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Reconstructing static spherically symmetric metrics in general relativity

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We present a general method to reconstruct static spherically symmetric metrics in general relativity based on the $1+1+2$ covariant approach. This method allows a more complete exploration of the properties of these metrics in the case of a generic fluid and in the presence of a scalar field. A number of new exact solutions are reconstructed in these cases.

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I. INTRODUCTION

It is well known that in solving the Einstein equations one can proceed in two ways. A first one in which the gravitational field is deduced assigning some symmetries for the metric and giving the thermodynamics of the sources, and a second one in which the gravitational field is assigned and one finds the sources that generate such field. Because of the structure of the Einstein equations, the second approach is normally much more simple than the first one: since the stress energy tensor contains in general derivatives of order lower than the Einstein tensor one needs to solve easier differential equations to find a solution.

In Friedmannian cosmology the idea to solve for the sources generating a certain behavior of the scale factor (i.e. the metric) is called *reconstruction* and has been exploited for a long time. The main advantage is that by assigning the desired evolution of the scale factor one can deduce in a relatively easy way the form of a certain unknown function in the model. For example, in the context of inflation one can reconstruct inflation potentials able to realize a given scenario and/or match observational constraints [1]. More recently, reconstruction methods have been used in the attempt to resolve the degeneracy implicit in the additional unknown functions appearing in extensions of general relativity [2]. Research in this sector has produced very interesting models, some of which are still under exploration.

It is then natural to ask if the reconstruction technique can be used as a solution generator in other frameworks, like the static spherically symmetric metrics. A quick scan through the key equations reveals that, differently from the Friedmannian case, in which one has to deal only with two equations (Friedmann and matter conservation equations), in the (nonvacuum) static spherically symmetric case one needs to deal with three equations, which bring in additional complications. In addition, even if the reconstruction technique can be performed it is difficult to control the physical properties of the solutions one tries to reconstruct.

The aim of this paper is to propose a more effective reconstruction technique in general relativity in the static spherically symmetric case. This technique is based on the $1+1+2$ covariant approach presented in [3]. The covariant approaches constitute a powerful method for investigating the properties of many different classes of spacetime in Einsteinian gravity. In particular, in cosmology the $1+3$ covariant approach has been very successful, leading to interesting insights in determining the features of Bianchi and nonhomogeneous cosmologies and their perturbations [4]. The $1+1+2$ formalism instead was applied to the study of linear perturbations of a Schwarzschild spacetime and to the interaction between electromagnetic radiation and gravitational waves [3]. More recently, both the covariant formalisms have proven to be particularly useful also in dealing with modified gravity [5–8].

The paper is organized in the following way: Section II summarizes the $1+1+2$ covariant approach; Sec. III the general equations for the most general class of spacetime that can admit spherical symmetry: the LRS-II spacetime; Sec. IV introduces new key variables on which the reconstruction will be based. Section V shows how to perform the reconstruction in the case of general relativity plus a perfect fluid. Section VI deals with the implementation of the energy conditions and the asymptotic flatness in the new reconstruction method. In Section VII we prove that the new formalism can be also used to attempt a direct resolution of the Einstein equations. In Section VII we consider the case in which matter is represented by a scalar field. Finally, Section IX is dedicated to the conclusions.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper; Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $-,+,+,+$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \qquad (1)$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{2}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd}. \tag{3}$$

The symmetrization and the antisymmetrization over the indexes of a tensor are defined as

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}). \tag{4}$$

Finally the Hilbert-Einstein action in the presence of matter is given by

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}[R + 2\mathcal{L}_m]. \tag{5}$$

II. $1+1+2$ COVARIANT APPROACH IN BRIEF

In the following we will make use of the $1+1+2$ approach that is optimized for the treatment of the standard static and spherically symmetric spacetime and more in general any LRS II spacetime [9]. In the $1+1+2$ approach we perform a threading decomposition of the spacetime manifold with respect to a timelike congruence and a selection of a special vector field in the orthogonal surfaces generated by this threading. In this way all the essential information in the system can be encoded in a set of kinematic and dynamic variables (the $1+1+2$ *variables*), which have the advantage to be physically well defined and to have a clear mathematical significance. The $1+1+2$ variables satisfy a set of evolution and constraint equations derived from the Bianchi and Ricci identities, forming a closed (but not always complete [4]) system for a chosen equation of state describing matter.

In the following we give a brief review of this formalism, before presenting the basic equations of the new reconstruction technique.

A. The $1+1+2$ decomposition

As mentioned above, the first step in the construction of the $1+1+2$ approach is the foliation of the spacetime. The foliation is obtained by defining a congruence of integral curves of the timelike vector field u^a so that the spacetime is split in 3-spaces V perpendicular to u^a. This decomposition allows us to define the projection tensor $h^a_b = g^a_b + u^a u_b$ which represents the metric of the 3-spaces. Using u_a and h_{ab} any tensorial object can be split according to the foliation. For example, a 4-vector X^a can be written as

$$X^a = \mathbb{X}u^a + \mathbb{X}^a, \qquad \mathbb{X} \equiv X^a u_a, \qquad \mathbb{X}^a \equiv X^{\langle a\rangle}, \tag{6}$$

and any symmetric 4-tensor

$$X^{ab} = \mathbb{X}_0 u^a u^b + \mathbb{X}_1 h^{ab} + 2\mathbb{X}^{(a}u^{b)} + \mathbb{X}^{ab}, \tag{7}$$

$$\mathbb{X}_0 \equiv X^{ab}u_a u_b, \qquad \mathbb{X}_1 \equiv \frac{1}{3}X^{ab}h_{ab}, \tag{8}$$

$$\mathbb{X}^a \equiv h^{ac}u^b X_{cb}, \qquad \mathbb{X}^{ab} = X^{\langle ab\rangle}. \tag{9}$$

The angle brackets above denote orthogonal projections of vectors and the orthogonally *projected symmetric trace-free* (PSTF) part of tensors [4]:

$$X^{\langle a\rangle} = h^a{}_b X^b, \qquad X^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}\right]X^{cd}. \tag{10}$$

At this point one performs a further split of the orthogonal 3-spaces via the choice of a special spacelike direction represented by e^a so that

$$e_a u^a = 0, \qquad e_a e^a = 1. \tag{11}$$

Then, as before, we can define the projection tensor

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \qquad N^a{}_a = 2, \tag{12}$$

which maps tensors onto the 2-surfaces W orthogonal to e^a *and* u^a. It is clear that $e^a N_{ab} = 0 = u^a N_{ab}$.

Using this decomposition any tensorial object can be split along u^a, e^a, and W. For example any 4-vector X^a can be irreducibly split as

$$X^a = \Xi_0 u^a + \Xi_1 e^a + \Xi_2^a, \tag{13}$$

$$\Xi_0 = \mathbb{X}, \qquad \Xi_1 \equiv -\mathbb{X}_a e^a, \qquad \Xi^a \equiv N^{ab}\mathbb{X}_b. \tag{14}$$

A similar decomposition can be performed for a symmetric 4-tensor, X^{ab}:

$$X^{ab} = \Xi_0 u^a u^b + \Xi_1 e^a e^b + \Xi_3 N^{ab} + 2\Xi_1^{(a}u^{b)} + 2\Xi_2^{(a}e^{b)} + 2\Xi_2 u^{(a}e^{b)} + \Xi^{ab}, \tag{15}$$

where

$$\Xi_0 = \mathbb{X}_0, \qquad \Xi_1 \equiv \mathbb{X}_1 + \mathbb{X}^{ab}N_{ab},$$

$$\Xi_2 \equiv \mathbb{X}^c e_c, \qquad \Xi_3 \equiv \mathbb{X}_1 - \frac{1}{2}\mathbb{X}^{ab}N_{ab},$$

$$\Xi_1^a \equiv \mathbb{X}^c e_c e^a, \qquad \Xi_2^a \equiv \mathbb{X}_b N^{ab} + \mathbb{X}_{cb}N^{ca}e^b,$$

$$\Xi^{ab} \equiv \mathbb{X}^{\{ab\}} \equiv \left(N^c{}_{(a}N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\mathbb{X}_{cd},$$

and the curly brackets denote the PSTF part of a tensor with respect to e^a.

The 2-surface W carries a natural 2-volume element, the Levi-Civita 2-tensor:

$$\varepsilon_{ab} \equiv \varepsilon_{abc}e^c = \eta_{dabc}e^c u^d, \tag{16}$$

where ε_{abc} and η_{abcd} are the Levi-Civita tensors in the 3-space and 4-space respectively.

The decomposition defined above can be also used to define a set of derivatives operators. Using the vector u^a the *covariant time derivative* for any tensor $X^{a..b}{}_{c..d}$ can be defined by

$$\dot{X}^{a..b}{}_{c..d} = u^e \nabla_e X^{a..b}{}_{c..d}. \tag{17}$$

The derivative in the spaces V are, instead, the fully orthogonally *projected covariant derivative D*,

$$D_e X^{a..b}{}_{c..d} = h^a{}_f h^p{}_c \ldots h^b{}_g h^q{}_d h^r{}_e \nabla_r X^{f..g}{}_{p..q}. \tag{18}$$

From this last operator one can deduce two other derivatives: the "hat derivative"

$$\hat{X}_{a..b}{}^{c..d} \equiv e^f D_f X_{a..b}{}^{c..d}, \tag{19}$$

which is the derivative along the e^a vector field in V and the "δ derivative"

$$\delta_f X_{a..b}{}^{c..d} \equiv N_a{}^h \ldots N_b{}^g N_i{}^c .. N_j{}^d N_f{}^p D_p X_{g..h}{}^{i..j}, \tag{20}$$

which is the projected derivative onto W.

B. Kinematical variables

We can now decompose the covariant derivative of e^a in the direction orthogonal to u^a into its irreducible parts:

$$\mathrm{D}_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab}, \tag{21}$$

where

$$a_b \equiv e^c \mathrm{D}_c e_b = \hat{e}_b, \tag{22}$$

$$\phi \equiv \delta_a e^a, \tag{23}$$

$$\xi \equiv \frac{1}{2}\varepsilon^{ab}\delta_a e_b, \tag{24}$$

$$\zeta_{ab} \equiv \delta_{\{a} e_{b\}}. \tag{25}$$

Therefore, for an observer who chooses e^a as a special direction in the spacetime, ϕ represents the expansion of the integral curves of the vector field e^a, ζ_{ab} is their distortion (i.e. the *shear of* e^a) and a^a the change of vector e_a along its integral curves (e.g. its *acceleration*). We can also interpret ξ as a representation of the "twisting" or rotation of the integral curves of e_a (i.e. the *vorticity* associated with e^a).

In order to complete the characterization of the kinematical variables we consider the $1+3$ kinematics variables

$$\mathrm{D}_a u_b = u_a \dot{u}_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{26}$$

together with the electric and magnetic parts of the Weyl curvature tensor C_{abcd}

$$E_{ab} = C_{abcd}u^c u^d, \tag{27}$$

$$H_{ab} = \frac{1}{2}\eta_{ade}C^{de}{}_{bc}u^c, \tag{28}$$

and we operate a further split associated with the choice of e^a:

$$\dot{u}^a = \mathcal{A}e^a + \mathcal{A}^a, \tag{29}$$

$$\omega^a = \frac{1}{2}\eta^{abc}\omega_{bc} = \Omega e^a + \Omega^a, \tag{30}$$

$$\sigma_{ab} = \Sigma\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Sigma_{(a}e_{b)} + \Sigma_{ab}, \tag{31}$$

$$E_{ab} = \mathcal{E}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab}, \tag{32}$$

$$H_{ab} = \mathcal{H}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}e_{b)} + \mathcal{H}_{ab}. \tag{33}$$

Therefore the kinematic variables of the $1+1+2$ formalism are eight scalars

$$\{\phi, \xi, \Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}\}, \tag{34}$$

six vectors

$$\{a^b, \mathcal{A}^a, \Omega^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a\}, \tag{35}$$

and four tensors

$$\{\zeta^{ab}, \Sigma^{ab}, \mathcal{E}^{ab}, \mathcal{H}^{ab}\}. \tag{36}$$

The full covariant derivatives of e^a and u^a in terms of these variables are given in [3,7].

C. Thermodynamics

Now, let us consider the matter stress energy tensor T^m_{ab}. It is known that upon foliation this tensor admits the irreducible decomposition

$$T^m_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}, \tag{37}$$

where u^a is the direction of a timelike observer, h_{ab} is the projected metric on V and μ, p, q, and π_{ab} denotes the standard matter density, pressure, heat flux, and anisotropic stress respectively.

As before, we can split the anisotropic $1+3$ fluid variables q^a and π_{ab} as

$$q^a = Qe^a + Q^a, \tag{38}$$

$$\pi_{ab} = \Pi\left[e_a e_b - \frac{1}{2}N_{ab}\right] + 2\Pi_{(a}e_{b)} + \Pi_{ab}. \tag{39}$$

The $1+1+2$ splitting of T_{ab} is, therefore,

$$T^m_{ab} = \mu u_a u_b + (p+\Pi)e_a e_b + \left(p - \frac{1}{2}\Pi\right)N_{ab} + 2Qe_{(a}u_{b)} + 2(Q_{(a} + \Pi_{(a})u_{b)} + \Pi_{ab}. \tag{40}$$

Thus matter is represented by four scalars:

$$\{\mu, p, Q, \Pi\}, \tag{41}$$

two vectors

$$\{Q^a, \Pi^a\}, \tag{42}$$

and one tensor Π^{ab}. Note that the three matter sources that appear in the Einstein equations as

$$\{\mu, p_r, p_\perp\}, \tag{43}$$

are connected to the $1+1+2$ matter scalars by $p_r = p + \Pi$, and $p_\perp = p - \frac{1}{2}\Pi$. In the following, for sake of brevity, we will only give the results of the thermodynamics in terms of $\{\mu, p, \Pi\}$.

III. $1+1+2$ EQUATIONS FOR STATIC AND SPHERICALLY SYMMETRIC SPACETIMES

The formalism given above is clearly able to describe in a natural way all locally rotationally symmetric (LRS) spacetime, e.g. space times in which one can define covariantly a unique, preferred spatial direction. In the following we are interested in the case of the isotropic, rotation free, static, and spherically symmetric spacetime. In this case *all* the $1+1+2$ vectors and tensors vanish as well as the variables Ω, ξ, $\mathcal{H}$, Θ, Σ, and Q. Thus one is left with the six scalars $\{\mathcal{A}, \phi, \mathcal{E}, \mu, p, \Pi\}$ and the set of $(1+1+2)$ equations, which describe spherically symmetric static spacetime, is [3,7]

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}, \tag{44}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right), \tag{45}$$

$$\hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\mathcal{A}, \tag{46}$$

$$\hat{\mathcal{A}} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{2}(\mu + 3p). \tag{47}$$

with the constraint

$$0 = -\mathcal{A}\phi + \frac{1}{3}(\mu + 3p) - \mathcal{E} + \frac{1}{2}\Pi. \tag{48}$$

In order to solve the equations above it is useful to define the Gaussian curvature K of W [3]

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2. \tag{49}$$

The propagation equation for K can be then written as

$$\hat{K} = -\phi K. \tag{50}$$

This last equation is the starting point for the choice of an affine parameter related to the hat derivative, which can lead to a simplification of the final $1+1+2$ equations. For our purposes a convenient parameter is the logarithmic space variable ρ such that $\hat{K} = K_{,\rho}\phi$. This operation allows us to make the ρ derivatives dimensionless. In this way (50) becomes

$$K_{,\rho} = -K, \tag{51}$$

and the other equations become

$$\phi\phi_{,\rho} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}, \tag{52}$$

$$\mathcal{E}_{,\rho} - \frac{1}{3}\mu_{,\rho} + \Pi_{,\rho} = -\frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right), \tag{53}$$

$$\phi(p_{,\rho} + \Pi_{,\rho}) = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\mathcal{A}, \tag{54}$$

$$\phi\mathcal{A}_{,\rho} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{2}(\mu + 3p), \tag{55}$$

$$\mathcal{A}\phi - \frac{1}{3}(\mu + 3p) + \mathcal{E} - \frac{1}{2}\Pi = 0, \tag{56}$$

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2. \tag{57}$$

Note that in the system above we cannot eliminate Eq. (57). This is due to the fact that using this new parameter Eq. (51) is decoupled. Therefore, there can be solutions that satisfy all the above differential equations, but not Eq. (57) (see [8] for an example in another framework).

The parameter ρ is designed in such a way to allow the definition of a new set of variables which will be the cornerstone of the present work. Another, maybe more natural choice, can be obtained requesting that $K \propto r^{-2}$ i.e. it is proportional to the inverse square of the *area radius* of W. Since Eq. (50) holds, one obtains that the hat derivative of any scalar X is

$$\hat{X} = \frac{1}{2} r\phi \frac{dX}{dr}. \tag{58}$$

However, all the affine parameters can be easily related via Eq. (50). Using this equation one finds immediately that $\rho = 2\ln(r/r_0)$, where r_0 is an arbitrary constant. In the following, without loss of generality, we will set r_0 to 1 i.e. we will assume r to be dimensionless. In the rest of this work we will perform the calculations in ρ, but we will give the final results in terms of r. This is due to the fact that in some cases integrations in r are easier to perform.

To conclude, it will be useful for our purposes also to connect directly the metric components to some of the quantities described above. In [3,7] the relation between $\mathcal{A}$, ϕ, K and the metric coefficients in r were found to be

$$\mathcal{A} = \frac{1}{2A\sqrt{B}} \frac{dA}{dr}, \tag{59}$$

$$\phi = \frac{2}{r\sqrt{B}}, \tag{60}$$

and correspond to

$$K = \frac{K_0}{r^2}. \tag{61}$$

In terms of ρ and using the same procedure one finds that for the line element

$$ds^2 = -A(\rho)dt^2 + B(\rho)d\rho^2 + C_0 e^{\rho}(d\theta^2 + \sin^2\theta d\phi^2), \tag{62}$$

we have

$$\mathcal{A} = \frac{1}{2A\sqrt{B}} \frac{dA}{d\rho}, \tag{63}$$

$$\phi = \frac{1}{\sqrt{B}}, \tag{64}$$

$$K = \frac{1}{C_0} e^{-\rho} = K_0 e^{-\rho}. \tag{65}$$

In general, for the metric

$$ds^2 = -A(p)dt^2 + B(p)dp^2 + C(p)(d\theta^2 + \sin^2\theta d\phi^2), \tag{66}$$

$\mathcal{A}$, ϕ, and K are given by

$$\mathcal{A} = \frac{1}{2A\sqrt{B}} \frac{dA}{dp}, \tag{67}$$

$$\phi = \frac{1}{C\sqrt{B}} \frac{dC}{dp}, \tag{68}$$

$$K = \frac{1}{C}. \tag{69}$$

This form of the variables highlights the connection between the $1+1+2$ formalism and the Takeno formalism [10].[1]

IV. NEW SET OF VARIABLES FOR THE STATIC SPHERICALLY SYMMETRIC CASE

Equations (52)–(57) characterize completely the static and spherically symmetric metrics in relativity, and we can use them to find solutions of Einstein theory with this symmetry. However, this system of equations can be further simplified to a set of dimensionless equations.

Let us the define the following variables

$$X = \frac{\phi_{,\rho}}{\phi}, \quad Y = \frac{\mathcal{A}}{\phi}, \quad \mathcal{K} = \frac{K}{\phi^2}, \quad E = \frac{\mathcal{E}}{\phi^2},$$
$$M = \frac{\mu}{\phi^2}, \quad \mathbb{P} = \frac{\Pi}{\phi^2}, \quad P = \frac{p}{\phi^2}, \tag{70}$$

and use the affine parameter ρ. Equations (52)–(57) take the form

$$Y_{,\rho} = M + 3P - 2Y(X + Y + 1), \tag{71}$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X), \tag{72}$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = -2Y(M + \mathbb{P}) - 2P(2X + Y) - \mathbb{P}(4X + 3), \tag{73}$$

with the constraints

$$2M + 2P + 2\mathbb{P} + 2X - 2Y + 1 = 0, \tag{74}$$

$$1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P} = 0, \tag{75}$$

[1]In fact, the covariant formalism we have presented with this choice of the affine parameter has similarities with a number of other approaches to the resolution of the Einstein equations, like the one in [11]. However there are important differences in terms of generality, physical meaning of the key variables, and the covariance and gauge invariance of the equations.

$$2M + 6P + 3\mathbb{P} - 6Y - 6E = 0. \tag{76}$$

These constraints allow the elimination of X and E as well as the elimination of one of Eqs. (71)–(73). This is possible because, due to the definition of $\mathcal{K}$, Eq. (72) is not decoupled from the system. The above system of equations will be the starting point of the reconstruction scheme we intend to propose.[2]

As a check of the new equations, let us consider first the case of standard Einstein gravity in vacuum. Equations (71)–(76) can be reduced to the differential equations:

$$Y_{,\rho} = -(1+Y)Y - XY, \tag{77}$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1+2X), \tag{78}$$

with the constraints

$$E = -Y, \tag{79}$$

$$1 + 2X - 2Y = 0, \tag{80}$$

$$1 - 4\mathcal{K} + 4Y = 0. \tag{81}$$

Implementing the constraints both the equation for Y and for $\mathcal{K}$ collapse to the Bernoulli equation

$$2Y_{,\rho} + Y + 4Y^2 = 0. \tag{82}$$

The integral of this equation can be written as

$$2\ln\left|\frac{Y}{1+4Y}\right| = \rho - \rho_0, \tag{83}$$

where ρ_0 is a constant. If we take the solution for which the absolute value is positive (which we will call the positive branch of the solution), (82) resolves to

$$Y = \frac{Y_0}{e^{\rho/2} - 4Y_0}, \tag{84}$$

where $Y_0 \neq 0$ is an integration constant. Here and in the following the subscript "0" is used for all the (integration) constants. Substituting this solution in (80) and using the definitions (70) we have

$$\phi = \phi_0 e^{-3\rho/4}\sqrt{e^{\rho/2} - 4Y_0}, \tag{85}$$

and, as a consequence,

$$\mathcal{A} = \frac{Y_0\phi_0 e^{-\frac{3\rho}{4}}}{\sqrt{e^{\rho/2} - 4\phi_0}}, \qquad K = \frac{\phi_0^2}{4}e^{-\rho}. \tag{86}$$

The solution above can be given also in terms of r

$$\phi = \frac{2\phi_0}{r}\sqrt{1 - \frac{4Y_0}{r}}, \qquad \mathcal{A} = \frac{\phi_0 Y_0}{r^2\sqrt{1 - \frac{4Y_0}{r}}}, \qquad K = \frac{\phi_0^2}{4r^2}, \tag{87}$$

which are exactly the solutions of [3] representing the Schwarzschild space-time.

Instead, taking the solution for which the absolute value in (83) is negative (the negative branch of the solution) we obtain

$$Y = -\frac{Y_0}{e^{\rho/2} + 4Y_0}, \tag{88}$$

which can be shown to correspond to negative mass Schwarzschild space-time [12]. The fact that Eq. (82) leads to two different solutions might appear surprising, but it is perfectly compatible with the Picard-Lindelöff theorem: given a specific boundary condition Eq. (82) gives always a unique solution. In the examples in the following sections it will happen that a given prescription in the variables $Y, X, \mathcal{K}$ might lead to more than a single solution. The main purpose of this work is to show the working of the reconstruction method rather than analyze the solutions obtained in detail, so for the sake of brevity in the following we will show only one of these solutions per example.

Finally the quantities X and Y can be related to the coefficients of the metric. Substituting the relations (59)–(61) and (67)–(69) in the definition (70), one has

$$Y = \frac{rA_{,r}}{4A}, \qquad X = \frac{1}{2}\left(\frac{rB_{,r}}{2B} - 1\right), \qquad \mathcal{K} = K_0 B, \tag{89}$$

in terms of r and

$$Y = \frac{A_{,\rho}}{A}, \qquad X = -\frac{B_{,\rho}}{2B}, \qquad \mathcal{K} = K_0 B e^{-\rho}, \tag{90}$$

in terms[3] of ρ. In the above formulas the constant K_0 is connected to ϕ_0 by the relation $K_0 = \phi_0^2/4$.

[2]In principle there is no reason not to extend this redefinition to a more complicated LRS metric, but here we will limit ourselves to the static spherical symmetric case leaving such tasks for future work.

[3]Note that a direct transformation of ρ in r does not map the corresponding expressions above one in the other. This happens because a change of radial "coordinate" involves also its differential in the line element. So in general

$$B(r)dr^2 = B(\rho)d\rho^2 \Rightarrow B(r) = B(\rho)e^{-\rho}. \tag{91}$$

V. THE CASE OF GENERAL RELATIVITY AND A GENERIC FLUID

Since we have checked the consistency of Eqs. (52)–(56) we can consider some elementary applications within the framework of GR. In particular, one can use Eqs. (71)–(75) to determine the matter distribution corresponding to a specific gravitational field and vice versa.

Solving[4] (71)–(75) for the variables $(M, P, P_{,\rho}, \mathbb{P})$ we obtain

$$M = \mathcal{K} - X - \frac{3}{4}, \tag{92}$$

$$\mathbb{P} = \frac{1}{3}[Y - 2Y^2 - X(2Y+1) - 2(\mathcal{K} + Y_{,\rho})], \tag{93}$$

$$P = \frac{1}{12}[-4\mathcal{K} + X(8Y+4) + 8Y_{,\rho} + 8Y(Y+1) + 3], \tag{94}$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = \mathcal{K}(1+2X) + Y_{,\rho}, \tag{95}$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1+2X). \tag{96}$$

In the equations above, (95) coincides with the sum of the derivatives of (93) and (94) once (96) has been substituted. Therefore we can exclude Eq. (95) and the remaining system is sufficient to solve for the thermodynamics once the metric is chosen. Specifically, starting from a metric element one can calculate the variables $(Y, X, \mathcal{K})$ and, using (70), the pressure distribution and the energy density.

Let us consider, for example, the Rindler metric proposed in [13]:

$$ds^2 = -K^2 dt^2 + \frac{dr^2}{K^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{97}$$

$$K^2 = 1 - \frac{2M}{r} - \Lambda r^2 + br. \tag{98}$$

This metric was proposed using the technique of spherical reduction [14] and it has been tested as a model for the Pioneer effect against the standard planetary data [13]. Using Eq. (89) one obtains

$$X = Y - \frac{1}{2}, \qquad Y = \frac{2be^{\rho} + \mu}{4(2be^{\rho} + e^{\rho/2} - \mu)}, \tag{99}$$

$$\mathcal{K} = \frac{e^{\rho/2}}{4(2be^{\rho} + e^{\rho/2} - \mu)}. \tag{100}$$

Substituting in Eqs. (71)–(74) one obtains immediately

$$\mu = -\frac{2b}{r}, \qquad p = \frac{4b}{3r}, \qquad \Pi = \frac{2b}{3r}, \tag{101}$$

or

$$\mu = -\frac{2b}{r}, \qquad p_r = \frac{2b}{r}, \qquad p_\perp = \frac{b}{r}, \tag{102}$$

$$c_s^2 = \frac{dp}{d\mu} = -\frac{2}{3}, \tag{103}$$

which is, modulus the choice of the conventions, the result of [14].

Another example can be made using directly relations between the variables $X, Y, \mathcal{K}$. For example, consider the simple choice

$$X = \beta Y^2, \qquad Y_{,\rho} = \alpha Y. \tag{104}$$

$\mathcal{K}$ can be obtained integrating (96) so that

$$Y = Y_0 e^{\alpha\rho}, \qquad X = \beta Y_0^2 e^{2\alpha\rho}, \tag{105}$$

$$\mathcal{K} = \mathcal{K}_0 \exp\left(-\rho - \frac{\beta}{\alpha} Y_0^2 e^{2\alpha\rho}\right). \tag{106}$$

Using the definitions in (70), one has immediately

$$\phi = \pm\sqrt{\frac{K_0}{\mathcal{K}_0}} \exp\left(\frac{\beta Y_0^2 e^{2\alpha\rho}}{2\alpha}\right), \tag{107}$$

$$\mathcal{A} = -2Y_0\sqrt{\frac{K_0}{\mathcal{K}_0}} \exp\left(\alpha\rho + \frac{\beta Y_0^2 e^{2\alpha\rho}}{2\alpha}\right). \tag{108}$$

Passing to the parameter r and using Eqs. (67)–(68) corresponds to the metric[5]

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{109a}$$

$$A = A_0 \exp\left(\frac{2Y_0 r^{2\alpha}}{\alpha}\right), \tag{109b}$$

[4]Equation (76) gives only the value of the variable E, which is not important in our setting and will be neglected in the following. Of course one could choose to substitute the variable Y with E. This formulation will lead to a different set of reconstruction equations and, possibly, to an additional set of exact solutions.

[5]One could also use directly the (90) or the (89) to obtain the same result. Here and in the following ϕ and $\mathcal{A}$ were given with the intent of making possible a check of the results also against the system (52)–(56) other than the Einstein equations. We have done these tests for every solution given in the present work.

$$B = \frac{4\mathcal{K}_0}{K_0 r^2}\exp\left(-\frac{\beta Y_0^2 r^{4\alpha}}{\alpha}\right), \tag{109c}$$

$$C = \frac{r^2}{K_0}, \tag{109d}$$

and a fluid whose thermodynamics is given by

$$\mu = \frac{K_0}{r^2} - \frac{K_0}{4\mathcal{K}_0}(4\beta Y_0^2 r^{4\alpha} + 3)e^{\frac{\beta Y_0^2 r^{4\alpha}}{\alpha}}, \tag{110a}$$

$$p = \frac{K_0}{12\mathcal{K}_0}\{4Y_0^2 r^{4\alpha}(\beta + 2\beta Y_0 r^{2\alpha} + 2) \\ 8Y_0 r^{2\alpha}(\alpha+1) + 3\}e^{\frac{\beta Y_0^2 r^{4\alpha}}{\alpha}} - \frac{K_0}{3r^2}, \tag{110b}$$

$$\Pi = -\frac{K_0 Y_0 r^{2\alpha}}{3\mathcal{K}_0}[Y_0 r^{2\alpha}(\beta + 2\beta Y_0 r^{2\alpha} + 2) \\ + 2\alpha - 1]e^{\frac{\beta Y_0^2 r^{4\alpha}}{\alpha}} - \frac{2K_0}{3r^2}. \tag{110c}$$

The metric coefficients we have obtained present no divergences at finite r and their behavior at $r = 0$ and $r = \infty$ depends on the values of the parameters α, β, and Y_0. Since Y_0 is an integration constant, we can consider four cases depending only on the values of α and β. Particularly interesting is the case $\alpha < 0, \beta < 0$ shown in Fig. 1. In the limit $r \to 0$ it might look like the singularity is absent because all the metric coefficients tend to zero. However the Ricci scalar and the Kretschmann invariant behave as (we assume for simplicity $\alpha = -1$, $\beta = -1$)

$$R \to r^{-6}\exp(-16r^{-4}), \tag{111}$$

and

$$K = R^{abcd}R_{abcd} \to r^{-12}\exp(-32r^{-4}), \tag{112}$$

which both diverge as $r \to 0$. This discordance can be interpreted thinking that, in spite of the regularity of the metric coefficients, the actual gravitational force related to the derivatives of the metric tensor is in fact divergent. This implies that a singularity is present in the point $r = 0$, but its nature is different from the standard Schwarzschild singularity.

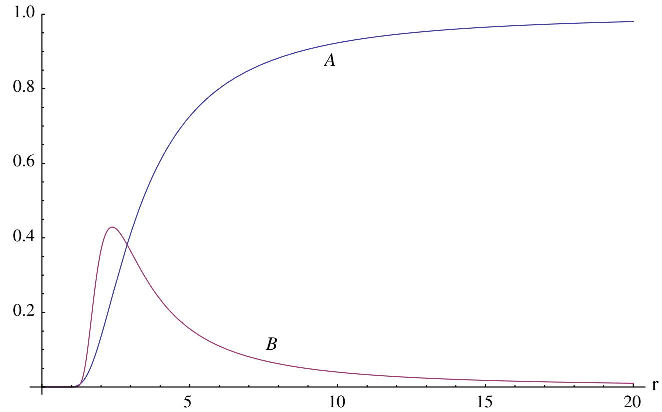
As another example, consider

$$X = -\beta\mathcal{K}, \qquad Y_{,\rho} = \alpha Y^2, \tag{113}$$

with $\beta > 0$. $\mathcal{K}$ can be obtained integrating (96) so that

$$Y = -\frac{1}{\alpha\rho + Y_0}, \qquad X = -\beta\frac{e^{\mathcal{K}_0}}{2\beta e^{\mathcal{K}_0} + e^{\rho}}, \tag{114}$$

$$\mathcal{K} = \frac{e^{\mathcal{K}_0}}{2\beta e^{\mathcal{K}_0} + e^{\rho}}. \tag{115}$$



(a) The coefficients of the metric (109).
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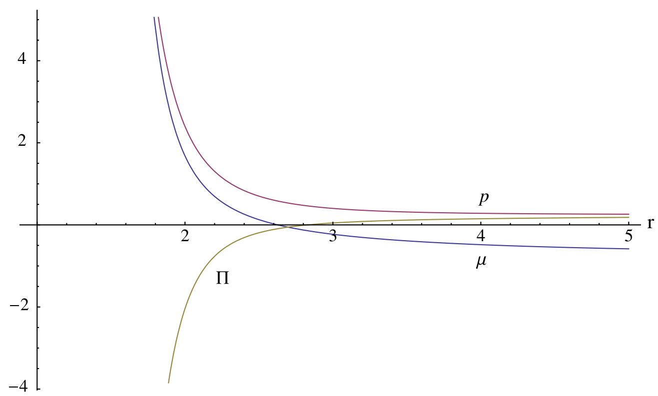


(b) The thermodynamic quantities (110) associated with (109).

FIG. 1 (color online). Graphs of the solution (109) in the case $\alpha < 0$, $\beta < 0$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

Using the definition of Y and $\mathcal{K}$ one has immediately

$$\phi = \pm\sqrt{K_0(e^{-\mathcal{K}_0} + 2\beta e^{-\rho})}, \tag{116}$$

$$\mathcal{A} = \frac{1}{\alpha\rho + Y_0}\sqrt{K_0(e^{-\mathcal{K}_0} + 2\beta e^{-\rho})}. \tag{117}$$

Passing to the parameter r and using Eqs. (67)–(68) corresponds to the metric

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{118a}$$

$$A = A_0|2\alpha\ln r + Y_0|^{-2/\alpha}, \tag{118b}$$

$$B = \frac{4e^{\mathcal{K}_0}}{K_0(r^2 + 2\beta e^{\mathcal{K}_0})}, \tag{118c}$$

$$C = \frac{r^2}{K_0}, \tag{118d}$$

and a fluid whose thermodynamics is given by

$$\mu = K_0\left(\frac{(\beta+2)}{2r^2} - \frac{3}{4}e^{-\mathcal{K}_0}\right), \tag{119a}$$

$$p = K_0\left[\frac{e^{-\mathcal{K}_0}}{4} - \frac{\beta+2}{6r^2} + \frac{2e^{-\mathcal{K}_0}r^2(\alpha - 2\alpha\ln r - Y_0 + 1) + \beta(Y_0 - 2(\alpha+1) + 2\alpha\ln r)}{3r^2(2\alpha\ln r + Y_0)^2}\right], \tag{119b}$$

$$\Pi = K_0\left[\frac{4\beta(\alpha + 2\alpha\ln r + Y_0 + 1) - r^2(2\alpha + 2\alpha\ln r + 2 + Y_0)}{3r^2 e^{-\mathcal{K}_0}(2\alpha\ln r + Y_0)^2} - \frac{(\beta+2)}{3r^2}\right]. \tag{119c}$$

The features of this solution for different values of the parameter α are shown in Figs. 2 and 3. At $r = 0$ the Ricci scalar and the Kretschmann diverge, thus despite the behavior of the metrics' coefficients (and in a similar way of the previous case) there is a singularity.

In the case $\alpha > 0$, the coefficient B is regular, whereas A presents a divergence. The analysis of the Ricci scalar and the Kretschmann invariant indicates that the divergence corresponds to an actual singularity. Thus in this case our solution seems to show a second singularity, which is naked. Note that the divergence of A is mirrored by a divergence of the anisotropic pressure Π.

In the case $\alpha < 0$, both coefficient are regular. However A presents a zero and this again implies the presence of a second singularity albeit of different nature (but still naked). The presence of this singular point is also accompanied by a divergence in the pressure p.

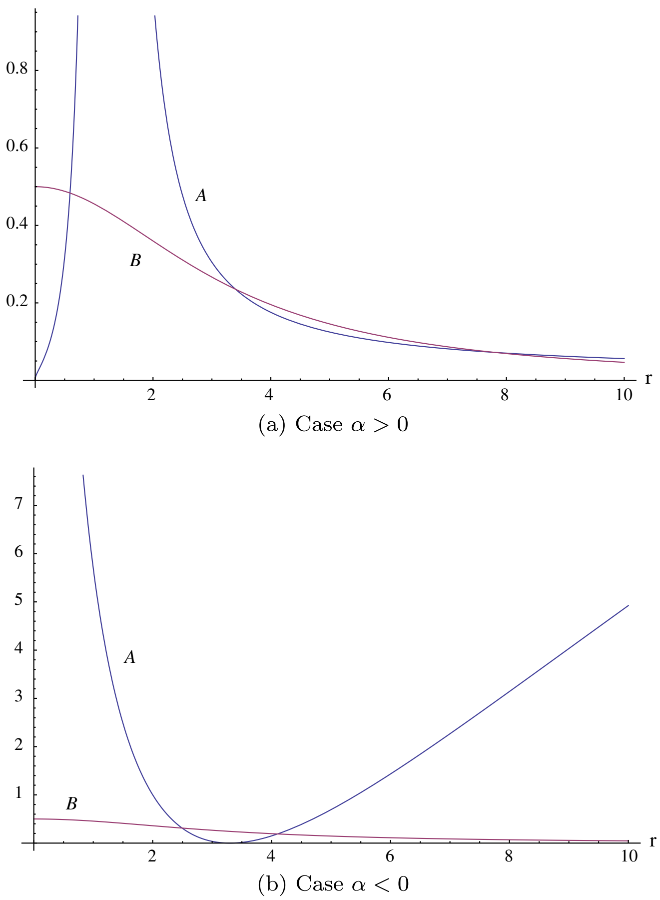


FIG. 2 (color online). The coefficients of the metric (118) for different values of the parameter α. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible and consistent with the choice in Fig. 3.

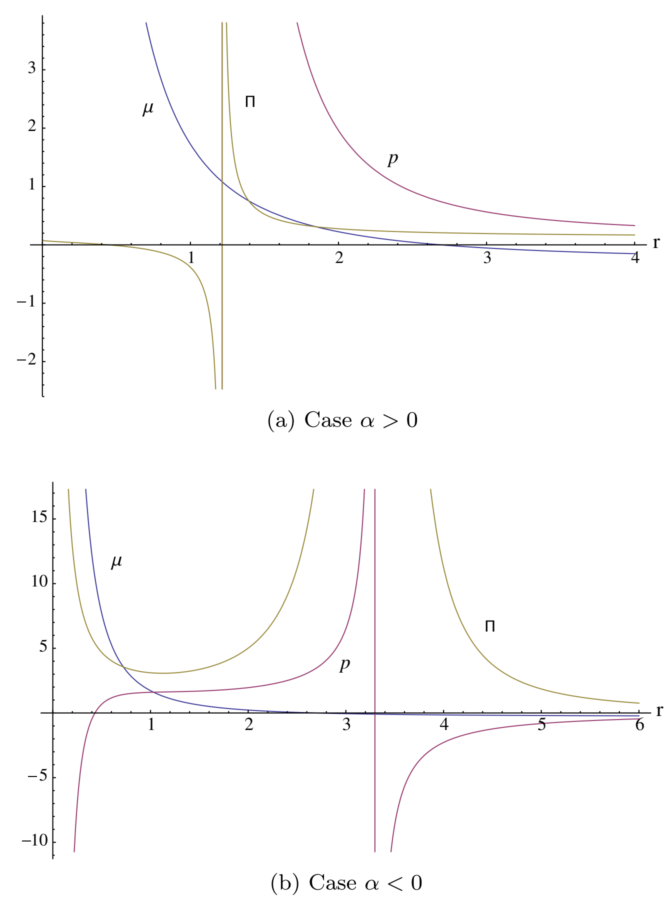


FIG. 3 (color online). The thermodynamic quantities (119) associated with (118) for different values of the parameter α. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible and consistent with the choice in Fig. 2.

VI. RECONSTRUCTING SOLUTIONS COMPLYING WITH SPECIFIC PHYSICAL CONSTRAINTS

The general technique proposed above allows one to find many other exact solutions and they contain even more complicated features. However, we would like to refine the method in order to control the physical properties of the solution. For example, one would like to generate solutions which satisfy a given energy condition or that is asymptotically flat.

Let us consider first the weak energy condition (WEC). In terms of the $1+1+2$ formalism, this condition can be easily stated. Consider an observer that moves with velocity $\bar{u}^a = \gamma(u^a + v^a) = \gamma(u^a + w_1^a + w_2^a)$ with respect to the frame u^a we have chosen. Here v^a is the relative (spatial) velocity measured by u^a so that $v^a u_a = 0$. The vectors w_1^a and w_2^a are the projection along and orthogonal to a given e^a and therefore $w_1^a u_a = 0 = w_2^a u_a$ and $w_1^a g_{ab} w_1^b = 0$. The observer $\bar{u}^a$ will measure an energy density

$$\bar{\mu} = T_{ab}\bar{u}^a\bar{u}^b = \mu + \gamma^2 w_1^2(\mu + p + \Pi) + \gamma^2 w_2^2\left(\mu + p - \frac{1}{2}\Pi\right), \tag{120}$$

where we have used the general $1+1+2$ splitting of the tensor T_{ab} (40). In this way imposing $\bar{\mu} \geq 0$ implies

$$\mu \geq 0, \qquad \mu + p_r = \mu + p + \Pi \geq 0, \qquad \mu + p_\perp = \mu + p - \frac{1}{2}\Pi \geq 0. \tag{121}$$

In terms of Eq. (70) and for $\phi^2 > 0$ the WEC can be written as

$$M \geq 0, \qquad M + P + \mathbb{P} \geq 0, \qquad M + P - \frac{1}{2}\mathbb{P} \geq 0, \tag{122}$$

and using Eqs. (92)–(96) this gives, for any X

$$Y \geq \frac{1}{2}(2X+1), \qquad \mathcal{K} \geq \frac{1}{4}(4X+3), \tag{123}$$

$$Y_{,\rho} \geq \frac{1}{2}(1 - 2\mathcal{K} - 2XY + X - 2Y^2 - Y), \tag{124}$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1+2X). \tag{125}$$

Let us see if we can reconstruct a solution that fulfills WEC. Choosing

$$\mathcal{K} = \frac{1}{4}(4X + 3 + \alpha), \qquad Y = \frac{1}{2}(2X + 1 + \beta), \tag{126}$$

and imposing

$$Y_{,\rho} = \frac{1}{2}(1 - 2\mathcal{K} - 2XY + X - 2Y^2 - Y) + \gamma, \tag{127}$$

it is easy to prove that for $\alpha = 3\beta$ and $\gamma = \frac{1}{4}(\beta + 3)$ the relations above are consistent with the WEC for any choice of X and $\beta \geq 0$ and the equations (127) and (125) coincide. Now substituting our ansatz on Y and $\mathcal{K}$ in either (127) or (96) we obtain

$$X_{,\rho} = -\frac{1}{4}(2X+1)(3 + 3\beta + 4), \tag{128}$$

which gives

$$\frac{1}{2(1+3\beta)}\ln\left|\frac{3(\beta+1) + 4X}{1+2X}\right| = \rho - \rho_0, \tag{129}$$

where ρ_0 is a constant. Choosing the negative branch of this solution gives

$$X = -\frac{1}{2} + \frac{3\beta+1}{2(e^{\frac{1}{2}(3\beta+1)\rho} + 2)}, \tag{130}$$

and therefore

$$Y = \frac{\beta}{2} + \frac{1+3\beta}{2(e^{\frac{1}{2}(3\beta+1)\rho} + 2)}, \tag{131}$$

$$\mathcal{K} = \frac{3\beta+1}{4} + \frac{1+3\beta}{2(e^{\frac{1}{2}(3\beta+1)\rho} + 2)}. \tag{132}$$

Using the definition of Y and $\mathcal{K}$ one has immediately

$$\phi = \pm 2\sqrt{\frac{K_0}{\mathcal{K}_0}} e^{-\frac{3}{4}(\beta+1)\rho}\sqrt{e^{\frac{1}{2}(3\beta+1)\rho} + 2}, \tag{133}$$

$$\mathcal{A} = \frac{\sqrt{K_0}[\beta e^{\frac{1}{4}(\beta+1)\rho} + (\beta+1)e^{-\frac{3}{4}(\beta+1)\rho}]}{\sqrt{\mathcal{K}_0(2 + e^{\frac{1}{2}(3\beta+1)\rho})}}. \tag{134}$$

Passing to the parameter r and using Eqs. (67) and (68) corresponds to the metric

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{135a}$$

$$A = \frac{A_0}{r^{\beta+1}}(2 + r^{3\beta+1}), \tag{135b}$$

$$B = \frac{4K_0}{\mathcal{K}_0}(2 + r^{3\beta+1})^{-1}, \tag{135c}$$

$$C = \frac{4r^2}{K_0}, \tag{135d}$$

and a fluid whose thermodynamics is given by

$$\mu = \frac{K_0}{4\mathcal{K}_0 r^3}[6\beta r^{-3\beta} - (1-4\mathcal{K}_0)r)], \tag{136a}$$

$$p = \frac{K_0 r^{-3\beta-3}}{12\mathcal{K}_0}\{[2\beta(\beta+1) - 4\mathcal{K}_0 + 1]r^{3\beta+1} + 2\beta(2\beta-1)\}, \tag{136b}$$

$$\Pi = -\frac{K_0 r^{-3\beta-3}}{6\mathcal{K}_0}\{[(\beta-2)\beta + 4\mathcal{K}_0 - 1]r^{3\beta+1} + 2\beta(\beta+1)\}. \tag{136c}$$

Note that the energy density is positive for $\mathcal{K}_0 > 1/4$. Calculating the rest of the quantities in (121) one obtains

$$\rho + p_r = \frac{\beta 4K_0(2 + r^{1+3\beta})}{2\mathcal{K}_0 r^{3\beta+1}}, \tag{137}$$

$$\rho + p_\perp = \frac{K_0}{\mathcal{K}_0 r^{3(\beta+1)}}[2(\beta^2 + 4\mathcal{K}_0 - 1)r^{3\beta+1} + \beta(\beta+3)], \tag{138}$$

which are also always positive for $\mathcal{K}_0 > 1/4$. The metric presents no divergences at finite $r \neq 0$, but in $r = 0$ and for $r \to \infty$ the analysis of the Ricci scalar and Kretschmann invariant shows the presence of singularities. It is interesting to notice that the solution (135) satisfies also the strong energy condition for $\mathcal{K}_0 > 0$:

$$\rho + 3p = \frac{2\beta K_0(\beta+1)}{\mathcal{K}_0 r^{3(\beta+1)}}(r^{3\beta+1} + 2). \tag{139}$$

A plot of the behavior of the metric coefficients and of the thermodynamic quantities is given in Fig. 4.

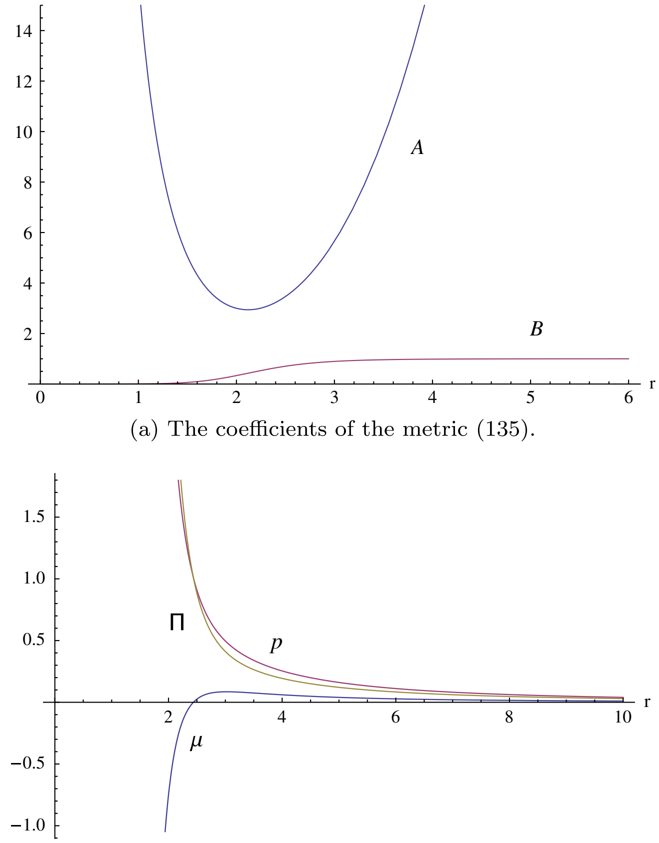
Let us now formulate the concept of asymptotic flatness in terms of our new variables. Loosely speaking, a metric is asymptotically flat if there exists at least a coordinate system in which, at a large value of the radial coordinate, the metric tends to Minkowski i.e. in which the Riemann tensor is identically zero. This implies that in this limit u_a, e_a, and N_{ab} are constant objects and $\mathcal{A}$ and ϕ have to be zero. Unfortunately our definition (70) is not useful to deduce the behavior of the new variables in this limit. However, looking at Eq. (90) it is clear that the conditions corresponding to asymptotic flatness are given by

$$Y \to 0, \qquad X \to 0, \qquad \mathcal{K} \to 0, \tag{140}$$

when $\rho \to \infty$. The behavior of the other variables can be derived as in [8] from the decomposition of the Riemann tensor. In particular, we have that $\{\mu, p, \Pi\} \to 0$ and as consequence $\mathcal{E} \to 0$. Since also $\phi \to 0$ this leaves three possibilities for $\{M, P, \mathbb{P}\}$[6]: they can go to zero or a constant (different from zero) or diverge. Only the second last case is compatible with the constraints, but this possibility also implies that $M < 0$. Assuming B positive and therefore ϕ real [see Eq. (68)], this means $\mu < 0$ which is not physically interesting. This means that no metric can be asymptotically flat and have a positive energy density if we write it in the parameter ρ. Of course, since the concept of asymptotic flatness depends on the choice of the coordinate/affine parameter, one can require that the solution is asymptotically flat in r:

$$\{Y, \mu, p, \Pi\} \to 0, \quad \mathcal{K} \to \frac{K_0}{\phi_0^2} = \frac{1}{4}, \quad X \to -\frac{1}{2} \tag{142}$$

for $\rho \to \infty$.



(a) The coefficients of the metric (135).

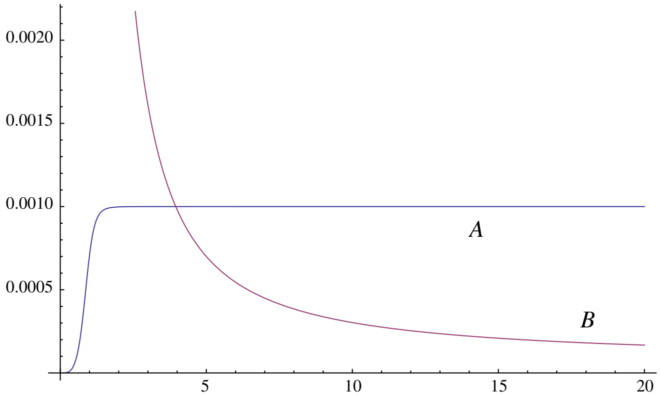
(b) The thermodynamic quantities (136) associated with (135).

FIG. 4 (color online). Graphs of the solution (135) in the case $\beta > 0$. The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

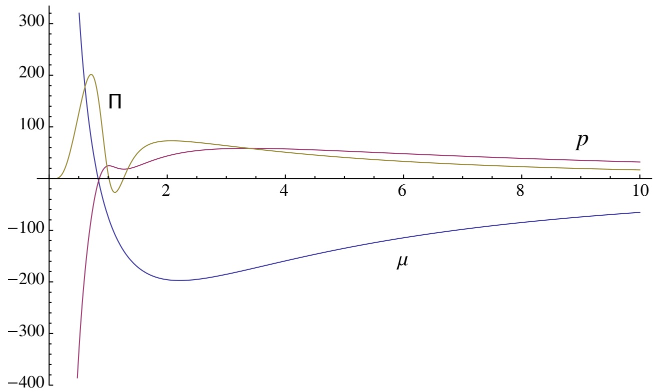
[6]Since (56) holds, one has

$$E \to \frac{1}{3}(M + 3P) + \frac{1}{2}\mathbb{P}, \tag{141}$$

which is essentially Eq. (76) in the limit $Y \to 0$. Therefore E has the same behavior of $\{M, P, \mathbb{P}\}$.



(a) The coefficients of the metric (147).



(b) The thermodynamic quantities (148) associated with (147).

FIG. 5 (color online). Graphs of the solution (147). The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

Let us test the above results reconstructing a solution which is asymptotically flat in r. Equation (142) gives us the prescription for X and Y. We can set for example

$$Y = \frac{1}{e^{\alpha^2\rho} + Y_0}, \qquad X = X_0 \exp(-\beta^2\rho) - \frac{1}{2}. \tag{143}$$

Equation (73) gives immediately

$$\mathcal{K} = \mathcal{K}_0 \exp\left(\frac{2X_0 e^{-\beta^2\rho}}{\beta^2}\right), \tag{144}$$

where, to fulfill (142), we must have $\mathcal{K}_0 = 1/4$. Using the definition of Y and $\mathcal{K}$ one has immediately

$$\phi = \pm 2\sqrt{\frac{K_0}{\mathcal{K}_0}} \exp\left(\frac{\rho}{2} - \frac{X_0 e^{-\beta^2\rho}}{\beta^2}\right), \tag{145}$$

$$\mathcal{A} = -\sqrt{\frac{K_0}{\mathcal{K}_0}} \frac{e^{-\frac{\rho}{2} - \frac{X_0 e^{-\beta^2\rho}}{\beta^2}}}{e^{\alpha^2\rho} + Y_0}. \tag{146}$$

Passing to the parameter r and using Eqs. (67) and (68) corresponds to the metric

$$ds^2 = -A dt^2 + B dr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{147a}$$

$$A = A_0 r^{\frac{4}{Y_0}} (r^{2\alpha^2} + Y_0)^{-\frac{2}{\alpha^2 Y_0}}, \tag{147b}$$

$$B = \frac{4\mathcal{K}_0}{K_0} \exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right), \tag{147c}$$

$$C = \frac{r^2}{K_0}, \tag{147d}$$

and a fluid whose thermodynamics is given by

$$\mu = \frac{K_0}{4\mathcal{K}_0 r^2}\left[4\mathcal{K}_0 - (4X_0 r^{-2\beta^2} + 1)\exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right)\right], \tag{148a}$$

$$p = \frac{K_0}{3\mathcal{K}_0 r^2}\left[X_0 r^{-2\beta^2}\left(\frac{2}{r^{2\alpha^2} + Y_0} - \mathcal{K}_0 \exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right) + 1\right) - \frac{(2\alpha^2 - 1) r^{2\alpha^2}}{(r^{2\alpha^2} + Y_0)^2} + \frac{(Y_0 + 2)}{(r^{2\alpha^2} + Y_0)^2} + \frac{1}{4}\right] \exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right), \tag{148b}$$

$$\Pi = \frac{K_0}{6\mathcal{K}_0 r^2}\left[4\mathcal{K}_0 \exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right) + 2X_0 r^{-2\beta^2}\left(1 + \frac{2}{r^{2\alpha^2} + Y_0}\right) + \frac{4(\alpha^2 Y_0 + 1)}{(r^{2\alpha^2} + Y_0)^2} - \frac{4(\alpha^2 + 1)}{r^{2\alpha^2} + Y_0} - 1\right] \exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right). \tag{148c}$$

Figure 5 shows the metric coefficient and the thermodynamics quantities. Apart from $r = 0$ in which a singularity is present, the metric coefficients are always regular. This implies the presence of a naked singularity. Also the thermodynamics does not present any irregularity. As $r \to \infty$, A and B go to a constant and the thermodynamics quantities approach zero. The metric is then asymptotically flat and the discussion above is confirmed.

VII. DIRECT RESOLUTION OF THE EINSTEIN EQUATIONS WITH A PERFECT FLUID

It is clear from the above examples that the method proposed above is very effective in terms of the reconstruction approach. One might therefore ask if the same is true for the inverse problem i.e. to obtain the metric from a certain matter configuration. This is achieved solving Eqs. (71)–(75), for the variables $(X, Y, \mathcal{K}, Y_{,\rho}, \mathcal{K}_{,\rho})$. The resulting equations are somewhat long and are given in Appendix A. Schematically one has

$$X = X(M, P, \mathbb{P}, P_{,\rho}, \mathbb{P}_{,\rho}), \tag{149}$$

$$Y = Y(M, P, \mathbb{P}, P_{,\rho}, \mathbb{P}_{,\rho}), \tag{150}$$

$$\mathcal{K} = \mathcal{K}(M, P, \mathbb{P}, P_{,\rho}, \mathbb{P}_{,\rho}), \tag{151}$$

$$Y_{,\rho} = f_1(M, P, \mathbb{P}, P_{,\rho}, \mathbb{P}_{,\rho}), \tag{152}$$

$$\mathcal{K}_{,\rho} = f_2(M, P, \mathbb{P}, P_{,\rho}, \mathbb{P}_{,\rho}). \tag{153}$$

An important point is that the above system of equations contains constraints because the ρ derivative of (150) and (151) must be equal to (152) and (153), respectively. We will call these constraints "Y constraint" and "$\mathcal{K}$ constraint." This feature implies that a general combination of functional forms for $(M, P, \mathbb{P})$ does not necessarily correspond to an actual solution of the system so that only a matter distribution with specific features is compatible with the symmetries we have imposed on the metric. However, it is easy to check that, as before, one of the equations above is redundant and can be eliminated, thus one can eliminate either the Y constraint or the $\mathcal{K}$ constraint.

Let us consider some examples. We start with the simple case

$$P_{,\rho} = 0, \qquad \mathbb{P}_{,\rho} = 0, \qquad M_{,\rho} = 0, \tag{154}$$

which corresponds to a mass distribution given by

$$P = P_0, \qquad \mathbb{P} = \mathbb{P}_0, \qquad M = M_0, \tag{155}$$

and we choose $P_0 = 1/12$, $\mathbb{P}_0 = 1/24$, and $M_0 = 1/8$.

Equations (71)–(74) are solved by

$$X = -\frac{1}{2}, \qquad Y = -\frac{1}{4}, \qquad \mathcal{K} = \frac{3}{8}, \tag{156}$$

and, using the definition of Y and $\mathcal{K}$, we can obtain $\mathcal{A}$ and ϕ

$$\mathcal{A} = \frac{\phi_0}{4} e^{-\rho/2}, \quad \phi = \phi_0 e^{-\rho/2}, \quad \mathcal{K} = \frac{3}{8}\phi_0^2 e^{-\rho}. \tag{157}$$

Converting to the parameter r, this corresponds to the thermodynamical variables

$$p = \frac{\phi_0^2}{12r^2}, \qquad \Pi = \frac{\phi_0^2}{24r^2}, \qquad \mu = \frac{\phi_0^2}{8r^2}, \tag{158}$$

and the metric

$$ds^2 = -A_0 r dt^2 + \frac{4}{\phi_0^2} dr^2 + \frac{8}{3\phi_0^2} r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{159}$$

Another way to obtain solutions is to assign only two conditions on the thermodynamics and to use the Y constraint to obtain the third one. For example, choosing

$$P_{,\rho} = \alpha M_{,\rho}, \qquad \mathbb{P}_{,\rho} = \beta M_{,\rho}, \tag{160}$$

where α and β are constants. Integrating we have

$$P = P_0 + \alpha M, \qquad \mathbb{P} = \mathbb{P}_0 + \beta M, \tag{161}$$

where P_0 and $\mathbb{P}_0$ are constants. This equation suggests that matter in this case has an equation of state that resembles the one of a barotropic fluid with the difference that in this case *both* the isotropic and anisotropic pressures are proportional to the energy density. Setting $P_0 = 0$, $\mathbb{P}_0 = 0$, $\alpha = -1/3$, and $\beta = -2/3$, the Y constraint gives the differential equation

$$8MM_{,\rho\rho} - 16M_{,\rho}^2 + 4MM_{,\rho} = 0, \tag{162}$$

which can be solved exactly to give

$$M = \frac{M_0 e^{\rho/2}}{e^{\rho/2} + 2}, \tag{163}$$

This implies immediately

$$X = \frac{1}{-2e^{\rho/2} - 4} - \frac{1}{2}, \qquad Y = \frac{1}{-2e^{\rho/2} - 4}, \tag{164}$$

$$\mathcal{K} = \frac{e^{\rho/2}(4M_0 + 1)}{4(e^{\rho/2} + 2)}, \tag{165}$$

where M_0 is a constant. Using the definition of Y and $\mathcal{K}$ we obtain

$$\mathcal{A} = -\frac{e^{-3\rho/4}\phi_0}{2\sqrt{e^{\rho/2}+2}}, \qquad \phi = \phi_0 e^{-3\rho/4}\sqrt{e^{\rho/2}+2}. \tag{166}$$

Converting to the parameter r, this corresponds to the thermodynamical variables

$$\mu = \frac{\phi_0^2}{8r^2}, \quad p = -\frac{M_0\phi_0^2}{3r^2}, \quad \Pi = -\frac{2M_0\phi_0^2}{3r^2}, \tag{167}$$

and the metric element

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{168a}$$

$$A = A_0\left(1+\frac{2}{r}\right), \tag{168b}$$

$$B = \frac{4}{\phi_0^2}\left(1+\frac{2}{r}\right)^{-1}, \tag{168c}$$

$$C = \frac{4}{\phi_0^2} r^2. \tag{168d}$$

This metric corresponds again to the negative mass Schwarzschild solution obtained in Sec. IV although in this case we are not in vacuum and $\mu > 0$. A singularity at $r = 0$ is evident from the behavior of the Kretschmann invariant and the Ricci scalar and the singularity is naked because it presents no divergences at finite $r \neq 0$. It is also obvious that the solution is asymptotically flat. A plot of the metric and the thermodynamic coefficients is given in Fig. 6.

As a final example let us consider the case

$$\mathbb{P}_{,\rho} = \beta P_{,\rho}, \qquad M_{,\rho} = \alpha M, \tag{169}$$

where α and β are constants. Integrating, we have

$$\mathbb{P} = M_0 + \beta P, \qquad M = \frac{\beta M_0}{\alpha}\exp(\alpha\rho). \tag{170}$$

Choosing $\alpha = 1$, $\beta = -1$, $M_0 = -1/4$, the Y constraint gives

$$\frac{3}{2}(e^\rho - 3)P_{,\rho} + 18P^2 - \frac{3}{4}P(e^{2\rho}+1) + \frac{1}{16}(e^{2\rho}-1) = 0, \tag{171}$$

which admits a solution in terms of generalized Laguerre polynomials

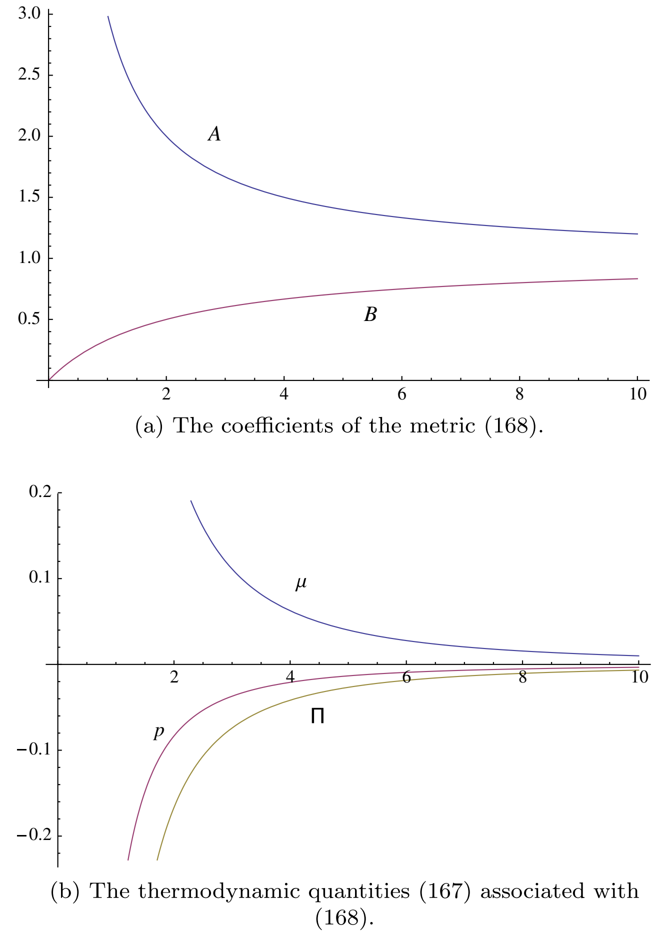


FIG. 6 (color online). Graphs of the solution (168). The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

$$P = \frac{(e^\rho - 1)L_{-1/2}^{1/2}(\frac{e^\rho}{2}) - e^\rho(e^\rho - 3)L_{-3/2}^{1/2}(\frac{e^\rho}{2})}{24L_{-1/2}^{1/2}(\frac{e^\rho}{2})}. \tag{172}$$

This implies

$$X = \frac{1}{4}\left[-\frac{e^\rho L_{-3/2}^{3/2}(\frac{e^\rho}{2})}{L_{-1/2}^{1/2}(\frac{e^\rho}{2})} - e^\rho - 2\right], \tag{173}$$

$$Y = -\frac{e^\rho L_{-3/2}^{3/2}(\frac{e^\rho}{2})}{4L_{-1/2}^{1/2}(\frac{e^\rho}{2})} - \frac{1}{4}, \tag{174}$$

$$\mathcal{K} = \frac{1}{4} - \frac{e^\rho L_{-3/2}^{3/2}(\frac{e^\rho}{2})}{4L_{-1/2}^{1/2}(\frac{e^\rho}{2})}. \tag{175}$$

As before from this result we obtain

$$\phi = \pm 2\sqrt{\frac{K_0 L^{1/2}_{-1/2}(\frac{e^\rho}{2})}{e^\rho L^{1/2}_{-1/2}(\frac{e^\rho}{2}) - e^{2\rho} L^{1/2}_{-1/2}(\frac{e^\rho}{2})}}, \tag{176}$$

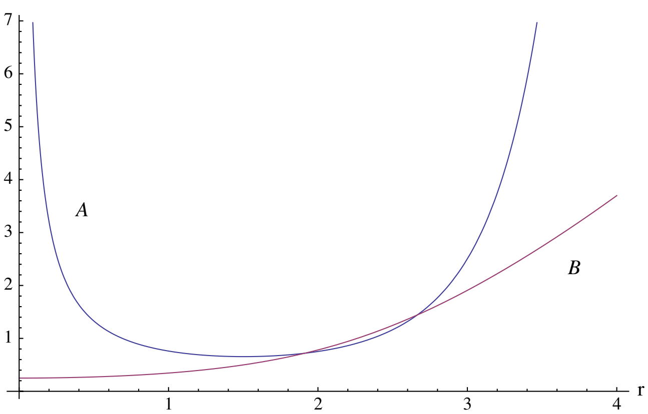
$$\mathcal{A} = \frac{2\sqrt{K_0}e^{-\rho/2}(e^\rho L^{3/2}_{-3/2}(\frac{e^\rho}{2}) + L^{1/2}_{-1/2}(\frac{e^\rho}{2}))}{\sqrt{L^{1/2}_{-1/2}(\frac{e^\rho}{2})}\sqrt{L^{1/2}_{-1/2}(\frac{e^\rho}{2}) - e^\rho L^{3/2}_{-3/2}(\frac{e^\rho}{2})}}. \tag{177}$$

Converting to the parameter r, this corresponds to the thermodynamical variables

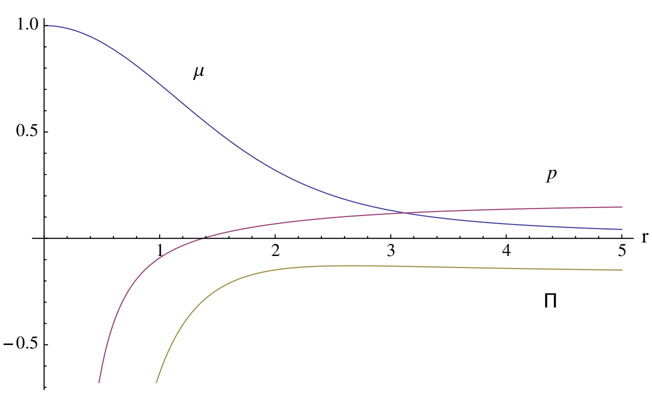
$$\mu = \frac{\sqrt{2}K_0 D_+(\frac{r}{\sqrt{2}})}{r}, \tag{178a}$$

$$p = -\frac{K_0(2\sqrt{2}D_+(\frac{r}{\sqrt{2}}) + r^3 - 3r)}{6r^3}, \tag{178b}$$

$$\Pi = -\frac{K_0(8\sqrt{2}D_+(\frac{r}{\sqrt{2}}) + r^3 - 3r)}{6r^3}, \tag{178c}$$



(a) The coefficients of the metric (180).



(b) The thermodynamic quantities (178) associated with (180).

FIG. 7 (color online). Graphs of the solution (180). The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

where

$$D_+ = e^{-x^2}\int_0^x e^{t^2}dt \tag{179}$$

is the Dawson function [15] and to the metric element

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \tag{180a}$$

$$A = \frac{A_0 L^{1/2}_{-1/2}(\frac{r^2}{2})}{r}, \tag{180b}$$

$$B = \frac{r}{4\sqrt{2}K_0 D_+(\frac{r}{\sqrt{2}})}, \tag{180c}$$

$$C = r^2. \tag{180d}$$

The behavior of the thermodynamical variables and of the metric coefficients are given in Fig. 7. As in the previous example, also in this case the metric coefficients and the thermodynamics is regular and we have a naked singularity.

VIII. RECONSTRUCTING METRICS IN THE PRESENCE OF A SCALAR FIELD

We will consider now the special case in which matter is represented by a real scalar field. The scalar field is, at first sight, easier to deal with than a general perfect fluid, because it presents less unknown variables i.e. the scalar field itself and the potential. In terms of our purposes, however, this fact also means an additional number of constraints, which increase the difficulty of the search for solutions.

In order to specialize the general equation of the previous section to the case of the scalar field we need to write the matter variables μ, p, Π in terms of the scalar field:

$$\mu^\sigma = \frac{1}{2}\hat{\sigma}^2 + V, \tag{181}$$

$$p^\sigma = -\frac{\hat{\sigma}^2}{6} - V, \tag{182}$$

$$\Pi^\sigma = \frac{2}{3}\hat{\sigma}^2. \tag{183}$$

In this way Eqs. (52)–(57) can be written as

$$2\hat{\phi} - 2\mathcal{A}\phi + \phi^2 = -2\hat{\sigma}^2, \tag{184}$$

$$\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi) = -V, \tag{185}$$

$$\hat{\hat{\sigma}} + 2(\mathcal{A} - \phi)\hat{\sigma} - V' = 0, \tag{186}$$

$$4K - \phi^2 - 4\mathcal{A}\phi = -2\phi^2\hat{\sigma}^2 + 4V, \tag{187}$$

$$3\mathcal{E} = -3\mathcal{A}\phi + \hat{\sigma}^2 - 2V, \quad (188)$$

$$\hat{K} = -\phi K. \quad (189)$$

Note that the third equation above is the Klein-Gordon equation, thus no additional equations are needed in this case.

For $\phi \neq 0$ and introducing the parameter ρ and the variables (70) one can write these equations as

$$2X - 2Y + 1 = -\Sigma^2, \quad (190)$$

$$Y_{,\rho} + XY + Y^2 + 1 = -\mathbb{V}, \quad (191)$$

$$\Sigma_{,\rho} + (X + Y + 1)\Sigma - \mathbb{V}_{,\sigma} = 0, \quad (192)$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X), \quad (193)$$

$$3E = -3Y + \Sigma^2 - 2\mathbb{V}, \quad (194)$$

where we have defined

$$\Sigma = \sigma_{,\rho}, \qquad \mathbb{V} = \mathbb{V}(\sigma, \phi) = \frac{V(\sigma)}{\phi^2}, \quad (195)$$

so that

$$V_{,\sigma} = \mathbb{V}_{,\sigma}\phi^2. \quad (196)$$

Using the above equations it is possible to apply the new reconstruction technique also to the scalar field case. As before, one can use the constraints to eliminate one equation of the system. We choose to eliminate the Klein-Gordon equation. The remaining equations can be written, for $\phi \neq 0$, as

$$\Sigma = \pm\sqrt{\frac{2Y - 2X - 1}{2}}, \quad (197)$$

$$\mathbb{V} = \frac{1}{2}(2\mathcal{K} - X - Y - 1), \quad (198)$$

$$2Y_{,\rho} + 2Y^2 + Y + X(2Y - 1) + \mathcal{K} + 1 = 0. \quad (199)$$

Note that in the system above the last equation does not depend directly on the scalar field. In addition, to have a real scalar field one has to guarantee that $2Y - 2X - 1 \geq 0$. This shows that the metrics that are compatible with scalar fields must satisfy some specific constraints that are independent from the form of the scalar field and its potential. Equations (197) and (198) can also be expressed directly in terms of the metric coeffients:

$$\Sigma = \sqrt{\frac{1}{2}\left(\frac{A_{,\rho}}{A} + \frac{B_{,\rho}}{B} - 1\right)}, \quad (200)$$

$$V(\sigma) = \frac{1}{4}\left(-\frac{A_{,\rho}}{AB} - \frac{2}{B} + \frac{B_{\rho}}{B^2} + 4K_0 e^{-\rho}\right). \quad (201)$$

These expression will be useful in the following.

The equations above make clear an additional difficulty of the reconstruction process: one has to integrate Eq. (200). This will be a major problem in the search for exact solutions: even if one manages to solve Eq. (199), there is still the possibility that Eq. (200) or equivalently (197) will not be easily integrable.

As a first example, we will reconstruct a solution for a real scalar field starting from an ansatz on the variables $X, Y, \mathcal{K}$ as in the previous sections. The real nature of the scalar field can be guaranteed by setting, for example,

$$X = -1 + Y \leq -\frac{1}{2} + Y. \quad (202)$$

Substituting into (192) and (193), integrating and choosing the positive branch of the solution, one obtains

$$Y = \frac{e^{\rho}\rho}{2e^{\rho}(\rho - 1) - 2}, \qquad \mathcal{K} = \frac{e^{\rho}}{2(1 - e^{\rho}\rho + e^{\rho})}. \quad (203)$$

Using the definition of Y and $\mathcal{K}$ one has immediately

$$\phi = \pm e^{-\rho}\sqrt{2K_0[1 - e^{\rho}(\rho - 1)]}, \quad (204)$$

$$\mathcal{A} = -\sqrt{\frac{\rho^2 K_0}{2 - 2e^{\rho}(\rho - 1)}}. \quad (205)$$

In terms of r and using Eqs. (67) and (68) corresponds to the metric

$$ds^2 = -Adt^2 + Bdr^2 + C(d\theta^2 + \sin^2\theta d\phi^2), \quad (206a)$$

$$A = A_0[r^2 - 2r^2 \ln r + 1], \quad (206b)$$

$$B = \frac{2r^2}{K_0[r^2 - 2r^2 \ln r + 1]}, \quad (206c)$$

$$C = \frac{r^2}{K_0}, \quad (206d)$$

and from Eqs. (197)–(198)

$$\sigma = \frac{\sqrt{2}}{2}\ln(r), \quad (207)$$

$$V = K_0 e^{-\sqrt{2}\sigma}(\sqrt{2}\sigma + 1). \quad (208)$$

The metric coefficients for this solution are rational functions of r. This implies that the divergences of B are zeros for A and the zeros of B are divergences of A, much in the same way of the Schwarzschild solution. The analysis of the Ricci scalar and the Kretschmann invariant shows that, like in the Schwarzschild case, there is no real singularity in the solution, apart from, of course, $r=0$. In this respect, therefore, the solution above seems to represent a spatially symmetric scalar field sourced solution embedded in an expanding homogeneous and isotropic space-time via a horizon structure. This case looks conceptually similar to the Swiss cheese models [4,16], but with the difference to not require special junction condition. Plots of the metric coefficients, the scalar field and the potential are given in Figure 8.

Another example can be given looking carefully at the structure of the constraint (199) and expressing A as function of B so that this constraint becomes an equation for the coefficient B which admits an analytical solution. For example, setting

$$\frac{A_{,\rho}}{A}+\frac{B_{,\rho}}{B}-1=-\frac{3}{1-e^{3\rho/2}}, \tag{209}$$

we have

$$A=\frac{A_0e^{-2\rho}(1-e^{3\rho/2})^2}{B(\rho)}. \tag{210}$$

Using the relations (89) the (199) give a solution for metric component B:

$$B=\frac{120e^{\rho}(e^{3\rho/2}-1)^2}{40B_0e^{4\rho}-60C_0(2e^{\rho}-4e^{5\rho/2}+2e^{4\rho})-3\rho_0(8e^{3\rho/2}-40e^{3\rho}+5)}. \tag{211}$$

Passing t to the parameter r one obtains for the metric coefficients

$$ds^2=-Adt^2+Bdr^2+C(d\theta^2+\sin^2\theta d\phi^2), \tag{212}$$

$$A=\frac{A_0(r^3-8)^2}{r^6B}=\frac{1}{12}A_0(B_0-3B_1)r^2+4A_0K_0+\frac{4A_0B_1}{r}-\frac{32A_0K_0}{5r^3}-\frac{16A_0B_1}{r^4}-\frac{32A_0K_0}{r^6}, \tag{213}$$

$$B=\frac{480(r^3-1)^2}{40B_0r^8-3[40C_0(r^4-r)^2+K_0(-40r^6+8r^3+5)]}, \tag{214}$$

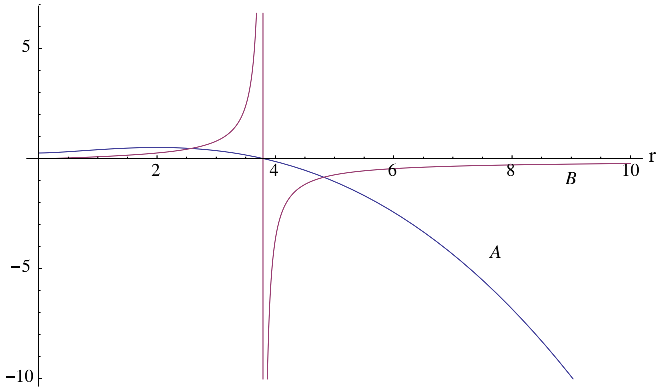
$$C=\frac{r^2}{K_0}, \tag{215}$$

and, using (200), one obtains

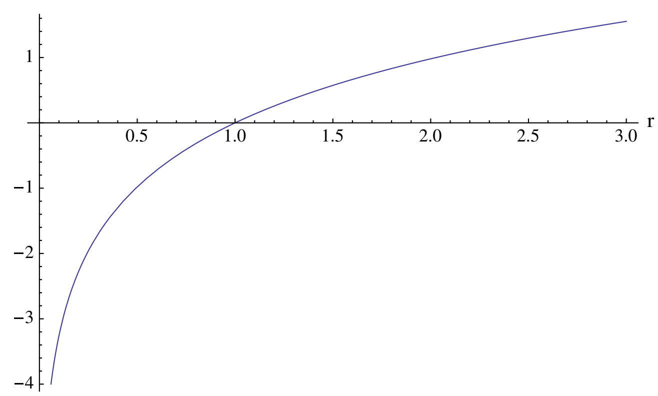
$$\sigma=2\sqrt{\frac{2}{3}}\arctan\left(\sqrt{r^3-1}\right), \tag{216}$$

$$V=20\csc^6\left(\frac{1}{2}\sqrt{\frac{3}{2}}\sigma\right)\left\{4(2B_0-3C_0)\cos\left(\sqrt{\frac{3}{2}}\sigma\right)+3C_0[\cos(\sqrt{6}\sigma)+3]\right\}$$
$$+\frac{3K_0\cos^{\frac{10}{3}}\left(\frac{1}{2}\sqrt{\frac{3}{2}}\sigma\right)}{640\sin^6\left(\frac{1}{2}\sqrt{\frac{3}{2}}\sigma\right)}\left[38\cos\left(\sqrt{\frac{3}{2}}\sigma\right)-5\cos(\sqrt{6}\sigma)+75\right]. \tag{217}$$

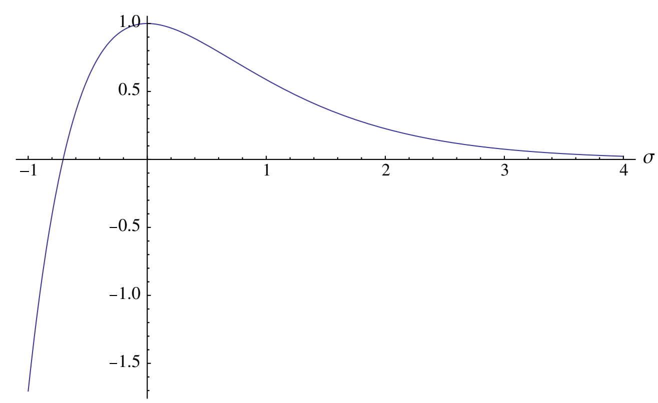
The metric we obtained is again characterized by a divergence in the coefficient B that corresponds to a zero in the coefficient A. The denominator of the coefficient B is a polynomial of the eighth order and consequently we can expect at most eight singular points. The Ricci scalar, the Kretschmann invariant, the scalar field, and its potential are finite in these points so that these singularities are probably related to the horizon structure of the



(a) The coefficients A and B of (206).

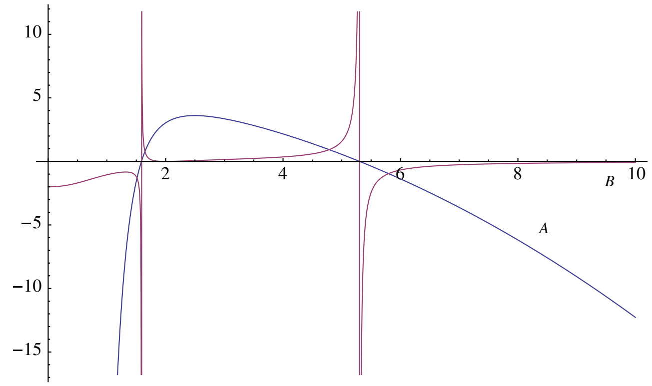


(b) The behavior of the scalar field (207) associated with the metric (206).

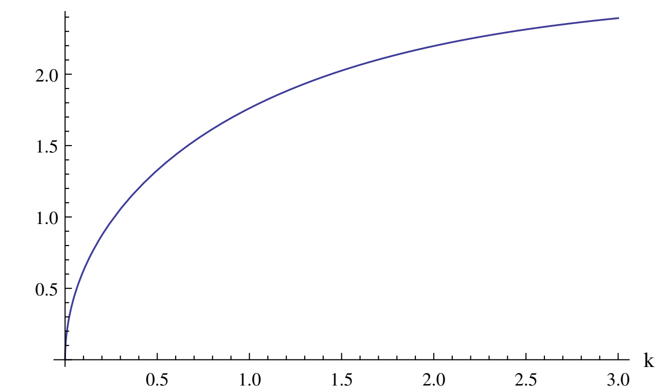


(c) The behavior of the potential (208) associated with the metric (206).

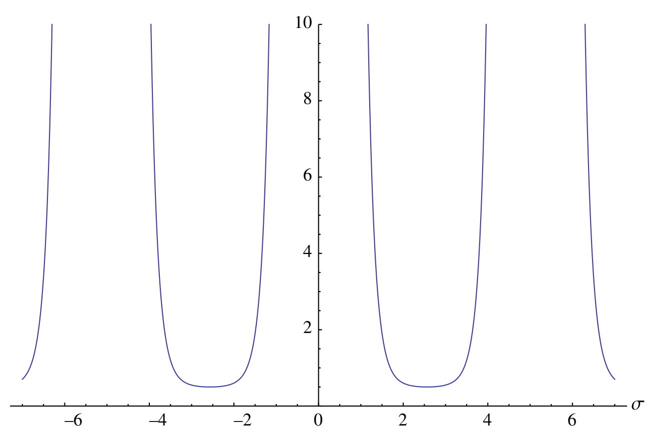
FIG. 8 (color online). Graphs of the solution (206). The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.



(a) The coefficients A and B of (212).



(b) The behavior of the scalar field (216) associated with the metric (212).



(c) The behavior of the potential (217) associated with the metric (212).

FIG. 9 (color online). Graphs of the solution (212). The values of the parameters have been chosen in such a way to make the features of the solution as clear as possible.

space-time. In Fig. 9, an example is given in which only two of these singularities are apparent. In $r = 0$ the actual presence of a singularity in the gravitational field is confirmed by the behavior of the Ricci scalar and the Kretschmann invariant.

IX. CONCLUSIONS

In this paper we have presented a new reconstruction technique that can be used to obtain static spherically symmetric solutions in general relativity. Defining a special affine parameter and new variables in the $1+1+2$

covariant approach framework, we have been able to obtain a new form of the covariant equations, which is more suitable for the application of the reconstruction paradigm.

As in cosmology this strategy has revealed itself very useful to obtain nontrivial results. In fact the new method can be also considered more efficient than its cosmological counterpart as one is able to achieve even more than one reconstructed solution from a given variable prescription. Another advantage of the method proposed is that it gives the possibility to encode naturally in the reconstruction process also some physically relevant constraints like the fulfillment of the energy conditions and the asymptotic flatness. This possibility makes the new method unique among the ones devised to obtain spherically symmetric solutions of the Einstein equations.

The set of variables devised for the reconstruction can also be used for the direct resolution of the Einstein equations in the presence of matter. In the new formalism the main obstacle to the achievement of a solution is the fulfillment of complicated first-order constraints. We have managed to solve these equations in some simple cases and in one example, we also found a solution in terms of Legendre polynomials. The possibility of considering solutions containing special functions increases enormously the richness of the physics of the reconstructed spacetime.

The new method has been also applied to the case of matter composed by a scalar field, finding some new exact solutions. When a scalar field is present, the search for exact solutions is further complicated because the behavior of the scalar field can be only found via an integration, and in many physically interesting cases such integration is very hard to achieve. Of course one can seek for numerical solutions for these cases, but they have not been considered here as we focused on exact solutions. Such result was somehow expected: exact static and spherically symmetric solutions sourced by a scalar field with nontrivial potential are very rare and very few are known so far. In this respect the new reconstruction method helps in the investigation of these solutions and reveals that they might reserve some surprises especially correlated to the assumption that the scalar field is real.

The solutions we have reconstructed as examples are only a small set of the ones that can be obtained with the new method. They have been chosen mainly in terms of their simplicity and obvious interesting features. Since it was not our purpose to give a complete analysis of their properties a very limited exploration of their features has been made in the text. A future work will be dedicated to a detailed analysis of these features. However, even using the little information obtained above it is possible to draw some interesting general conclusion. The first concerns the form of the metric coefficients A and B. The "classic" static and spherically symmetric metric in which the parameter $A = B^{-1}$ does not seem to be common in our results. However, from the results above it is evident that metrics in which $A = f(r)B^{-\alpha}$ with $\alpha > 0$ and $f(r)$ regular seems to have a special physical meaning as it prevents the formation of naked singularities. The second is related to the horizon structures that cover the singularities. We use here the wording "horizon structure" because the determination of the actual presence and the nature of the horizons in the solution we found would require an investigation that is way beyond the purposes of this work (see [17] for a review on these topics). While we are left with many open questions on the actual nature of these structures they also show interesting features. For example in the solution (206)–(208) the horizon was found to be the junction between the spherically symmetric solution and an expanding space-time. This result suggests a new potential way to approach the problem of "embedding" static and spherically symmetric spacetime in cosmological manifolds, which definitely deserves further study.

ACKNOWLEDGMENTS

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APPENDIX: EQUATIONS (71)–(75) SOLVED IN TERMS OF THE GEOMETRIC VARIABLES

Solving (71)–(75) for the variables $(X, Y, \mathcal{K}, Y_{,\rho}, \mathcal{K}_{,\rho})$ one obtains

$$X = -\frac{P(4M + 4\mathbb{P} + 1) + 2(M + \mathbb{P})^2 + M + 2P^2 + 2(P_{,\rho} + \mathbb{P}_{,\rho} + 2\mathbb{P})}{2(M + 3P + 3\mathbb{P})}, \tag{A1}$$

$$Y = \frac{P(4M + 8\mathbb{P} + 2) + \mathbb{P}(4M + 4\mathbb{P} - 1) + 4P^2 - 2(P_{,\rho} + \mathbb{P}_{,\rho})}{2(M + 3P + 3\mathbb{P})}, \tag{A2}$$

$$\mathcal{K} = \frac{P(4M - 8\mathbb{P} + 7) + 4\mathbb{P}(M - \mathbb{P}) + M - 4P^2 - 4(P_{,\rho} + \mathbb{P}_{,\rho}) + \mathbb{P}}{4(M + 3P + 3\mathbb{P})}, \tag{A3}$$

$$
\begin{aligned}
Y_{,\rho} = {} & \frac{M^3(8\mathbb{P}+2)+2P^2(4M^2+M(11-12\mathbb{P})+10P_{,\rho}+10\mathbb{P}_{,\rho}+3(7-8\mathbb{P})\mathbb{P}-7)}{4(M+3P+3\mathbb{P})^2} \\
& + \frac{M^2[8\mathbb{P}^2-4(P_{,\rho}+\mathbb{P}_{,\rho})6\mathbb{P}]+P^3(-8M-32\mathbb{P}+22)-8P^4}{4(M+3P+3\mathbb{P})^2} \\
& + \frac{P[8M^3+2M^2(8\mathbb{P}+9)+2M(8P_{,\rho}+8\mathbb{P}_{,\rho}+4(4-3\mathbb{P})\mathbb{P}-1)]}{4(M+3P+3\mathbb{P})^2} \\
& + \frac{P\mathbb{P}(-32\mathbb{P}^2+40P_{,\rho}+40\mathbb{P}_{,\rho}+18\mathbb{P}+5)+22(P_{,\rho}+\mathbb{P}_{,\rho})}{4(M+3P+3\mathbb{P})^2} \\
& + \frac{M[2(P_{,\rho}+\mathbb{P}_{,\rho})+\mathbb{P}(16P_{,\rho}+16\mathbb{P}_{,\rho}+2(5-4\mathbb{P})\mathbb{P}+1)]}{4(M+3P+3\mathbb{P})^2} \\
& - \frac{[2\mathbb{P}^2-4(P_{,\rho}+\mathbb{P}_{,\rho})+\mathbb{P}][\mathbb{P}(4\mathbb{P}-1)-2(P_{,\rho}+\mathbb{P}_{,\rho})]}{4(M+3P+3\mathbb{P})^2},
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
\mathcal{K}_{,\rho} = {} & -\frac{2[P(2M+2\mathbb{P}-1)+(M+\mathbb{P})^2+P^2]+2(P_{,\rho}+\mathbb{P}_{,\rho})+\mathbb{P}}{4(M+3P+3\mathbb{P})^2} \\
& \times \frac{P(-4M+8\mathbb{P}-7)-M(4\mathbb{P}+1)+4P^2+4(P_{,\rho}+\mathbb{P}_{,\rho})+\mathbb{P}(4\mathbb{P}-1)}{4(M+3P+3\mathbb{P})^2}.
\end{aligned} \tag{A5}
$$

These equations are completely equivalent to the full Einstein equations when the staticity and spherical symmetry have been imposed.

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