \in N$ ).

Since (3.2) is obtained from rewriting the geodesic equation in first order form we have  $E^1 = 0$  and  $Q^1 = 1$ , hence we obtain  $\mathbf{T}_n^+(\mathbf{W}) = F^1(\mathbf{W}) = \dot{U} = \mathbf{T}_n^-(\mathbf{W})$ . Now let  $\mathbf{W}(\tau)$  be a geodesic with initial value  $\mathbf{W}(0) = (U_0, \dot{U}_0, V_0, \dot{V}_0, X_0, \dot{X}_0, Y_0, \dot{Y}_0)$  with  $U_0 < 0$ , which has  $\dot{U}(\tau) \geq 0$  before it reaches  $N$ . (This is the only relevant case since if  $U_0 < 0$  the geodesic can reach  $N$  at all only if  $\dot{U} \geq 0$  just before it reaches  $N$  and the cases for  $U_0 > 0$  are completely analogous.) Assume that  $\dot{U} = 0$  at  $N$ , then reversing this geodesic, one obtains a geodesic, which starts in  $N$ , is tangential to  $N$  and leaves it—a contradiction to  $N$  being totally geodesic. Consequently  $\dot{U} > 0$  at  $N$ , hence  $\mathbf{T}_n^+(\mathbf{W}) > 0$ ,  $\mathbf{T}_n^-(\mathbf{W}) > 0$ , and we have unique solutions in the sense of Filippov by corollary A.5.  $\square$

### 3.4. The geodesic equation in nonexpanding impulsive waves

Our next goal is thus to explicitly write the geodesic equation for nonexpanding impulsive waves in the continuous form and to see that we can apply theorem 3.3. The continuous line element of nonexpanding impulsive waves is given by equation (2.5). Using the relation

$$Z = \frac{1}{\sqrt{2}}(X + iY), \quad \bar{Z} = \frac{1}{\sqrt{2}}(X - iY), \quad (3.5)$$

we obtain its real form

$$ds^2 = \omega^{-2} (U, V, X^k) \left[ g_{ij} (U, X^k) dX^i dX^j - 2 dU dV \right], \quad (3.6)$$

$i, j = 2, 3$  with  $X^2 \equiv X$ ,  $X^3 \equiv Y$ , where

$$g_{ij} = \delta_{ij} + 2U_+ H_{,ij} + U_+^2 \delta^{kl} H_{,ik} H_{,jl}, \quad (3.7)$$

$$\omega = 1 + \frac{\Lambda}{12} (\delta_{ij} X^i X^j - 2UV - 2U_+ G), \quad (3.8)$$

$$G = H - X^i H_{,i}. \quad (3.9)$$

Recall that by Rademacher's theorem, (locally) Lipschitz continuous functions are differentiable almost everywhere with derivatives belonging (locally) to  $L^\infty$ . Taking derivatives of the metric coefficients (e.g.  $U_+$ ,  $U_+^2$ ) will always be understood in this sense. The Christoffel symbols for the metric (3.6) are

$$\begin{aligned} \Gamma_{VV}^U &= 0, & \Gamma_{VU}^U &= 0, & \Gamma_{Vi}^U &= 0, \\ \Gamma_{UU}^U &= -2 \frac{\omega, U}{\omega}, & \Gamma_{Ui}^U &= -\frac{\omega, i}{\omega}, & \Gamma_{ij}^U &= -g_{ij} \frac{\omega, V}{\omega}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Gamma_{VV}^V &= -2 \frac{\omega, V}{\omega}, & \Gamma_{VU}^V &= 0, & \Gamma_{Vi}^V &= -\frac{\omega, i}{\omega}, \\ \Gamma_{UU}^V &= 0, & \Gamma_{Uj}^V &= 0, & \Gamma_{jk}^V &= -g_{jk} \frac{\omega, U}{\omega} + \frac{1}{2} g_{jk, U}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \Gamma_{VV}^i &= 0, & \Gamma_{VU}^i &= -g^{ij} \frac{\omega, j}{\omega}, & \Gamma_{Vk}^i &= -\delta_k^i \frac{\omega, V}{\omega}, \\ \Gamma_{UU}^i &= 0, & \Gamma_{Uk}^i &= \frac{1}{2} g^{ij} g_{jk, U} - \delta_k^i \frac{\omega, U}{\omega}, \\ \Gamma_{kl}^i &= {}^s \Gamma_{kl}^i - \frac{1}{\omega} (\delta_k^i \omega, l + \delta_l^i \omega, k - g^{ij} g_{kl} \omega, j), \end{aligned} \quad (3.12)$$

where  ${}^s \Gamma_{kl}^i$  denotes the Christoffel symbols of the 'spatial' metric  $g_{ij}$ . The equations of geodesics in this case thus take the explicit form

$$\begin{aligned} \ddot{U} - 2 \frac{\omega, U}{\omega} \dot{U}^2 - g_{ij} \frac{\omega, V}{\omega} \dot{X}^i \dot{X}^j - 2 \frac{\omega, i}{\omega} \dot{U} \dot{X}^i &= 0, \\ \ddot{V} - 2 \frac{\omega, V}{\omega} \dot{V}^2 - \left( g_{jk} \frac{\omega, U}{\omega} - \frac{1}{2} g_{jk, U} \right) \dot{X}^j \dot{X}^k - 2 \frac{\omega, i}{\omega} \dot{V} \dot{X}^i &= 0, \\ \ddot{X}^i + \left[ {}^s \Gamma_{kl}^i - \frac{1}{\omega} (\delta_k^i \omega, l + \delta_l^i \omega, k - g^{ij} g_{kl} \omega, j) \right] \dot{X}^k \dot{X}^l - 2 g^{ij} \frac{\omega, j}{\omega} \dot{V} \dot{U} \\ - 2 \delta_k^i \frac{\omega, V}{\omega} \dot{V} \dot{X}^k + \left( g^{ij} g_{jk, U} - 2 \delta_k^i \frac{\omega, U}{\omega} \right) \dot{U} \dot{X}^k &= 0. \end{aligned} \quad (3.13)$$

To explicitly see that (3.13) is of the form (3.2) we treat the cases  $\Lambda = 0$  and  $\Lambda \neq 0$  separately.

### 3.5. Geodesics in impulsive $pp$ -waves

In the  $pp$ -wave case, i.e.,  $\Lambda = 0$  and hence  $\omega = 1$ , the geodesic equations (3.13) simplify to

$$\ddot{U} = 0, \quad \ddot{V} + \frac{1}{2}g_{jk,U}\dot{X}^j\dot{X}^k = 0, \quad \ddot{X}^i + {}^s\Gamma_{kl}^i\dot{X}^k\dot{X}^l + g^{ij}g_{jk,U}\dot{U}\dot{X}^k = 0. \quad (3.14)$$

Recall that the main (technical) obstacle in case of general  $\Lambda$  was that  $U$  could not be used as an affine parameter along the geodesics. However, in the case  $\Lambda = 0$  we have  $\ddot{U} = 0$  and hence  $U$  can be used to parametrize the geodesics, significantly simplifying our task. Indeed, using  $U$  as an affine parameter and thus setting  $\dot{U} = 1$  the geodesic equations take the explicit form

$$\begin{aligned} \ddot{V} &= - \left[ (U_+),_U H_{,ij} + \frac{1}{2}(U_+^2),_U \delta^{mn}H_{,im}H_{,jn} \right] \dot{X}^i\dot{X}^j, \\ \ddot{X}^i &= -g^{ij} \left[ U_+ H_{,jkl} + U_+^2 \delta^{mn}H_{,jm}H_{,kln} \right] \dot{X}^k\dot{X}^l \\ &\quad - 2g^{ij} \left[ (U_+),_U H_{,jk} + \frac{1}{2}(U_+^2),_U \delta^{mn}H_{,jm}H_{,kn} \right] \dot{X}^k, \end{aligned} \quad (3.15)$$

where the inverse spatial metric is given by

$$g^{ij} = D^{-1}g_{pq}(\delta^{ij}\delta^{pq} - \delta^{ip}\delta^{jq}), \quad D \equiv \det g_{ij} = g_{22}g_{33} - (g_{23})^2. \quad (3.16)$$

Now we extract  $\Theta(U)$  from (3.15) to put it into the form (3.2). To this end we write the kinks as  $U_+ = \Theta(U)U$  and  $U_+^2 = \Theta(U)U^2$ , respectively, and (recalling that by (3.1)  $W^5 = X = X^2$ ,  $W^7 = Y = X^3$ ,  $W^6 = \hat{X} = \dot{X}^2$ , and  $W^8 = \hat{Y} = \dot{X}^3$ ) we obtain

$$\begin{aligned} E^1 &= 0, \quad F^1 = \hat{U}, \quad Q^1 = 1, \\ E^2 &= 0, \quad F^2 = 0, \\ E^3 &= 0, \quad F^3 = \hat{V}, \quad Q^3 = 1, \\ E^4 &= -(H_{,ij} + U\delta^{kl}H_{,ik}H_{,jl})\dot{X}^i\dot{X}^j, \quad P^4 = 1, \quad F^4 = 0, \\ E^a &= 0, \quad F^a = \hat{X}^i, \quad Q^a = 1, \quad \text{for } a = 5 \text{ with } i = 2, \text{ and } a = 7 \text{ with } i = 3, \\ E^a &= -(\delta^{ij}\delta^{pq} - \delta^{ip}\delta^{jq})(\delta_{pq} + 2UH_{,pq} + U^2\delta^{rs}H_{,pr}H_{,qs}) \\ &\quad \times \left[ (UH_{,jkl} + U^2\delta^{mn}H_{,jm}H_{,kln})\dot{X}^k\dot{X}^l + 2(H_{,jk} + U\delta^{mn}H_{,jm}H_{,kn})\dot{X}^k \right] \\ &\quad \text{for } a = 6 \text{ with } i = 2, \text{ and } a = 8 \text{ with } i = 3, \\ F^6 &= 0 = F^8, \\ P^6 &= P^8 = D = \left[ 1 + (H_{,22} + H_{,33})U_+ + (H_{,22}H_{,33} - H_{,23}^2)U_+^2 \right]^2. \end{aligned} \quad (3.17)$$

Of course,  $P^a$  is irrelevant whenever  $E^a = 0$ , and similarly  $Q^a$  whenever  $F^a = 0$ .

Since in addition all Christoffel symbols of the form  $\Gamma_{\mu\nu}^U$  vanish, the hyperplane  $\{U = 0\}$  is totally geodesic and we may apply theorem 3.3. It thus follows that, provided  $H$  is smooth, the geodesic equation for data with  $U \neq 0$  in impulsive  $pp$ -waves is uniquely solvable in the sense of Filippov, and the geodesics are  $C^1$ -curves. Those which hit the wave surface also cross it. Since the background spacetime off the shock surface, i.e., Minkowski space is clearly complete we also have a completeness result. Note, however, that we do not obtain any additional information on the geodesics which lie within the null surface  $N$ .

**Theorem 3.4.** *Consider the impulsive pp-wave spacetime (2.5) with  $\Lambda = 0$  and smooth  $H$ . Given initial data off the wave surface  $\{U = 0\}$  the geodesic equation possesses global unique  $C^1$ -geodesics (in the sense of Filippov). In particular, these geodesics are complete.*

This theorem applies e.g. to plane waves, where  $H$  is quadratic in  $X, Y$ , as well as to pp-waves with  $H$  being a higher polynomial [45]. However, in many physically interesting models  $H$  will be non-smooth, possessing poles at the axis  $\{\rho = 0\}$ , cf section 2.4. In such a case theorem 3.3 can still be applied, but some care is needed. Indeed, if a geodesic starts in the (background) region where  $U < 0$  and hits the wave surface at  $U = 0$  with some  $\rho(U = 0) > 0$  we may work on an open subset of the spacetime with a small neighborhood of  $\rho = 0, U \geq 0$  removed. There the metric is locally Lipschitz and theorem 3.3 still applies to guarantee that the geodesic continues into the region  $U > 0$  and exists as a unique,  $C^1$ -solution in the sense of Filippov at least for small positive  $U$ . Of course, later on it might run into the singularity at  $U > 0, \rho = 0$  which, however, is just a coordinate singularity introduced in the Minkowski background via the transformation (2.3). Hence the geodesic will be complete. Also we can argue analogously for geodesics starting with positive  $U$  and running towards the wave surface. The only geodesics which do *not* allow for such an application of theorem 3.3 are those which *directly head* at the curvature singularity at  $U = 0, \rho = 0$ , which is in complete agreement with physical expectations. Summing up we have the following result:

**Theorem 3.5.** *Consider the impulsive pp-wave spacetime (2.5) with  $\Lambda = 0$  and  $H$  smooth off  $\rho = 0$ . Then given initial data off the wave surface  $\{U = 0\}$  the geodesic equation possesses locally defined unique  $C^1$ -solutions (in the sense of Filippov). Moreover, all geodesics starting at  $U \neq 0$  and not directly heading towards  $U = 0, \rho = 0$  are complete.*

As already said in section 3.2 the fact that  $U$  can serve as an affine parameter along the geodesics in impulsive pp-waves makes it possible to employ the (simpler) solution concept of Carathéodory [56, chapter 1], see [51]. Since in general Filippov and Carathéodory solutions do not agree we face the question of compatibility of theorem 3.4 with the result in [51]. However, it is easily seen (from the fact that for  $U \neq 0$  the Filippov set-valued map satisfies  $\mathcal{F}[f](U) = \{f(U)\}$ , where  $f$  is the right-hand side of the ODE) that for the equations (3.14) any Carathéodory solution is a Filippov solution and vice versa. Hence the geodesics agree in both approaches, a fact that is also clearly visible from the explicit junction conditions (4.4) derived below in section 4.

### 3.6. Geodesics in the general case with $\Lambda \neq 0$

With any value of the cosmological constant  $\Lambda$ , the function  $\omega$  takes the general form (3.8). The corresponding equations of geodesics (3.13), representing motion caused by impulsive gravitational waves propagating in the (anti-)de Sitter universe, are thus given by a considerably more complex system than (3.14) valid in a flat Minkowski background. Nevertheless, it is again possible to put (3.13) into the form (3.2). A straightforward but somewhat lengthy calculation reveals that

$$\begin{aligned}
E^1 &= 0, \quad F^1 = \hat{U}, \quad Q^1 = 1, \\
E^2 &= -\frac{\Lambda}{3} \left[ G \hat{U}^2 + \left( H_{,ij} + \frac{1}{2} U \delta^{kl} H_{,ik} H_{,jl} \right) U^2 \hat{X}^i \hat{X}^j - U \hat{U} H_{,ki} X^k \hat{X}^i \right], \quad P^2 = \omega, \\
F^2 &= -\frac{\Lambda}{3} \left[ V \hat{U}^2 + \frac{1}{2} U \delta_{ij} \hat{X}^i \hat{X}^j - \hat{U} \delta_{ik} \hat{X}^i X^k \right], \quad Q^2 = \omega, \\
E^3 &= 0, \quad F^3 = \hat{V}, \quad Q^3 = 1, \\
E^4 &= - \left[ \left( H_{,ij} + U \delta^{kl} H_{,ik} H_{,jl} \right) \left( 1 + \frac{\Lambda}{12} \delta_{mn} X^m X^n \right) + \frac{\Lambda}{6} \left( G \delta_{ij} + U (V + G) H_{,ij} \right) \right] \hat{X}^i \hat{X}^j \\
&\quad + \frac{\Lambda}{3} U \hat{V} H_{,ki} X^k \hat{X}^i, \quad P^4 = \omega, \\
F^4 &= -\frac{\Lambda}{3} \left[ U \hat{V}^2 + \frac{1}{2} V \delta_{ij} \hat{X}^i \hat{X}^j - \hat{V} \delta_{ik} \hat{X}^i X^k \right], \quad Q^4 = \omega, \\
E^a &= 0, \quad F^a = \hat{X}^i, \quad Q^a = 1, \quad \text{for } a = 5 \text{ with } i = 2, \text{ and } a = 7 \text{ with } i = 3, \quad (3.18) \\
E^a &= - \left( \delta^{ij} \delta^{pq} - \delta^{ip} \delta^{jq} \right) g_{pq}^+ \\
&\quad \times \left\{ \left[ \omega_+ \left( U H_{,jkl} + U^2 \delta^{mn} H_{,jm} H_{,kln} \right) \right. \right. \\
&\quad \left. \left. + \frac{\Lambda}{6} \left( \left( 2 U H_{,kl} + U^2 \delta^{mn} H_{,km} H_{,ln} \right) \delta_{js} X^s - G_{,j} g_{kl}^+ U \right) \right] \hat{X}^k \hat{X}^l \right. \\
&\quad \left. + 2 \omega_+ \left( H_{,jk} + U \delta^{mn} H_{,jm} H_{,kn} \right) \hat{U} \hat{X}^k + \frac{\Lambda}{3} G_{,j} U \hat{U} \hat{V} \right\} \\
&\quad - \frac{\Lambda}{6} \left( 2 U H_{,pq} + U^2 \delta^{mn} H_{,pm} H_{,qn} \right) \left( \delta^{pq} X^i - \delta^{ip} X^q \right) \left( \delta_{kl} \hat{X}^k \hat{X}^l - 2 \hat{U} \hat{V} \right) \\
&\quad + \frac{\Lambda}{3} \left\{ U H_{,kl} X^k \hat{X}^l - G \hat{U} + D \left[ \delta_{jk} X^j \hat{X}^k - \left( U \hat{V} + \hat{U} V \right) + U H_{,kl} X^k \hat{X}^l - G \hat{U} \right] \right\} \hat{X}^i \\
&\quad \text{for } a = 6 \text{ with } i = 2, \text{ and } a = 8 \text{ with } i = 3, \\
F^a &= \frac{\Lambda}{3} \left[ X^i \left( \hat{U} \hat{V} - \frac{1}{2} \delta_{kl} \hat{X}^k \hat{X}^l \right) + \hat{X}^i \left( \delta_{jk} X^j \hat{X}^k - \left( U \hat{V} + \hat{U} V \right) \right) \right] \\
&\quad \text{for } a = 6 \text{ with } i = 2, \text{ and } a = 8 \text{ with } i = 3, \\
P^6 &= P^8 = Q^6 = Q^8 = D \omega,
\end{aligned}$$

where

$$\begin{aligned}
g_{ij}^+ &= \delta_{ij} + 2 U H_{,ij} + U^2 \delta^{kl} H_{,ik} H_{,jl}, \\
\omega_+ &= 1 + \frac{\Lambda}{12} \left( \delta_{ij} X^i X^j - 2 U V - 2 U G \right), \\
D &= \left[ \left( H_{,22} + H_{,33} \right) + U \left( H_{,22} H_{,33} - H_{,23}^2 \right) \right] \\
&\quad \times \left[ 2 U + U^2 \left( H_{,22} + H_{,33} \right) + U^3 \left( H_{,22} H_{,33} - H_{,23}^2 \right) \right], \quad (3.19)
\end{aligned}$$

are smooth functions, polynomial in  $U$ . Of course, for  $\Lambda = 0$  these expressions reduce to those presented in the previous section.

Also, the null hypersurface given by  $U = 0$  is again totally geodesic:  $\omega_{,V}$  is proportional to  $U$ , and so the geodesic equation (3.13) with initial data  $U = 0$  and  $\dot{U} = 0$  allows for the solution  $U \equiv 0$ . By uniqueness this implies that  $\{U = 0\}$  is totally geodesic in the background



(anti-)de Sitter spacetime. Suppose now that in the impulsive wave spacetime such a geodesic leaves the hypersurface for  $\tau > \tau_0$ , it also would be a geodesic in the background for  $\tau > \tau_0$ . However, by continuity of the tangent vector this implies  $\dot{U}(\tau_0) = 0$  for a geodesic in the background, contradicting the above.

So for smooth  $H$  we obtain a result directly generalizing theorem 3.4:

**Theorem 3.6.** *Consider the nonexpanding impulsive wave spacetime (2.5) with arbitrary  $\Lambda$  and smooth  $H$ . Given initial data off the wave surface  $\{U = 0\}$  the geodesic equation possesses global unique  $C^1$ -geodesics (in the sense of Filippov). In particular, these geodesics are complete.*

Also the physically more relevant models with singular  $H$  can be dealt with similarly to the case of vanishing  $\Lambda$ . Observe from (2.19) that in the present case the singularities are localized on the wave-front at  $z = \pm 1$ . Hence we also obtain a generalization of theorem 3.5:

**Theorem 3.7.** *Consider the nonexpanding impulsive wave spacetime (2.5) with arbitrary  $\Lambda$  and  $H$  smooth off  $z = \pm 1$ . Then given initial data off the wave surface  $\{U = 0\}$  the geodesic equation possesses locally defined unique  $C^1$ -solutions (in the sense of Filippov). Moreover, all geodesics starting with  $U \neq 0$  and not directly heading towards  $U = 0$ ,  $z = \pm 1$  are complete.*

In view of the discussion in section 2.5 the two singularities occur at  $Z_2 = 0 = Z_3$  and  $Z_4 = \pm a$ . These are the north and south poles of a spherical impulsive wave in de Sitter space and the vertices of hyperboloidal waves in anti-de Sitter space.

#### 4. The $C^1$ -matching

In this section we apply the results of sections 3.5 and 3.6 to explicitly derive the form of the geodesics in nonexpanding impulsive gravitational waves by appropriately matching the geodesics of the background. We start, however, with a general remark.

**Remark 4.1** (The philosophy of the matching).

- (1) Observe that the matching is only justified *after* we have gained sufficient knowledge on the geodesics of the entire spacetime: the geodesics heading towards the wave surface *cross* it, are *unique* and of  $C^1$ -regularity.
- (2) However, one may consider the following more general situation where such a procedure is possible: assume we have a  $C^{0,1}$ -metric and the spacetime is separated by a hypersurface  $N$  into two parts  $D^+$  and  $D^-$  such that  $g|_{D^\pm} \in C^\infty$ . Then, in particular, one has (unique) classical (smooth) geodesics on both sides. Now provided the right-hand side of the geodesic equation (written as a first order system, cf (3.2)) satisfies  $\mathbf{T}_n^\pm > 0$ , these geodesics combine to unique ( $C^1$ -)solutions in the sense of Filippov. Hence the global geodesics can be computed simply by matching the background geodesics in a  $C^1$ -manner without the need to go into the details of Filippov's theory.

To explicitly carry out the  $C^1$ -matching procedure in our case we start with the unique globally defined  $C^1$ -geodesics in the continuous metric (3.6). Transforming them into coordinate systems (2.1) well adapted to the background spacetimes separately on either side of

the wave surface we derive explicit matching conditions for the position and the velocity of the geodesics of the background across  $\{U = 0\}$ .

To begin with we write the background spacetimes of constant curvature (2.1) in the real coordinates

$$x = \frac{1}{\sqrt{2}}(\eta + \bar{\eta}), \quad y = \frac{1}{\sqrt{2}i}(\eta - \bar{\eta}). \quad (4.1)$$

Using the real spatial variables (3.5) and (4.1), the transformation (2.7), which relates the continuous line element (2.5) to the constant curvature background spacetimes (2.1), can be expressed as

$$\begin{aligned} \mathcal{U} &= U, \\ \mathcal{V} &= V + \Theta H + \frac{1}{2}U_+ \left( (H_{,X})^2 + (H_{,Y})^2 \right), \\ x &= X + U_+ H_{,X}, \\ y &= Y + U_+ H_{,Y}, \end{aligned} \quad (4.2)$$

where  $H = H(X, Y)$ . Now we consider geodesics

$$U = U(\tau), \quad V = V(\tau), \quad X = X(\tau), \quad Y = Y(\tau), \quad (4.3)$$

in the continuous metric (3.6). By the results of section 3.6 they are unique globally defined  $C^1$ -curves. In particular, positions and velocities at the instant of interaction with the impulse are *equal on both sides*. Hence by employing the transformation (4.2) and its derivative separately in the region  $U > 0$  and  $U < 0$  we can express the refraction formulae for the geodesics crossing the impulsive hypersurface  $U = 0$  as

$$\begin{aligned} \mathcal{U}_i^- = 0 = \mathcal{U}_i^+, \quad \dot{\mathcal{U}}_i^- = \dot{\mathcal{U}}_i^+, \\ \mathcal{V}_i^- = \mathcal{V}_i^+ - H_i, \quad \dot{\mathcal{V}}_i^- = \dot{\mathcal{V}}_i^+ - H_{i,X} \dot{x}_i^+ - H_{i,Y} \dot{y}_i^+ + \frac{1}{2} \left( (H_{i,X})^2 + (H_{i,Y})^2 \right) \dot{\mathcal{U}}_i^+, \\ x_i^- = x_i^+, \quad \dot{x}_i^- = \dot{x}_i^+ - H_{i,X} \dot{\mathcal{U}}_i^+, \\ y_i^- = y_i^+, \quad \dot{y}_i^- = \dot{y}_i^+ - H_{i,Y} \dot{\mathcal{U}}_i^+. \end{aligned} \quad (4.4)$$

Here the subscript  $i$  denotes the values of the respective quantities at the instant when the geodesics interact with the impulse at  $U = 0$  (note that  $H_{i,X} = (H_{,X})_i$ ), while the superscripts  $+$  and  $-$  denote the values of the positions and velocities of the geodesics as they approach the impulse from the region  $\{U < 0\}$  resp.  $\{U > 0\}$ . Interestingly, these relations do not explicitly depend on the cosmological constant  $\Lambda$  since the conformally flat coordinates are used. Moreover they clearly reduce to the conditions derived in [51] in the case  $\Lambda = 0$ .

However, to better understand the influence of the (anti-)de Sitter background, it is convenient to employ the five-dimensional formalism (see section 2.5 and [50]). Specifically, we can work with the metric (2.14) which is related to (2.11) by transformation (2.16).

Defining the evaluation of the conformal factor on either side by  $\Omega_i^\pm = 1 + \frac{1}{12}\Lambda \left( (x_i^\pm)^2 + (y_i^\pm)^2 \right) = 2a/(Z_{4i}^\pm + a)$  we find, using the fact  $x_i^- = x_i^+$  and  $y_i^- = y_i^+$ , that  $\Omega_i^- = \Omega_i^+$  and we may just denote it as  $\Omega_i$ . So we obtain from (4.4)

$$\tilde{\mathcal{U}}_i^- = 0 = \tilde{\mathcal{U}}_i^+, \quad \tilde{\mathcal{V}}_i^- = \tilde{\mathcal{V}}_i^+ - \frac{H_i}{\Omega_i}, \quad Z_{2i}^- = Z_{2i}^+, \quad Z_{3i}^- = Z_{3i}^+, \quad Z_{4i}^- = Z_{4i}^+, \quad (4.5)$$

which are in fact the Penrose junction conditions (2.2) in five dimensions.

Moreover,  $\dot{\Omega}_i^- = \dot{\Omega}_i^+ + \frac{1}{2\epsilon a^2} G_i \Omega_i \dot{U}_i^+$ , where for  $G_i = G_i^\pm = H_i - H_{i,X} \Omega_i Z_{2i}^\pm - H_{i,Y} \Omega_i Z_{3i}^\pm$  (see (3.9) with (4.2) and (2.16)) we have adopted a convention analogous to that for  $\Omega_i$ . In this way we obtain for the velocities

$$\begin{aligned}\dot{U}_i^- &= \dot{U}_i^+, \\ \dot{V}_i^- &= \dot{V}_i^+ + 2p \dot{U}_i^+ - H_{i,X} \dot{Z}_{2i}^+ - H_{i,Y} \dot{Z}_{3i}^+ - \frac{G_i}{2a} \dot{Z}_{4i}^+, \\ \dot{Z}_{2i}^- &= \dot{Z}_{2i}^+ - \dot{U}_i^+ \left( H_{i,X} + \frac{G_i}{2\epsilon a^2} Z_{2i}^+ \right), \\ \dot{Z}_{3i}^- &= \dot{Z}_{3i}^+ - \dot{U}_i^+ \left( H_{i,Y} + \frac{G_i}{2\epsilon a^2} Z_{3i}^+ \right), \\ \dot{Z}_{4i}^- &= \dot{Z}_{4i}^+ - \dot{U}_i^+ \frac{G_i}{\epsilon a \Omega_i},\end{aligned}\tag{4.6}$$

where

$$p \equiv \frac{1}{4} \left[ (H_{i,X})^2 + (H_{i,Y})^2 - \frac{G_i}{\epsilon a^2} \left( \dot{V}_i^+ - \frac{H_i}{\Omega_i} \right) \right].\tag{4.7}$$

The above expressions can also be rewritten in the five-dimensional Minkowski coordinates as

$$\begin{aligned}Z_{0i}^- &= Z_{0i}^+ - \frac{H_i}{\sqrt{2} \Omega_i}, & Z_{1i}^- &= Z_{1i}^+ + \frac{H_i}{\sqrt{2} \Omega_i}, \\ Z_{2i}^- &= Z_{2i}^+, & Z_{3i}^- &= Z_{3i}^+, & Z_{4i}^- &= Z_{4i}^+, \end{aligned}\tag{4.8}$$

for the positions and

$$\begin{aligned}\dot{Z}_{0i}^- &= (1+p) \dot{Z}_{0i}^+ + p \dot{Z}_{1i}^+ - \frac{1}{\sqrt{2}} \left( H_{i,X} \dot{Z}_{2i}^+ + H_{i,Y} \dot{Z}_{3i}^+ + \frac{G_i}{2a} \dot{Z}_{4i}^+ \right), \\ \dot{Z}_{1i}^- &= -p \dot{Z}_{0i}^+ + (1-p) \dot{Z}_{1i}^+ + \frac{1}{\sqrt{2}} \left( H_{i,X} \dot{Z}_{2i}^+ + H_{i,Y} \dot{Z}_{3i}^+ + \frac{G_i}{2a} \dot{Z}_{4i}^+ \right), \\ \dot{Z}_{2i}^- &= \dot{Z}_{2i}^+ - \frac{1}{\sqrt{2}} \left( \dot{Z}_{0i}^+ + \dot{Z}_{1i}^+ \right) \left( H_{i,X} + \frac{G_i}{2\epsilon a^2} Z_{2i}^+ \right), \\ \dot{Z}_{3i}^- &= \dot{Z}_{3i}^+ - \frac{1}{\sqrt{2}} \left( \dot{Z}_{0i}^+ + \dot{Z}_{1i}^+ \right) \left( H_{i,Y} + \frac{G_i}{2\epsilon a^2} Z_{3i}^+ \right), \\ \dot{Z}_{4i}^- &= \dot{Z}_{4i}^+ - \frac{1}{\sqrt{2}} \left( \dot{Z}_{0i}^+ + \dot{Z}_{1i}^+ \right) \frac{G_i}{\epsilon a \Omega_i},\end{aligned}\tag{4.9}$$

for the velocities, where

$$p \equiv \frac{1}{4} \left[ (H_{i,X})^2 + (H_{i,Y})^2 + \frac{\sqrt{2} G_i}{\epsilon a^2} \left( Z_{1i}^+ + \frac{H_i}{\sqrt{2} \Omega_i} \right) \right].\tag{4.10}$$

The matching conditions derived above clearly demonstrate the following behavior of the geodesics: *as seen in the ‘halves’ of the background (anti-)de Sitter spacetimes* in front and behind the impulsive wave, they are *refracted* in all directions but the one normal to the wave surface, see (4.4), (4.6), (4.9). Additionally, they suffer a *jump* in the  $\mathcal{V}$ -coordinate (4.4)

(respectively the  $\tilde{V}$ -coordinate (4.5), respectively the corresponding  $Z_0$ - and  $Z_1$ -components (4.8)).

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### Appendix. Filippov solutions

In this appendix we briefly recall the essentials of Filippov’s solution concept [56] for ordinary differential equations with discontinuous right-hand side. Due to our exclusive interest in geodesic equations we only discuss the autonomous case. For a general pedagogical introduction into the topic see e.g. [57].

Consider a  $d$ -dimensional system of the form

$$\dot{x}(t) = f(x(t)) \quad (t \in I), \quad (\text{A.1})$$

where  $I$  is some interval in  $\mathbb{R}$ ,  $D \subseteq \mathbb{R}^d$  and  $f: D \rightarrow \mathbb{R}^d$  is given. Peano’s classical existence theorem needs the right-hand side  $f$  to be continuous. Without this assumption, however, one has to change the solution concept since then  $x$  is not continuously differentiable in general. The idea of Filippov’s approach is to average the values of the right-hand side in a neighborhood of points of discontinuity. Formally one associates to the right-hand side  $f$  a set-valued map  $\mathcal{F}[f]: D \rightarrow \mathcal{B}(D)$  (the collection of all non-empty, closed and convex subsets of  $D$ ) via

$$\mathcal{F}[f](x) \equiv \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \text{co}(f(B_\delta(x) \setminus S)).$$

Here  $\text{co}(A)$  denotes the closed convex *hull* of a set  $A$ , i.e., the smallest convex set containing  $A$ . Moreover,  $B_\delta(x)$  denotes the closed Euclidean *ball* around  $x$  of radius  $\delta$ , and  $\mu$  is the Lebesgue measure on  $\mathbb{R}^d$ . Hence  $\mathcal{F}[f](x)$  is given as the intersection of convex hulls of images of shrinking closed balls around  $x$ , while ignoring sets  $S$  of measure zero. Observe that the value of  $\mathcal{F}[f]$  at  $x$  is independent of the value of  $f$  at  $x$ , but at points of continuity we have  $\mathcal{F}[f](x) = \{f(x)\}$ . Finally one replaces the differential equation (A.1) by the *differential inclusion*

$$\dot{x}(t) \in \mathcal{F}[f](x(t)), \quad (\text{A.2})$$

in this way prescribing a *range of values* for  $\dot{x}$  rather than a single value. Now we are in the position to define the notion of a Filippov solution.

**Definition A.1.** A Filippov solution of (A.1) on an interval  $[a, b] \subseteq I$  is an absolutely continuous curve  $x: [a, b] \rightarrow D$ , that satisfies (A.2) almost everywhere.

Recall that a curve  $x: [a, b] \rightarrow \mathbb{R}^d$  is said to be *absolutely continuous* if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that for all collections of non-overlapping intervals  $([a_i, b_i])_{i=1}^m$  in  $[a, b]$

with  $\sum_{i=1}^m (b_i - a_i) < \delta$  we have that  $\sum_{i=1}^m \|x(b_i) - x(a_i)\| < \epsilon$ . Moreover, recall that an absolutely continuous curve is differentiable almost everywhere.

Of course, any classical, i.e.,  $C^2$ -solution is a Filippov solution but the latter exist under much more general conditions. In fact, Filippov in [56] has developed a complete theory of ordinary differential equations based on this solution concept which has been found to be widely applicable e.g. in non-smooth mechanics. Here we just state two simple results suitable for our purpose.

**Theorem A.2.** (Existence, [56, theorem 7.8]) *If  $f: D \rightarrow \mathbb{R}^d$  is bounded, then for each  $(t_0, x_0) \in I \times D$  there is a Filippov solution  $x$  of (A.1) with  $x(t_0) = x_0$ .*

Just to give a simple example we consider an ODE with the Heaviside function as the right-hand side.

**Example A.3.** Let  $\Theta \in L^\infty(\mathbb{R})$  denote the Heaviside function, defined by  $\Theta(u) = 0$  if  $u < 0$  and  $\Theta(u) = 1$  for  $u > 0$ . Recall that  $\Theta$  as a class in  $L^\infty(\mathbb{R})$  has no value assigned at 0 and the Filippov set-valued map does not depend on the value of the function at a single point. So we easily obtain

$$\mathcal{F}[\Theta](u) = \begin{cases} \{0\} & u < 0, \\ [0, 1] & u = 0, \\ \{1\} & u > 0, \end{cases}$$

since  $\Theta$  is continuous at  $u \neq 0$  and  $[0, 1]$  is the smallest closed, convex set containing 0 and 1. We now consider the differential inclusion  $\dot{u}(t) \in \mathcal{F}[\Theta](u(t))$  with initial condition  $u(0) = u_0$ . If  $u_0 < 0$ , then  $u(t) := u_0$  ( $t \in [0, \infty)$ ) is the unique Filippov solution with this initial condition. Similarly, if  $u_0 > 0$ , then  $u(t) := u_0 + t$  ( $t \in [0, \infty)$ ) is the unique Filippov solution. If  $u_0 = 0$ , however, the functions  $u_1(t) := 0$ ,  $u_2(t) := t$  ( $t \in [0, \infty)$ ) are both Filippov solutions starting at 0.

Uniqueness, in general, is more difficult to achieve. Recall that already classical uniqueness theorems use a Lipschitz condition. While one-sided Lipschitz conditions can be used to prove one-sided uniqueness of Filippov solutions [56, section 10.1], they turn out to be ill-suited for piecewise continuous right-hand sides (cf e.g. [57, page 53]), which are our main interest. Hence we will resort to results derived in [56, section 10.2] and assume that  $D \subseteq \mathbb{R}^d$  is connected and separated by a smooth surface  $N$  into two domains  $D^+$  and  $D^-$ . Let  $f$  and  $\frac{df}{dx^i}$  ( $i = 1, \dots, d$ ) be continuous in  $D^+$  and  $D^-$  up to the boundary  $N$ . Denote by  $f^+$  (respectively  $f^-$ ) the extensions of  $f|_{D^+}$  (respectively  $f|_{D^-}$ ) to the boundary. Then set  $h(x) \equiv f^+(x) - f^-(x)$  for  $x \in N$  and let  $f_n^+$ ,  $f_n^-$ ,  $h_n$  be the projections of  $f^+$ ,  $f^-$ ,  $h$  onto the normal to  $N$  directed from  $D^-$  to  $D^+$  at the points of  $N$ .

**Theorem A.4.** (Sufficient conditions for uniqueness, [56, lemma 10.2]) *If for  $x_0 \in N$  we have  $f_n^+(x_0) > 0$ , then in the domain  $D^+$  there exists a unique Filippov solution of (A.1) starting at  $x_0$ . Analogous assertions hold for  $D^-$  and  $f_n^-(x_0) < 0$ .*

Actually we will make use of the following result.

**Corollary A.5.** ([56, corollary 10.1]) *On the region of the surface  $N$ , where  $f_n^+ > 0$  and  $f_n^- > 0$  the solutions pass from  $D^-$  to  $D^+$  and uniqueness is not violated.*

We conclude this appendix with an example and a remark relevant to our work.

**Example A.6.** We consider the following variant of example A.3: Let

$$f(u) = \begin{cases} 1 & u < 0, \\ 2 & u > 0, \end{cases} \quad \text{then} \quad \mathcal{F}[f](u) = \begin{cases} \{1\} & u < 0, \\ [1, 2] & u = 0, \\ \{2\} & u > 0. \end{cases}$$

We consider  $\dot{u}(t) \in \mathcal{F}[f](u(t))$  with  $u(0) = u_0 < 0$ , then

$$u(t) = \begin{cases} u_0 + t & t \leq -u_0, \\ 2(u_0 + t) & t \geq -u_0, \end{cases}$$

is the unique Filippov solution by corollary A.5. Indeed, we have  $D^- = (-\infty, 0)$ ,  $D^+ = (0, \infty)$ ,  $N = \{0\}$  and  $f_n^- = f^- = 1$ ,  $f_n^+ = f^+ = 2 > 0$ .

For a general function  $f$  it may be difficult to calculate the Filippov associated map  $\mathcal{F}[f]$ , hence a calculus to compute (respectively bound) Filippov set-valued maps has been developed, see [61]. For example one may prove that for a real-valued continuous function  $g$  and a real-valued locally bounded function  $h$  we have  $\mathcal{F}[gh](x) = g(x)\mathcal{F}[h](x)$ . In our setting this shows how to compute the Filippov set-valued map of the right-hand side  $\mathbf{T}$  in (3.2).

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