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Non-minimally coupled condensate cosmologies: a phase space analysis

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Abstract

We present an analysis of the phase space of cosmological models based on a non-minimal coupling between the geometry and a fermionic condensate. We observe that the strong constraint coming from the Dirac equations allows a detailed design of the cosmology of these models, and at the same time guarantees an evolution towards a state indistinguishable from general relativistic cosmological models. In this light, we show in detail how the use of some specific potentials can naturally reproduce a phase of accelerated expansion. In particular, we find for the first time that an exponential potential is able to induce two de Sitter phases separated by a power law expansion, which could be an interesting model for the unification of an inflationary phase and a dark energy era.

Keywords: fermion condensate, cosmology, dynamical systems approach, dark energy, non-minimal coupling
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1. Introduction

One of the fundamental initial hypotheses in the formulation of any relativistic theory of gravitation is the way in which matter couples to geometry. In the first prototype of these theories, general relativity (GR), one chooses the so-called minimal coupling; that is, matter and geometry only couple via the determinant of the metric tensor which, in the definition of the matter action, multiplies the matter Lagrangian density.

However, it has been known for a long time now that in the semiclassical approach to quantum gravity, also called quantum field theory on curved spacetimes [1], the appearance of non-minimal couplings (NMC) is inevitable if one renormalizes the quantum stress energy tensor of the matter fields [2]. NMCs also appear in the low energy limit of a number of fundamental theories in particle physics, such as the tree-level action of string theory [3] and the attempt to more naturally include Mach's principle in the framework of Einstein's gravitation, as in Brans-Dicke theory [4]. The study of non-minimally coupled theories has accompanied the development of our understanding of Einstein's gravitational theory, but only in the last thirty years has the potential importance of the role of NMCs in inflationary and dark energy models become clear.

In spite of these interesting features, the introduction of NMCs also brings some fundamental problems, which have so far remained unsolved. For example, it is clear that, since NMCs prescribe a different way in which these matter fields couple to gravity (in addition to the one given by the Einstein equations), the presence of NMCs violates the strong equivalence principle. It has also become clear that some types of NMCs can be responsible for the appearance of ghosts and tachyons in the particle spectrum of gravitational theories [5]. Last but not least, the NMC is often modeled using a scalar field, whose nature has so far remained obscure[3].

In this paper we will explore the possibility that the scalar field typical of NMCs is not fundamental, but rather is represented by a fermion condensate.

The idea that scalar fields can be non-fundamental and therefore composed by other fields (like spinors) was already proposed in the context of particle physics by Weinberg with specific reference to the Higgs field [8] (see also [9]).

Compared to scalar fields, spinors have received much less attention; only as recently as the 1990s were first full spinor cosmological solutions considered [10]. Since then, the non-trivial properties of minimally coupled spinors have been used to model both inflation [11] and, with the discovery of cosmic acceleration, dark energy [12–14].

A number of works have also considered the possibility that the main action of the fermions in the cosmological context is in the form of a condensate [15, 16]. The formation of fermion condensates (also in terms of Bardeen–Cooper–Schrieffer (BCS) theory) has proven to be fundamentally related to the vacuum state(s) of quantum chromodynamics (QCD) [17]. The motivation behind the exploration of this topic stems from the observation that, within standard Big Bang cosmology, the dynamics of the early universe have been dominated by the primordial quark-gluon plasma (QGP). As the universe expands, QGP goes through a phase transition connected to the breaking of chiral symmetry that involves the formation of condensates of scalar pion-like (fermion–antifermion) particles. This is considered, together with the electroweak phase transition, to be the chief mechanism by which quarks acquire mass [18]. In this case, therefore, a possible candidate for the field we shall consider is a quark[4]. A realization of this mechanism, based on supersymmetric theories, has been also given in [13] considering the condensation of N_f flavors in a $SU(N_c)$ gauge theory. Formation of condensates at late time is also thought to be possible via the mediation of the torsion so that other fields, like neutrinos, can condensate as well [14]. In this paper, however, we will

[3] Only recently, with the discovery of the Higgs field, has a non-minimally coupled theory of this field with the geometry been considered [6] which might work as a model for inflation. The minimally coupled Higgs field is known not to be a good candidate in this respect [7].

[4] It is important to stress, however, that in this paper, as in the majority of papers on this topic, we will consider a classical spinor field (i.e., a set of four complex-valued spacetime functions) that transform according to the spinor representation of the Lorentz group. This is already a strong approximation, as the real spin 1/2 fermions are described by quantum spinor fields, and there is no classical limit for fundamental quantum Fermi fields. A way out of this difficulty is to imagine that classical spinors arise from an effective description of more complex quantum system [12].

neglect torsion and we will proceed with the idea that the condensate is generated by the breaking of the chiral symmetry in the QGP setting in the early universe. The use of NMC coupling with the gravitation interaction appears natural in this setting as it classically approximates the quantum corrections nature of the gravitational interaction.

Our analysis shows that, at early times, a fermion condensae could combine with the NMC to originate accelerated expansion independently from the condensate potential. The potential becomes relevant, instead, at late time. In fact, we will see that the condensate is forced to go monotonically to zero by the Dirac equation. Therefore, a suitable choice of the potential can lead to the emergence of an effective cosmological constant. Since in this state the condensate is effectively a (null) constant, it turns out that in this dynamical state, the theory is indistinguishable from general relativity at the pure Friedmann level.

We will also see that the Dirac equation constrains the behavior of the condensate in such a way that it becomes straightforward to control the properties of the cosmological models. Such 'model design' naturally realizes a feature which was sought for a long time in the context of fourth-order gravity [19].

The analysis we will perform is based on the dynamical systems approach (DSA). Defined by the work of Collins, Wainwright and Ellis, this technique is a precious tool for the understanding of complex cosmological models in the framework of GR [20, 21], as well as extensions of Einstein theory [22]. DSA consists of recasting the cosmological equations as an autonomous system of differential equations for some tailored variables that also carry physical meaning. Using DSA, one is able to unfold in a relatively easy way a number of important aspects of cosmological models, including the general behavior of the cosmology as well as the evolution of the shear and the occurrence of bounces. In some cases in which free functions appear in the theory, the DSA has helped to select the structure of such functions [25].

In the following, we will explore the phase space of cosmological models in which a non-minimally coupled fermion condensate exists, and in which the fermions have a self-interaction potential which depends only on the condensate itself. We will also consider an additional form of matter described by a perfect fluid. Although the method is completely general, we will choose some simple, specific forms of this potential as examples. Among other results, our analysis shows that these models, as other 'scalar tensor' models, can present the phenomenon of the 'isotropization' to GR [26], and at the same time show some peculiar properties which might be used as a framework for dark energy models and/or inflation.

The paper will be divided in the following way. Section 2 is a brief review of the general details of the properties of the theory we will consider in this paper and its key equations in the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. Some exact results in vacuum are also presented. Section 3 deals with the dynamical systems analysis in both the presence and absence of a perfect fluid, and for different potentials. Finally, section 4 gives our conclusions.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper; Latin and Greek indices run from 0 to 3. The symbol ∇ represents the Levi–Civita covariant derivative associated with a metric tensor, g_{ij}. We use the $+, -, -, -$ signature for the metric tensor, and the Riemann tensor is defined by

$$R^{\mathrm{d}}{}_{cab} = \partial_a \Gamma^{\mathrm{d}}_{bc} - \partial_b \Gamma^{\mathrm{d}}_{ac} + \Gamma^{\mathrm{d}}_{ap}\Gamma^{p}_{bc} - \Gamma^{h}_{bp}\Gamma^{p}_{ac};\,, \tag{1}$$

where the Γ^c_{ab} are the Christoffel symbols associated with the metric g_{ij}, defined by

$$\nabla_{\partial_a}\partial_b = \Gamma_{ab}{}^{c}\,\partial_c\,. \tag{2}$$

The Ricci tensor is obtained by contracting the *first* and the *third* index via the metric g_{ab}:

$$R_{ab} = R^{c}{}_{acb}\,. \tag{3}$$

2. The $(1+\epsilon\bar{\psi}\psi)$ *R*-theory in the FLRW metric

Let us consider an action of the form (also proposed in a more complicated form in [16, 29])

$$S = \int \sqrt{|g|}\Big[\big(1 + \epsilon\bar{\psi}\psi\big)R - L_D\Big]\mathrm{d}s, \tag{4}$$

where the Einstein–Hilbert term is non–minimally coupled to the condensate, $\bar{\psi}\psi$, of a Dirac field whose Lagrangian has the form

$$L_D = \frac{i}{2}\Big(\bar{\psi}\Gamma^i D_i\psi - D_i\bar{\psi}\Gamma^i\psi\Big) - m\bar{\psi}\psi + V(\bar{\psi}\psi), \tag{5}$$

where a fermionic self–interaction potential, $V(\bar{\psi}\psi)$, is present. In equation (4), ϵ indicates a suitable constant parameter. In equation (5) we have $\Gamma^i := e^i_\mu\gamma^\mu\,\gamma^\mu$ representing Dirac matrices and $e_i^{\ \mu}$ a tetrad field such that $g_{ij} = e_i^{\ \mu} e_j^{\ \nu}\eta_{\mu\nu}$ and the D_i indicate covariant derivatives of the spinor field

$$D_i\psi = \partial_i\psi - \Omega_i\psi, \tag{6}$$

$$D_i\bar{\psi} = \partial_i\bar{\psi} + \bar{\psi}\Omega_i, \tag{7}$$

where

$$\Omega_i = -\frac{1}{4}g_{ij}\Big(\Gamma^j_{pq} - e^j_\mu\partial_p e^\mu_q\Big)\Gamma^p\Gamma^q. \tag{8}$$

Introducing for simplicity the notation $\varphi := \bar{\psi}\psi$, it is easily seen that action (4) yields Einstein–like field equations of the form

$$(1 + \epsilon\varphi)\bigg(R_{ij} - \frac{1}{2}Rg_{ij}\bigg) = \Sigma_{ij} + \epsilon\Big(\nabla_i\nabla_j\varphi - g_{ij}g^{pq}\nabla_p\nabla_q\varphi\Big), \tag{9}$$

where

$$\Sigma_{ij} = \frac{i}{4}\Big(\bar{\psi}\Gamma_{(i}D_{j)}\psi - D_{(i}\bar{\psi}\Gamma_{j)}\psi\Big) - \frac{1}{2}L_D\,g_{ij}, \tag{10}$$

is the energy–momentum tensor of the Dirac field, obtained by variation of the Dirac Lagrangian, L_D, with respect to the tetrad field. At the same time, from action (4) we derive Dirac equations for the spinor field of the form

$$i\Gamma^i D_i\psi - m\psi + V'(\varphi)\psi - \epsilon\psi R = 0, \tag{11}$$

$$iD_i\bar{\psi}\Gamma^i + m\bar{\psi} - V'(\varphi)\bar{\psi} + \epsilon\bar{\psi}R = 0, \tag{12}$$

where $V' = \frac{\mathrm{d}V}{\mathrm{d}\varphi}$. Making use of equations (11) and (12) we can express the energy–momentum tensor (10) as

$$\Sigma_{ij} = \frac{i}{4}\left(\bar{\psi}\Gamma_{(i}D_{j)}\psi - D_{(i}\bar{\psi}\Gamma_{j)}\psi\right) - \frac{\epsilon}{2}\varphi R\, g_{ij} - \frac{1}{2}V(\varphi)g_{ij} + \frac{1}{2}\varphi V'(\varphi)g_{ij}. \tag{13}$$

Now, let us consider a spatially flat FLRW metric tensor,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{14}$$

The tetrad field associated with metric (14) is expressed as

$$e_0^\mu = \delta_0^\mu, \qquad e_A^\mu = a(t)\delta_A^\mu, \qquad A = 1, 2, 3. \tag{15}$$

From this, it is easily seen that the $\Gamma^i = e_\mu^i\gamma^\mu$ matrices are given by

$$\Gamma^0 = \gamma^0, \quad \Gamma^A = \frac{1}{a(t)}\delta_\mu^A\gamma^\mu. \tag{16}$$

It is also easy to see that the coefficients of the spin connection are

$$\Omega_0 = 0, \qquad \Omega_A = \frac{\dot{a}}{2}\gamma^A\gamma^0, \qquad A = 1, 2, 3. \tag{17}$$

Due to equations (14)–(17), in metric (14) the Dirac equations (11) and (12) assume the form

$$\dot{\psi} + \frac{3}{2}\frac{\dot{a}}{a}\psi + im\gamma^0\psi - V'(\varphi)\gamma^0\psi + i\epsilon R\gamma^0\psi = 0, \tag{18}$$

$$\dot{\bar{\psi}} + \frac{3}{2}\frac{\dot{a}}{a}\bar{\psi} - im\bar{\psi}\gamma^0 + V'(\varphi)\bar{\psi}\gamma^0 - i\epsilon R\bar{\psi}\gamma^0 = 0. \tag{19}$$

From equations (18) and (19) we derive the evolution law for the scalar field, $\varphi = \bar{\psi}\psi$:

$$\dot{\varphi} + 3\frac{\dot{a}}{a}\varphi = 0, \tag{20}$$

thus obtaining the final relation

$$\varphi = \frac{\varphi_0}{a^3}. \tag{21}$$

This result, found in [12] and successively in [23], is independent of the form of the gravitational action and constitutes a very tight constraint on the entire theory. Since one knows that $\varphi \to 0$ when the scale factor grows, the non-minimal coupling can be used as a 'switch' at the action level to regulate the onset of the different terms of the Lagrangian. This is not possible in standard scalar tensor theories because in those cases, the behavior of the scalar field is described by a Klein–Gordon equation, whose solution is in general much more complicated.

Making use again of equations (14)–(19), it is a straightforward matter to verify that the non-vanishing components of the fermionic energy–momentum tensor (13) are represented by

$$\left(\Sigma_D\right)_{00} = \frac{m}{2}\varphi - \frac{1}{2}V(\varphi), \tag{22}$$

$$\left(\Sigma_D\right)_{AA} = \frac{\epsilon}{2}\varphi R a^2 + \frac{1}{2}V(\varphi)a^2 - \frac{1}{2}\varphi V'(\varphi)a^2 \qquad A = 1, 2, 3. \tag{23}$$

Inserting the content of equations (22) and (23) into equation (9), the Einstein–like equations assume the following expression:

$$(1+\epsilon\varphi)3\left(\frac{\dot a}{a}\right)^2=\frac{m}{2}\varphi-3\epsilon\frac{\dot a}{a}\dot\varphi-\frac{1}{2}V(\varphi), \tag{24}$$

$$(1+\epsilon\varphi)\left[2\frac{\ddot a}{a}+\left(\frac{\dot a}{a}\right)^2\right]=-\frac{\epsilon}{2}\varphi R-\epsilon\ddot\varphi-2\epsilon\frac{\dot a}{a}\dot\varphi-\frac{1}{2}V(\varphi)+\frac{1}{2}\varphi V'(\varphi). \tag{25}$$

We can replace equation (25) by the equivalent Raychaudhuri equation,

$$(1+\epsilon\varphi)6\frac{\ddot a}{a}=-\frac{3}{2}\epsilon\varphi R-3\epsilon\ddot\varphi-3\epsilon\frac{\dot a}{a}\dot\varphi-\frac{m}{2}\varphi-V(\varphi)+\frac{3}{2}\varphi V'(\varphi). \tag{26}$$

In the absence of the self–interaction potential ($V(\varphi)=0$), by inserting (21) into (24) we obtain the final differential equation,

$$\dot a^2=\frac{m}{2}\left[\frac{3a}{\varphi_0}-\frac{6\epsilon}{a^2}\right]^{-1}. \tag{27}$$

For values $a\ll 1$, equation (27) can be approximated to

$$\dot a^2=-\frac{ma^2}{12\epsilon}. \tag{28}$$

This solution for $\epsilon<0$ admits a de Sitter solution, $a(t)=a_0\exp(\lambda t)$ with $\lambda:=\sqrt{\frac{m}{-12\epsilon}}$, that can describe inflationary models or dark energy eras. Instead, for $\epsilon>0$, the real part of the solution of (28) becomes oscillatory. In the case $a\gg 1$ equation (27) can be approximated to

$$\dot a^2=\frac{m\varphi_0}{6a}, \tag{29}$$

which possesses a Friedmann solution, $a(t)=\left[\frac{3}{2}(\lambda t+c)\right]^{\frac{2}{3}}$ with $\lambda:=\sqrt{\frac{m\varphi_0}{6}}$. This means that in this case, the theory can describe a transition from a dark energy era (a period characterized by accelerated expansion) to a Friedmann one (a period characterized by a decelerated expansion).

In order to add a perfect fluid to our cosmological model, we suppose a barotropic perfect fluid assigned, with equation of state $p=w\rho$ ($w\in[0,1]$) and standard conservation law

$$\dot\rho+3\frac{\dot a}{a}(\rho+p)=0, \tag{30}$$

yielding the relation

$$\rho=\frac{\rho_0}{a^{3(1+w)}}. \tag{31}$$

The field equations (24)–(26) become

$$(1+\epsilon\varphi)3\left(\frac{\dot a}{a}\right)^2=\rho+\frac{m}{2}\varphi-\frac{1}{2}V(\varphi)-3\epsilon\frac{\dot a}{a}\dot\varphi, \tag{32}$$

$$(1+\epsilon\varphi)\left[2\frac{\ddot a}{a}+\left(\frac{\dot a}{a}\right)^2\right]=-p-\frac{\epsilon}{2}\varphi R-\frac{1}{2}V(\varphi)+\frac{1}{2}\varphi V'(\varphi)-\epsilon\ddot\varphi-2\epsilon\frac{\dot a}{a}\dot\varphi, \tag{33}$$

and

$$(1+\epsilon\varphi)6\frac{\ddot{a}}{a} = -(\rho+3p) - \frac{3}{2}\epsilon\varphi R - 3\epsilon\ddot{\varphi} - 3\epsilon\frac{\dot{a}}{a}\dot{\varphi} - \frac{m}{2}\varphi - V(\varphi) + \frac{3}{2}\varphi V'(\varphi). \tag{34}$$

The complete system of equations is therefore

$$3(1-2\epsilon\varphi)\left(\frac{\dot{a}}{a}\right)^2 = \rho + \frac{m}{2}\varphi - \frac{1}{2}V(\varphi), \tag{35}$$

$$6(1-2\epsilon\varphi)\frac{\ddot{a}}{a} = -(\rho+3p) - 18\epsilon\varphi\left(\frac{\dot{a}}{a}\right)^2 - \frac{m}{2}\varphi - V(\varphi) + \frac{3}{2}\varphi V'(\varphi), \tag{36}$$

$$\dot{\varphi} + 3\frac{\dot{a}}{a}\varphi = 0, \tag{37}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0, \tag{38}$$

where we have already substituted the evolution equation for the scalar field (20) and the expression of the Ricci scalar in flat FLRW spacetime, $R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$. These equations will be the starting point of our analysis.

3. Dynamical systems analysis

Let us now analyze the cosmology deriving from equations (35)–(38), using DSA. This method consists of rewriting the cosmological equations written above in terms of some dimensionless variables (including the time variable) that play the same role of the Ω parameters in standard Friedmannian cosmology, and therefore carry a precise physical meaning.

We choose the variables

$$\Omega = \frac{\rho}{3H^2(1-2\epsilon\varphi)}, \quad X = \frac{\epsilon\varphi}{1-2\epsilon\varphi}, \quad Y = \frac{V(\varphi)}{6H^2(1-2\epsilon\varphi)},$$
$$M = \frac{m\varphi}{6H^2(1-2\epsilon\varphi)}, \tag{39}$$

where $H = \frac{\dot{a}}{a}$ and the logarithmic time $\mathcal{N} = \ln a$. In terms of X, Y, Ω, and M, the cosmological equations can be written as

$$\frac{dX(\mathcal{N})}{d\mathcal{N}} = -3X(1+2X), \tag{40}$$

$$\frac{dY(\mathcal{N})}{d\mathcal{N}} = Y[M + (3w+1)\Omega - 3\mathcal{V} + 2] + Y^2[2 - 3\mathcal{V}], \tag{41}$$

$$\frac{dM(\mathcal{N})}{d\mathcal{N}} = M[M + (3w+1)\Omega + Y(2-3\mathcal{V}) - 1], \tag{42}$$

$$\frac{d\Omega(\mathcal{N})}{d\mathcal{N}} = \Omega[M - 3w - 3Y\mathcal{V} + 2Y - 1 + (3w+1)\Omega], \tag{43}$$

with the constraint

$$1 - M + Y - \Omega = 0, \tag{44}$$

coming from (35). In the above equations, the characteristic function, $\mathcal{V}$,

$$\mathcal{V} = \mathcal{V}\left(\frac{X}{2X\epsilon + \epsilon}\right) = \left.\frac{\varphi V'(\varphi)}{V(\varphi)}\right|_{\varphi=\varphi(X,\epsilon)}, \tag{45}$$

contains the information on the form of the potential $V(\varphi)$ in (35)–(38).

Note that the variable X is zero for $\varphi \to 0$, $X = -1/2$ for $\varphi \to \infty$, and $X \to \pm\infty$ when $\varphi \to (1/2\epsilon)^{\pm}$. Thus the 'X direction' represents the complete evolution of the field, φ, and while the fixed points with X coordinate $-1/2$ will be unphysical, the ones in $X = 0$ will effectively represent GR. In this way, every time there is an attractor on the $X = 0$ axis it is a signal that the model converges to GR. This is a typical feature of theories which have a 'scalar tensor' structure. In addition, since (21) holds, we also have $X = -1/2$ for $a \to 0$ (i.e., 'early time') and $X = 0$ for $a \to \infty$ (i.e., 'late time'), so the fixed points on these invariant submanifolds characterize early and late time solutions for the theory[5].

The above system admits five invariant submanifolds ($X = 0$, $X = -1/2$, $Y = 0$, $M = 0$, $\Omega = 0$), so there can be no global attractor in these cosmologies. However, the origin of the coordinate axes can be an attractor for a large set of initial conditions. This also means that for these models, GR can be an attractor for the same set of these conditions. This feature is not at all common in scalar tensor gravity: the set of initial conditions is usually much smaller [24]. The solutions associated with the fixed points of the above system when $\beta \neq 0$ can be found using equations

$$a = a_0(t - t_0)^{1/\beta}, \tag{46}$$

$$\rho = \rho_0(t - t_0)^{-3(1+w)/\beta}, \tag{47}$$

$$\varphi = \varphi_0(t - t_0)^{-3/\beta}, \tag{48}$$

$$2\beta = 2 + M_0 + 6X_0 - \left(3\mathcal{V}_0 - 2\right)Y_0 + (1 + 3w)\Omega_0, \tag{49}$$

where the subscript '0' refers to the value of the variable at the fixed point. In the following section, we will make some illustrative choices for V, and as a consequence, for $\mathcal{V}$.

3.1. DSA in the absence of perfect fluid(s)

We first analyze the case of the absence of perfect fluid ($\Omega = 0$), so that the dynamical system outlined above loses an equation. Implementing (44) to eliminate Y, one obtains

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -3X(1 + 2X), \tag{50}$$

[5] It is clear that this definition holds only when the solution for the scale factor has a monotonic character which is not obvious in NMC cosmologies. However, the orbits plotted in the phase space represent only evolutions in which a is monotonic. Should a change its character, its derivative would be zero and consequently H would be zero, but this can happen only asymptotically. In this respect, the above definitions of 'early time' and 'late time' can be used without confusion.

Table 1. The fixed points and the solutions of the purely fermion NMC model with $V(\varphi) = V_0\varphi^\alpha$.

Point	(X, M, Y)	Scale Factor	Condensate
$\mathcal{A}$	$\left(-\frac{1}{2}, 0, -1\right)$	$a = a_0(t - t_0)^{\frac{2}{3(\alpha-1)}}$	$\varphi \to \infty$
$\mathcal{B}$	$\left(-\frac{1}{2}, 1, 0\right)$	$a = a_0 \exp\left(\sqrt{-\frac{m}{12\epsilon}}(t - t_0)\right)$	$\varphi \to \infty$
$\mathcal{C}$	$(0, 0, -1)$	$a = a_0(t - t_0)^{\frac{2}{3\alpha}}$	$\varphi = 0$
$\mathcal{D}$	$(0, 1, 0)$	$a = a_0(t - t_0)^{2/3}$	$\varphi = 0$

$$\frac{\mathrm{d}M(\mathcal{N})}{\mathrm{d}\mathcal{N}} = M^2 + M[(M - 1)(2 - 3\mathcal{V}) - 1], \tag{51}$$

$$Y = M - 1. \tag{52}$$

The phase space has dimension two and can be easily plotted. We will analyze this system in two cases: $V(\varphi) = V_0\varphi^\alpha$ and $V(\varphi) = V_0 \exp(-\lambda\varphi)$.

3.1.1. The case $V(\varphi) = V_0\varphi^\alpha$. As a first example, let us consider the potential $V(\varphi) = V_0\varphi^\alpha$. This type of potential is the one most commonly used in the treatment of interacting fermions [32]. Because of the (21) at early times ($a \to 0$), the scalar field will have high values and, depending on the sign of parameter α, the potential will be negligible or dominant. The converse happens at late times ($a \to \infty$). This allows a certain degree of control on the cosmological model.

In terms of the above variables, the considered potential is characterized by $\mathcal{V} = \alpha$ and we have

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -3X(1 + 2X), \tag{53}$$

$$\frac{\mathrm{d}M(\mathcal{N})}{\mathrm{d}\mathcal{N}} = M^2 + M[(M - 1)(2 - 3\alpha) - 1]. \tag{54}$$

This dynamical system has four fixed points shown in table 1. Their coordinates do not depend on the parameters of the system and therefore we can expect them to be present in other cases together with other fixed points. The solutions associated with the fixed points correspond to power law behaviors for the scale factor, whose character depends on the choice of α. The character of the scale factor depends on α for the points $\mathcal{A}$ and $\mathcal{C}$. For $\alpha < 0$, both these points always represent a contraction. For $0 < \alpha < 2/3$, $\mathcal{A}$ represents a contraction[6] and $\mathcal{C}$ an accelerated expansion. For $2/3 < \alpha < 1$, $\mathcal{A}$ represents a contraction and $\mathcal{C}$ a decelerated expansion. For $1 < \alpha < 5/3$, $\mathcal{A}$ represents an accelerated expansion (power law inflation) and $\mathcal{C}$ a decelerated expansion. Finally for $\alpha > 5/3$, both $\mathcal{A}$ and $\mathcal{C}$ represent a decelerated expansion.

Note that only two ($\mathcal{C}$, $\mathcal{D}$) of the fixed points in table 1 can represent physical solutions of the cosmological equations (i.e., they correspond to actual solutions of the cosmological

[6] The fact that this point is associated with to a contraction might appear to be in contrast with the statement made above in which along an orbit, H does not change sign. However, looking at the stability, one can see that for $\alpha < 1$, $\mathcal{A}$ is a saddle and therefore an orbit approaching it simply implies a slowing of the expansion rate whose magnitude depends on the distance between the orbit and the fixed point.

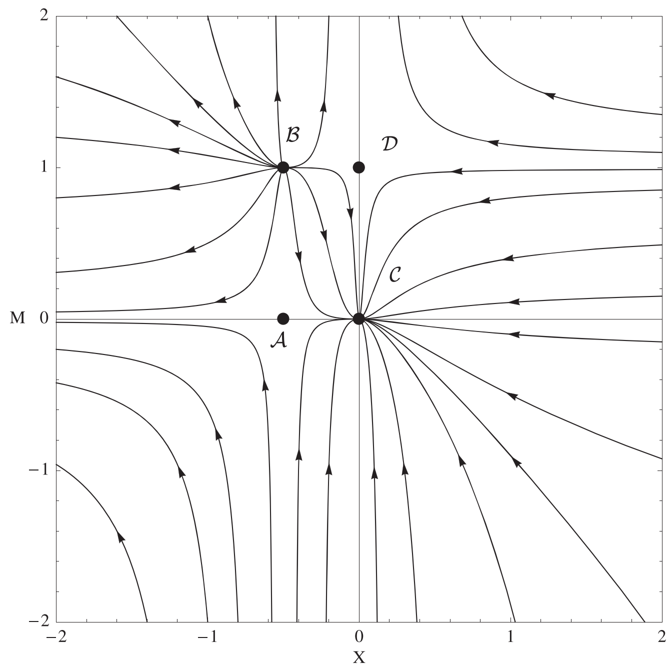

Figure 1. Phase space of the purely fermion NMC model with $V(\varphi) = V_0\varphi^{\alpha}$ for $\alpha < 1$.

equations). In particular, this happens for $\alpha < 1$ for $\mathcal{C}$ and for $\alpha > 1$ for $\mathcal{D}$. For the other points, it is easy to see that these behaviors for the scale factor can be obtained by (35) in the right approximation. In the case of point $\mathcal{A}$, the solution $a = a_0(t - t_0)^{\frac{2}{3(\alpha-1)}}$ can be obtained from (35) in the approximation $\varphi \gg 1$ and $m \approx 0$, which is suggested by its coordinates. The same happens for the solution associated with point $\mathcal{B}$: it can be derived assuming $\varphi \gg 1$ and $V(\varphi) \ll 1$. As for the physical points, the approximated solutions are consistent with the theory only for specific intervals of parameter α. For example, for point $\mathcal{B}$, the conditions $\varphi \gg 1$ and $V(\varphi) \ll 1$ are consistent only if $\alpha < 1$.

The stability of these fixed points can be found using the Hartman–Grobman (HG) theorem, and it is given together with their coordinates in table 2. Note that the critical values for the change in stability coincide with the values related to the above-mentioned issues of the consistency of the solutions. Of all the fixed points, only $\mathcal{C}$ and $\mathcal{D}$ can be attractors. Since both the fixed points lay on the $X = 0$ invariant submanifold and this submanifold represents states indistinguishable from GR, the fact that these points are attractors implies that a set of initial conditions exists for which the theory essentially evolves towards GR. These initial conditions are given by

$$X_0 > -\frac{1}{2}, \quad M_0 < 1, \quad \alpha < 1, \tag{55}$$

$$X_0 > -\frac{1}{2}, \quad M_0 > 0, \quad \alpha > 1. \tag{56}$$

Some examples of phase space are plotted in figures 1 and 2.

Table 2. The fixed points and their stability of the purely fermion NMC model with $V(\varphi) = V_0\varphi^\alpha$. Here R = repeller, A = attractor, and S = saddle point.

Point	(X, M, Y)	Eigenvalues	Stability $\alpha > 1$	Stability $\alpha < 1$
$\mathcal{A}$	$\left(-\frac{1}{2}, 0, -1\right)$	$[3, 3(\alpha - 1)]$	R	S
$\mathcal{B}$	$\left(-\frac{1}{2}, 1, 0\right)$	$[3, 3(1 - \alpha)]$	S	R
$\mathcal{C}$	$(0, 0, -1)$	$[-3, 3(\alpha - 1)]$	S	A
$\mathcal{D}$	$(0, 1, 0)$	$[-3, 3(1 - \alpha)]$	A	S

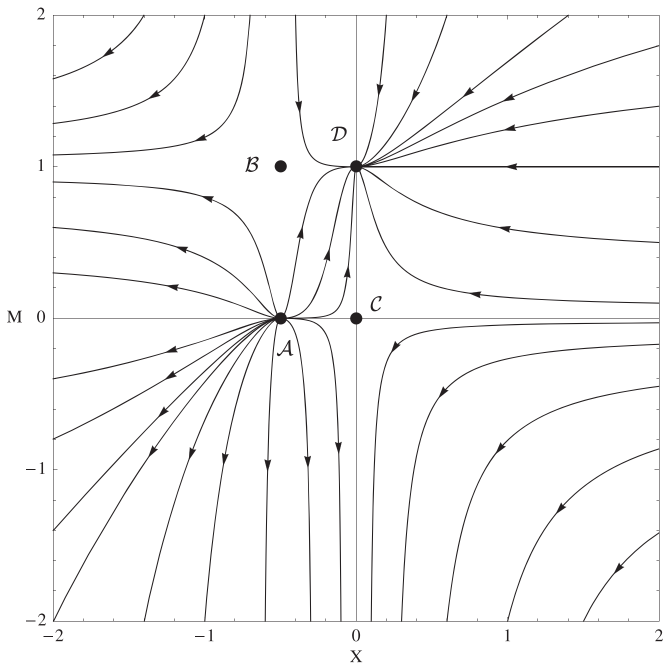

Figure 2. Phase space of the purely fermion NMC model with $V(\varphi) = V_0\varphi^\alpha$ for $\alpha > 1$.

3.1.2. The case $V(\varphi) = V_0 \exp(-\lambda\varphi)$. Our interest in this type of potential is due to the fact that in the limit $\varphi \to 0$, it becomes a cosmological constant term, introducing in this way a dynamical realization of the cosmological constant related to the condensate. In fact, since equation (21) holds, we know that at early time the potential will be irrelevant, reducing the theory to the case analyzed in (27), whereas at late time the potential effectively becomes a constant. Let us see how this behavior is realized in terms of the phase space.

For $V(\varphi) = V_0 \exp(-\lambda\varphi)$, we have $\mathcal{V} = -\frac{\lambda X}{(2X+1)\epsilon}$, and the associated dynamical system becomes

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -3X(1 + 2X), \tag{57}$$

Table 3. The fixed points and the solutions of the purely fermion NMC model with $V(\varphi) = V_0 \exp(-\lambda\varphi)$.

Point	(X, M, Y)	Scale Factor	Condensate
$\mathcal{A}$	$\left(-\frac{1}{2}, 0, -1\right)$	N/A	N/A
$\mathcal{B}$	$\left(-\frac{1}{2}, 1, 0\right)$	$a = a_0 \exp\left(\sqrt{-\frac{m}{12\epsilon}}(t - t_0)\right)$	$\varphi \to \infty$
$\mathcal{C}$	$(0, 0, -1)$	$a = a_0 \exp\left(\sqrt{-\frac{V_0}{6}}(t - t_0)\right)$	$\varphi = 0$
$\mathcal{D}$	$(0, 1, 0)$	$a = a_0(t - t_0)^{2/3}$	$\varphi = 0$

Table 4. The fixed points and their stability in the purely fermion NMC model with $V(\varphi) = V_0 \exp(-\lambda\varphi)$. Here S = saddle, A = attractor, and SN = saddle node. The saddle nodes change orientation (but not character) with the sign of $\epsilon\lambda$.

Point	(X, M, Y)	Eigenvalues	Stability
$\mathcal{A}$	$\left(-\frac{1}{2}, 0, -1\right)$	$\left[\frac{3\lambda}{2\epsilon}, 0\right]$	SN
$\mathcal{B}$	$\left(-\frac{1}{2}, 1, 0\right)$	$\left[-\frac{3\lambda}{2\epsilon}, 0\right]$	SN
$\mathcal{C}$	$(0, 0, -1)$	$[-3, -3]$	S
$\mathcal{D}$	$(0, 1, 0)$	$[-3, 3]$	S

$$\frac{\mathrm{d}M(\mathcal{N})}{\mathrm{d}\mathcal{N}} = 3M(M-1)\left(1 + \frac{\lambda X}{\epsilon(1+2X)}\right), \tag{58}$$

$$Y = M - 1. \tag{59}$$

Note that in this case, the above system presents a singularity in the line $X = -1/2$. In the variables we have used, this is in fact the most common case. The singularity, however, is purely a result of our parametrization and has no correspondence in the actual equations. Therefore, we can try to work around the singularity using a change of coordinates.

Setting for example[7] $\mathrm{d}\mathcal{M} = \mathrm{d}\mathcal{N}(1 - 2\epsilon\varphi)$, one obtains

$$\frac{\mathrm{d}X(\mathcal{M})}{\mathrm{d}\mathcal{M}} = -3X(1+2X)^2 \tag{60}$$

$$\frac{\mathrm{d}M(\mathcal{M})}{\mathrm{d}\mathcal{M}} = \frac{3}{\epsilon}M(M-1)[\lambda X + \epsilon(1+2X)] \tag{61}$$

$$Y = M - 1. \tag{62}$$

The system (57)–(59) has four fixed points, shown in table 3.

Note that for the point $\mathcal{A}$, the exponent (49) is divergent. In fact, looking at the variable definitions, we can see that this point corresponds to an inconsistent relation between the

[7] One might be tempted to redefine the dimensionless time in this way and treat the entire system in this way. However, in general the sign of the quantity $1 - 2\epsilon\varphi$ can be subject to a change, and this might lead to problems, in the interpretation of the orientation of the flow. One could, of course, use the factor $(1 - 2\epsilon\varphi)^2$ to ensure the monotonicity of the time coordinate. This choice would lead to a system in which a fixed sub manifold ($X = 1/2$) would appear. This would not change the topology of the flow, but it would complicate the DSA. For this reason, we will keep our original choice, using the analysis of the original system as a control tool.

quantities in the cosmological equations. As we will see, the point is always unstable and therefore it poses no issue for understanding the dynamics of the cosmology. For the other solutions, the same type of reasoning given in the previous section holds: we have two points which are unphysical ($\mathcal{A}$, $\mathcal{B}$) and two which are physical ($\mathcal{C}$, $\mathcal{D}$). The difference is that parameter λ does not have the same weight as parameter α in the previous case. This can be understood by considering that the change in the power law potential due to a change in α is a more radical modification of the change of the 'time constant' of the exponential potential. Note also that the solutions associated with the fixed points are the same, except for the one associated with point $\mathcal{C}$. This result is expected since this point is the only one that expresses a potential-dominated solution.

The stability of these fixed points can be found using the HG theorem, and it is given together with their coordinates in table 4. As expected from the considerations on the potential, point $\mathcal{C}$ (the potential dominated fixed point) is always an attractor. In addition, since points $\mathcal{A}$ and $\mathcal{B}$ have one zero eigenvalue, their stability has to be determined using the central manifold theorem (CMT) [30, 31].

Let us consider, for example, point $\mathcal{A}$. One can write the system (60)–(61) in the form

$$\frac{\mathrm{d}X(\mathcal{M})}{\mathrm{d}\mathcal{M}} = CX + F(X, M) \tag{63}$$

$$\frac{\mathrm{d}M(\mathcal{M})}{\mathrm{d}\mathcal{M}} = PM + G(X, M), \tag{64}$$

where C and P correspond to the linear part of the equation and F and G to the non-linear part. The CMT says that the behavior of the fixed points is determined by the solution, h, of the equation,

$$h'(X)[CX + F(X, h(X)] - [Ph(X) + G(X, h(X))] = 0. \tag{65}$$

In the case of $\mathcal{A}$, we have

$$C = 0, \tag{66}$$

$$P = -\frac{3\lambda}{2\epsilon}, \tag{67}$$

$$F = 6X^2 - 12X^3, \tag{68}$$

$$G = \frac{3\,M[2(\lambda + 2\epsilon)X(M - 1) - \lambda M]}{2\epsilon}, \tag{69}$$

and (65) can be integrated exactly to give

$$h(X) = \frac{1 - 2X}{Xe^{\frac{\lambda}{4X\epsilon}} - 2X + 1}. \tag{70}$$

This function describes the center manifold for the non-hyperbolic fixed point, $\mathcal{A}$. The same procedure can be used for point $\mathcal{B}$, obtaining

$$h(X) = -\frac{Xe^{\frac{\lambda}{4X\epsilon}}}{Xe^{\frac{\lambda}{4X\epsilon}} + 2X - 1}. \tag{71}$$

In this way, the phase space can be completely characterized. Unexpectedly, it turns out that the only structural differentiation of the phase space is given by the 'orientation' on the saddle

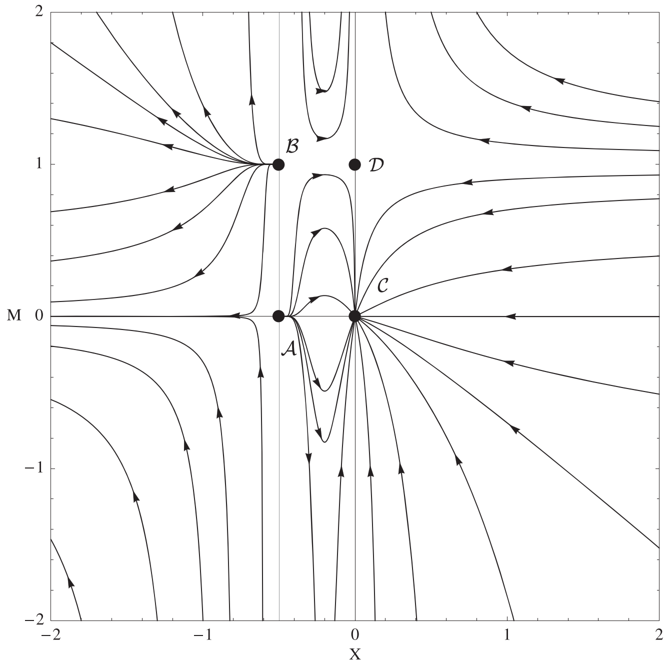

Figure 3. Phase space of the purely fermion NMC model with $V(\varphi) = V_0 \exp(-\lambda\varphi)$ for $\epsilon > 0$.

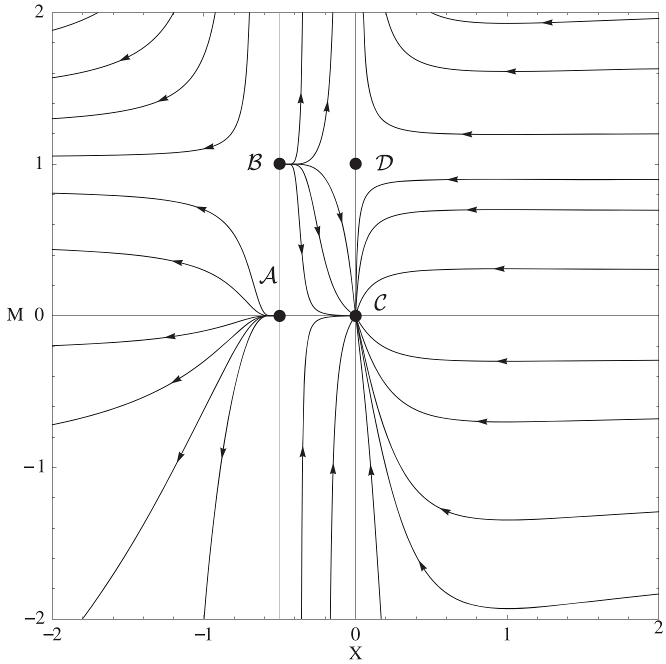

Figure 4. Phase space of the purely fermion NMC model with $V(\varphi) = V_0 \exp(-\lambda\varphi)$ for $\epsilon < 0$.

Table 5. The fixed points and the solutions of the NMC model with matter and $V(\varphi) = V_0 \exp(-\lambda\varphi)$.

Point	(Ω, X, M, Y)	Scale Factor
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	N/A
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$a = a_0 \exp\left(\sqrt{-\frac{m}{12\epsilon}}(t - t_0)\right)$
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$a = \begin{cases} a_0 \exp\left[\dfrac{\rho_0}{\varphi_0}(t - t_0)\right] & w = 0 \\ a_0(t - t_0)^{2/3w} & w \neq 0 \end{cases}$
$\mathcal{D}$	$(0, 0, 0, -1)$	$a = a_0 \exp\left(\sqrt{-\frac{V_0}{6}}(t - t_0)\right)$
$\mathcal{E}$	$(0, 0, 1, 0)$	$a = a_0(t - t_0)^{2/3}$
$\mathcal{F}$	$(1, 0, 0, 0)$	$a = a_0(t - t_0)^{\frac{2}{3(1+w)}}$
$\mathcal{L}$	$\left(\Omega_0, -\frac{1}{2}, 1 - \Omega_0, 0\right)$	$a = \begin{cases} a_0 \exp\left[\rho_0\varphi_0(t - t_0)\right] & w = 0 \\ a_0(t - t_0)^{2/3w\Omega_0} & w \neq 0 \end{cases}$

nodes points $(\mathcal{A}, \mathcal{B})$, whereas the others remain unchanged. The key parameter for the stability change is again ϵ. Samples of its structure can be seen in figures 3 and 4.

3.2. DSA in the presence of a perfect fluid

Let us now analyze the cosmology deriving from action (4) in the presence of a perfect fluid. We are specifically interested in seeing how the presence of an additional perfect fluid affects the action of the non-minimal coupling. We choose, therefore, two different potentials: $V = V_0 \exp(-\lambda\varphi)$ and $V = V_0(\varphi^2 + V_1)^\gamma$. This choice is motivated by the fact that we want to explore potentials able to give rise to a cosmological term at late time.

3.2.1. The case $V = V_0 \exp(-\lambda\varphi)$. As in the case without a perfect fluid, the characteristic function is given by $\mathcal{V} = -\frac{\lambda X}{(2X+1)\epsilon}$. The general system (40)–(44) takes the form

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -6X^2 - 3X, \tag{72}$$

$$\frac{\mathrm{d}Y(\mathcal{N})}{\mathrm{d}\mathcal{N}} = Y\left[M + (3w+1)\Omega + \frac{3\lambda X}{(2X+1)\epsilon} + 2\right] + Y^2\left[2 + \frac{3\lambda X}{(2X+1)\epsilon}\right], \tag{73}$$

$$\frac{\mathrm{d}M(\mathcal{N})}{\mathrm{d}\mathcal{N}} = M^2 + M\left[(3w+1)\Omega + Y\left(2 + \frac{3\lambda X}{(2X+1)\epsilon}\right) - 1\right], \tag{74}$$

$$\frac{\mathrm{d}\Omega(\mathcal{N})}{\mathrm{d}\mathcal{N}} = \Omega\left[M - 3w + 3Y\frac{\lambda X}{(2X+1)\epsilon} + 2Y - 1\right] + (3w+1)\Omega^2, \tag{75}$$

or implementing the constraint (44) to eliminate Y,

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -6X^2 - 3X, \tag{76}$$

Table 6. The fixed points and the solutions of the NMC model with matter and $V(\varphi) = V_0 \exp(-\lambda\varphi)$.

Point	(Ω, X, M, Y)	Condensate	Energy Density
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	N/A	N/A
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$\varphi \to \infty$	$\rho = 0$
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$\varphi \to \infty$	$\rho = \begin{cases} 0 & w = 0 \\ \rho_0 (t - t_0)^{-2\frac{(1+w)}{w}} & w \neq 0 \end{cases}$
$\mathcal{D}$	$(0, 0, 0, -1)$	$\varphi = 0$	$\rho = 0$
$\mathcal{E}$	$(0, 0, 1, 0)$	$\varphi = 0$	$\rho = \rho_0 (t - t_0)^{-2(1+w)}$
$\mathcal{F}$	$(1, 0, 0, 0)$	$\varphi = 0$	$\rho = \rho_0 (t - t_0)^{-2}$
$\mathcal{L}$	$\left(\Omega_0, -\frac{1}{2}, 1 - \Omega_0, 0\right)$	$\varphi \to \infty$	$\rho = \begin{cases} 0 & w = 0 \\ \rho_0 (t - t_0)^{-2\frac{(1+w)}{w\Omega_0}} & w \neq 0 \end{cases}$

Table 7. The fixed points and their stability of the NMC model with matter and $V(\varphi) = V_0 \exp(-\lambda\varphi)$. Here A = attractor, S = saddle point, and SN = saddle node bifurcation (non hyperbolic).

Point	(Ω, X, M, Y)	Eigenvalues	Stability
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	$\left[\frac{3\lambda}{2\epsilon}, \frac{3\lambda}{2\epsilon}, 0\right]$	SN
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$\left[-\frac{3\lambda}{2\epsilon}, 0, 0\right]$	SN
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$\left[-\frac{3\lambda}{2\epsilon}, 0, 0\right]$	SN
$\mathcal{D}$	$(0, 0, 0, -1)$	$[-3, -3 - 3(1 + w)]$	A
$\mathcal{E}$	$(0, 0, 1, 0)$	$[-3, 3, -3w]$	S
$\mathcal{F}$	$(1, 0, 0, 0)$	$[-3, 3w, 3(1 + w)]$	S
$\mathcal{L}$	$\left(\Omega_0, -\frac{1}{2}, 1 - \Omega_0, 0\right)$	$\left[0, 0, \frac{3\lambda}{2\epsilon}\right]$	SN

$$\frac{\mathrm{d}M(\mathcal{N})}{\mathrm{d}\mathcal{N}} = 3\,M\left[\frac{\lambda X(M + \Omega - 1)}{\epsilon(2X + 1)} + M + (w + 1)\Omega - 1\right], \tag{77}$$

$$\frac{\mathrm{d}\Omega(\mathcal{N})}{\mathrm{d}\mathcal{N}} = 3\Omega\left[\frac{\lambda X(M + \Omega - 1)}{\epsilon(2X + 1)} + M + (w + 1)(\Omega - 1)\right], \tag{78}$$

$$Y = \Omega + M - 1. \tag{79}$$

As in the $\Omega = 0$ case, we perform a change of variables to work around the singularities of the system. Setting again $\mathrm{d}\mathcal{M} = \mathrm{d}\mathcal{N}(1 - 2\epsilon\phi)$, we obtain

$$\frac{\mathrm{d}X(\mathcal{M})}{\mathrm{d}\mathcal{M}} = -3X(1 + 2X)^2, \tag{80}$$

$$\frac{\mathrm{d}M(\mathcal{M})}{\mathrm{d}\mathcal{M}} = \frac{3}{\epsilon}M\left[\lambda X(M + \Omega - 1) + \epsilon M(2X + 1) + \epsilon(w + 1)(\Omega - 1)(2X + 1)\right], \tag{81}$$

$$\frac{d\Omega(\mathcal{M})}{d\mathcal{M}} = \frac{3}{\epsilon}\Omega[\lambda X(M + \Omega - 1) + \epsilon M(2X + 1) + \epsilon(w + 1)(\Omega - 1)(2X + 1)], \tag{82}$$

$$Y = \Omega + M - 1. \tag{83}$$

Both systems admit six fixed points and a one-dimensional fixed subspace given in tables 5 and 6 whose coordinates are independent of the parameters of the system. The set of fixed points includes the ones we have found in the $\Omega = 0$ case, plus two new fixed points which represent states in which the matter contributions are dominant. One of these points ($\mathcal{F}$) is associated with standard Friedmann cosmologies. The other ($\mathcal{C}$) corresponds to an exponential in the case of dust ($w = 0$) and a power law otherwise. This is an unusual behavior, as a pressureless component is normally not expected to produce a de Sitter phase. However, since the condensate also acts as a form of dust (see (21)) in the cosmology, the theory does not 'recognize' the presence of matter and shows a phenomenology typical of the $\Omega = 0$ case.

As usual, the stability has to be calculated using the HG theorem for the hyperbolic points and the CMT for the non-hyperbolic points. Only the stability of the non-hyperbolic fixed points depends on the values of ϵ. The results are shown in table 7.

Because of the higher dimensionality of the phase space, the characteristic equation for the determination of the stability of the non hyperbolic fixed manifolds generalizes to

$$\mathbf{h}'(X)[CX + F(X, \mathbf{h}(X))] - [\mathbf{P}\cdot\mathbf{h}(X) + \mathbf{G}(X, \mathbf{h}(X))] = 0, \tag{84}$$

where now $\mathbf{h}(X) = (h_1(X), h_2(X))$. Let us consider as an example the case of point $\mathcal{A}$. In this case, we have

$$C = -24, \tag{85}$$

$$\mathbf{P} = \left\{ -6(w + 1) - \frac{3\lambda}{2\epsilon}, -\frac{3\lambda}{2\epsilon} - 6 \right\}, \tag{86}$$

$$F = -6\left\{(2X + 5)X^2 + 1\right\}, \tag{87}$$

$$\mathbf{G} = \left\{G_1, G_2\right\}, \tag{88}$$

$$G_1 = \frac{3\Omega}{2\epsilon}\{\lambda[2X(M + \Omega - 1) + M + \Omega] + 4\epsilon[M(X + 1) + (w + 1)(\Omega X - X + \Omega)]\}, \tag{89}$$

$$\begin{aligned} G_2 = \frac{3}{2\epsilon}\Big\{ &\lambda\Big[M^2 - \Omega^2 + 2MX(M + \Omega - 1)\Big] \\ &+ 4\epsilon\Big[M^2 - \Omega^2 + MX(M + w\Omega + \Omega - 1)\Big]\Big\}. \end{aligned} \tag{90}$$

Equations (84) cannot be solved exactly. A standard approach in this case is a resolution by series. Using the approximation

$$h_1(X) = aX^2 + bX^3 + \ldots, \quad h_2(X) = dX^2 + eX^3 + \ldots, \tag{91}$$

one solution for the leading coefficients is

$$a = -\frac{\begin{bmatrix}(\lambda + 4\epsilon)\left\{\lambda^2(5w - 48) + 16\left[w\left(w^2 - 21w + 83\right) + 40\right]\epsilon^2\right. \\ \left. + 16\lambda\left(w^2 - 16w + 30\right)\epsilon\right\}\end{bmatrix}}{32\epsilon[\lambda + 4(w - 7)\epsilon]\left[\lambda(w - 4) + 2\left(w^2 - 6w - 8\right)\epsilon\right]}, \tag{92}$$

$$d = \frac{\lambda^2(5w - 48) + 16\left[w\left(w^2 - 21w + 83\right) + 40\right]\epsilon^2 + 16\lambda\left(w^2 - 16w + 30\right)\epsilon}{32\epsilon\left[\lambda(w - 4) + 2\left(w^2 - 6w - 8\right)\epsilon\right]}. \tag{93}$$

The form of these coefficients is not unique, but it can be proven to be topologically equivalent to any other solution found in this way [30]. The sign of a and d specifies in general the stability of the fixed points. However, since the lower-order term of the two expansions is even, the points represent a saddle node bifurcation, and they are unstable. For points $\mathcal{B}$ and $\mathcal{C}$, the determination of the stability is complicated by the fact that these points present two zero eigenvalues. However, only one of them is due to its non-hyperbolic character, while the other one represents the fact that it belongs to line $\mathcal{L}$.

The entire line $\mathcal{L}$ lies on the invariant submanifold $X = -1/2$, and therefore it is unstable for all the initial conditions with $X \neq -1/2$. In the invariant submanifold $X = -1/2$, its stability can be characterized by the projection of the system in the invariant submanifold itself. However, the submanifold $X = -1/2$ is irrelevant in the physical point of view, and the above considerations combined with the considerations of the $\Omega = 0$ case are sufficient to characterize the line as unstable and points $\mathcal{B}$ and $\mathcal{C}$ as saddle node bifurcations.

Since the phase space is three-dimensional, plotting the phase space and deducing geometrical information for it is not as easy as in the previous section. For this reason, here and in the following subsection we will limit ourselves to giving analytical considerations.

3.2.2. The case $V = V_0\left(\varphi^2 + V_1\right)^\gamma$. The potential $V = V_0(\varphi^2 + V_1)^\gamma$ is a simple extension of the power law potential considered in the previous section. It was chosen because of its relevance in inflationary scenarios in the framework of the scalar field. At early times ($a \to 0$), this potential coincides with the pure power law considered in the $\Omega = 0$ case. However, at late time ($a \to \infty$), it generates a cosmological term related to the value of the constant, V_1.

In this case, $\mathcal{V} = \frac{2\gamma X^2}{X^2 + V_1(2X\epsilon + \epsilon)^2}$ and the general system (40)–(44) takes the form

$$\frac{\mathrm{d}X(\mathcal{N})}{\mathrm{d}\mathcal{N}} = -3X(1 + 2X), \tag{94}$$

$$\begin{aligned}\frac{\mathrm{d}Y(\mathcal{N})}{\mathrm{d}\mathcal{N}} = {} & Y\left[M + (3w + 1)\Omega - \frac{6\gamma X^2}{X^2 + V_1(2X\epsilon + \epsilon)^2} + 2\right] \\ & + 2Y^2\left[1 - \frac{3\gamma X^2}{X^2 + V_1(2X\epsilon + \epsilon)^2}\right],\end{aligned} \tag{95}$$

Table 8. The fixed points and the solutions of the NMC model with matter and $V = V_0(\varphi^2 + V_1)^\gamma$.

Point	(Ω, X, M, Y)	Scale Factor
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	$a = a_0(t - t_0)^{2/3(1--2\gamma)}$
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$a = a_0 \exp\left(\sqrt{-\frac{m}{12\epsilon}}(t - t_0)\right)$
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$a = \begin{cases} a_0 \exp\left[\dfrac{\rho_0}{\varphi_0}(t - t_0)\right] & w = 0 \\ a_0(t - t_0)^{2/3w} & w \neq 0 \end{cases}$
$\mathcal{D}$	$(0, 0, 0, -1)$	$a = a_0 \exp\left(i\frac{\lvert V_1\rvert}{\sqrt{6}}(t - t_0)\right)$
$\mathcal{E}$	$(0, 0, 1, 0)$	$a = a_0(t - t_0)^{2/3}$
$\mathcal{F}$	$(1, 0, 0, 0)$	$a = a_0(t - t_0)^{\frac{2}{3(1+w)}}$

Table 9. The fixed points and the solutions of the NMC model with matter and $V = V_0(\varphi^2 + V_1)^\gamma$.

Point	(Ω, X, M, Y)	Condensate	Energy Density
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	$\varphi \to \infty$	$\rho = 0$
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$\varphi \to \infty$	$\rho = 0$
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$\varphi \to \infty$	$\rho = \begin{cases} 0 & w = 0 \\ \rho_0(t - t_0)^{-2\frac{(1+w)}{w}} & w \neq 0 \end{cases}$
$\mathcal{D}$	$(0, 0, 0, -1)$	$\varphi = 0$	$\rho = 0$
$\mathcal{E}$	$(0, 0, 1, 0)$	$\varphi = 0$	$\rho = 0$
$\mathcal{F}$	$(1, 0, 0, 0)$	$\varphi = 0$	$\rho = \rho_0(t - t_0)^{-2}$

$$\frac{dM(\mathcal{N})}{d\mathcal{N}} = M^2 + M\left\{(3w + 1)\Omega + 2Y\left[1 - \frac{3\gamma X^2}{X^2 + V_1(2X\epsilon + \epsilon)^2}\right] - 1\right\}, \tag{96}$$

$$\frac{d\Omega(\mathcal{N})}{d\mathcal{N}} = \Omega\left\{M - 3w + 2Y\left[1 - \frac{3\gamma X^2}{X^2 + V_1(2X\epsilon + \epsilon)^2}\right] - 1\right\} + (3w + 1)\Omega^2, \tag{97}$$

or implementing the constraint (44) to eliminate Y,

$$\frac{dX(\mathcal{N})}{d\mathcal{N}} = -3X(1 + 2X), \tag{98}$$

$$\frac{dM(\mathcal{N})}{d\mathcal{N}} = 3M\left[M - \frac{2\gamma X^2(M + \Omega - 1)}{X^2 + V_1(2X\epsilon + \epsilon)^2} + (w + 1)\Omega - 1\right], \tag{99}$$

Table 10. The fixed points and their stability of the NMC model with matter and $V = V_0(\varphi^2 + V_1)^\gamma$. Here R = repeller, A = attractor and S = saddle point.

Point	(Ω, X, M, Y)	Eigenvalues	$\gamma < 1/2$	$1/2 < \gamma < 1$	$\gamma > 1$
$\mathcal{A}$	$\left(0, -\frac{1}{2}, 0, -1\right)$	$[3, -3(1-2\gamma), -3(1+w-2\gamma)]$	S	R if $0 \leqslant w < 2\gamma - 1$	R
			S	S if $2\gamma - 1 < w \leqslant 1$	
$\mathcal{B}$	$\left(0, -\frac{1}{2}, 1, 0\right)$	$[3, -3w, 3(1-2\gamma)]$	S	S	S
$\mathcal{C}$	$\left(1, -\frac{1}{2}, 0, 0\right)$	$[3, 3w, -3(1+w-2\gamma)]$	R	R if $2\gamma - 1w \leqslant 1$	S
			S	S if $0 \leqslant w < 2\gamma - 1$	
$\mathcal{D}$	$(0, 0, 0, -1)$	$[-3, -3, -3(1+w)]$	A	A	A
$\mathcal{E}$	$(0, 0, 1, 0)$	$[-3, 3, -3w]$	S	S	S
$\mathcal{F}$	$(1, 0, 0, 0)$	$[-3, -3w, 3(1+w)]$	S	S	S

$$\frac{d\Omega(\mathcal{N})}{d\mathcal{N}} = 3\Omega\left[M - \frac{2\gamma X^2(M+\Omega-1)}{X^2 + V_1(2X\epsilon + \epsilon)^2} + (w+1)(\Omega-1)\right], \tag{100}$$

$$Y = \Omega + M - 1. \tag{101}$$

The fixed points of the above system are shown in tables 8 and 9. In addition to the fixed points found in the previous section, we find two more fixed points, whose coordinates are also independent of the parameters of the system. The character of the solutions associated with fixed point $\mathcal{A}$ depends on parameter γ. For $\gamma < 1/6$, $\mathcal{A}$ is associated with decelerated expansion solutions, whereas for $1/6 < \gamma < 1/2$ it represents and accelerated expansion solution. The values $\gamma > 1/2$ are associated instead with a contraction solution whose meaning is the same as the one in the power law potential given in the previous section. Apart from $\mathcal{D}$, none of the fixed points represents an exact solution for the cosmological equations, so that the behavior given in tables 8 and 9 represents approximations of the general integral. The same caveats given in the previous sections hold here. Compared to the power law potential examined in the $\Omega = 0$ case, in this case a second exponential solution appears as a late time solution as, expected. However, upon substitution in the cosmological equations, this solution does not constitute a de Sitter phase, but rather an oscillatory solution.

Using the HG theorem, one can find the stability of the fixed points which is given in table 10. The stability of the fixed points depends on all the parameters of the system, including the nature of the perfect fluid considered. Of all the points, only $\mathcal{D}$ is an attractor for a large set of initial conditions and coincides effectively with a cosmology indistinguishable from a GR-based one. The remaining points are always unstable, so the fact that they do not represent physical solutions is irrelevant for the analysis of the cosmologies.

Finally, also in this case, the dimensionality of the phase space, prevents an effective graphical description of the phase space and we will not include that here.

4. Conclusions

In this paper, we have analyzed the dynamic of the cosmology of a metric theory of gravity, in which a condensate of fermions couples non-minimally with the geometry. One of the most interesting features of this model is the fact that the Dirac equations constrain the behavior of the condensate in a very strict way allowing the use of the non–minimal coupling as a switch for additional terms in the action. For example, consider the case of an action, $\varphi^2 R + \varphi R^2$. It is clear that since $\varphi \propto a^{-3}$, the Hilbert–Einstein term will be dominant at early time and the R^2 term will become important at late time. Instead, an action $\varphi R + \varphi^2 R^2$ will behave in the opposite way. In other words, using this feature of the condensate, one can regulate the influence of non–minimally coupled terms in a way what is impossible in the standard scalar tensor and higher–order gravity. In this sense, we can speak of 'design' of the actions in the case of theories with condensate NMC.

Our contribution in this framework is an analysis of the details of the cosmology via the DSA. The proposed formulation allows the exploration of a model which includes massive fermions, a perfect fluid, and a completely general self–interaction potential for the condensate. All these models present some common fixed points, although the solutions for the scale factor associated with them can differ strikingly.

The phase space analysis also highlights a somewhat expected feature: since the attractors of the theory are always a $\varphi = 0$ state, this kind of theory always evolves towards states which are indistinguishable from GR. In fact, since the solutions corresponding to the fixed points are characterized by a constant (null) condensate, for the orbits that are close to this point the condensate is small and evolves very slowly—so slowly, in fact, that its characteristic time of change can be bigger than the age of the Universe. It is clear that in such a case, no-observation on the exact Friedmannian cosmology can reveal the presence of such coupling. This is a common phenomenon in the cosmology of non-minimally coupled theories and has been found in different ways by one of the authors of this paper, as well as other researchers [24, 26]. This result, added to careful design of the interaction potential as in the case of the exponential potential, can offer a natural, dynamic way to approximate Λ(CDM) universes at late time. Of course this 'degeneracy' between non-minimally coupled theories and GR exists only at the level of the pure Friedmann cosmology. Judging from what happens in the case of the standard scalar tensor cosmologies, it is likely that looking into the evolution of the cosmological perturbations will reveal clearly that we are not dealing with GR. In fact, it is known that modification of GR dramatically impacts the behavior of the cosmological perturbations (see, e.g., [27, 28] and references, therein). These differences would probably also be evident in other cases, like black hole properties perturbations, etc. These issues have never been analyzed in this perspective, even in the case of scalar tensor gravity, and it would be outside the purpose of the present work to present them in details. We leave this task for a series of future works.

Notwithstanding the general formulation of the DSA equations, the present work specifically focused on two different types of potential: power law and exponential law in both the absence and presence of a perfect fluid. The choice of these potentials is motivated by both the standard form of the potential considered in fermionic self-interaction and the attempt to model an inflationary or dark energy phase.

In the absence of a perfect fluid, the structure of the phase space makes clear that some interesting orbits are present in the sector $-1/2 < X < 0$. In particular, for the case of a power law potential, $V = V_0\varphi^\alpha$ and $\alpha < 1$, the cosmology can start from an unstable de Sitter state ($\mathcal{B}$) and evolve through a Friedmannian ($t^{2/3}$) behavior ($\mathcal{D}$) or a reduction of the expansion rate (via $\mathcal{A}$) to approach a power law evolution ($\mathcal{C}$). In particular, for $0 < \alpha < 2/3$, the final power law phase of $\mathcal{C}$ can be a power law inflation so that the orbits $\mathcal{B} \to \mathcal{D} \to \mathcal{C}$ inludes an inflation era, a transition to a Friedmannian cosmology, and the onset of a (power law) dark energy era. This transition to dark energy domination was already found in [16] using a different approach. For $\alpha > 1$, instead, the cosmology can start with a power law behavior ($\mathcal{A}$) and evolve towards to a Friedmannian behavior ($\mathcal{D}$) either via an intermediate de Sitter ($\mathcal{B}$) or a power law decelerated expansion ($\mathcal{C}$). For $1 < \alpha < 5/3$, the power law phase of $\mathcal{A}$ corresponds to accelerated expansion (power law inflation). In this sense, the orbits $\mathcal{A} \to \mathcal{B} \to \mathcal{D}$ can describe the graceful exit from a (de Sitter) inflationary phase.

In the case of an exponential potential, one of the fixed points ($\mathcal{A}$) does not represent a solution for the system, but it is always unstable and does not constitute an issue for the understanding of the dynamics of the model. The orbits in the phase space of this case can contain up to two de Sitter phases and one Friedmannian phase, and one of the de Sitter phases ($\mathcal{C}$) is an attractor for every value of the parameters (although not for all the orbits/ initial conditions). Particularly interesting are the orbits in which the de Sitter phase, Friedmann phase, de Sitter phase sequence is realized. It is worth to stressing again that from the properties of the behavior of the condensates, the appearance of such behavior could be inferred already the action level.

As expected, the inclusion of a perfect fluid adds to the degrees of freedom of the cosmology, and this change is reflected in both an increase of the dimensionality of the phase space and the appearance of additional fixed points. We have considered in this case the exponential potential and a generalization of the power law potential considered in the previous cases.

In the case of an exponential potential in the presence of a perfect fluid, the picture that emerges is considerably more complicated than the purely fermionic one ($\Omega = 0$). On top of the $\Omega = 0$ fixed points, other points appear together with a line of fixed points which are associated with power law and exponential solutions. The cosmology, however, still contains two de Sitter phases other than a proper Friedmann phase, and it cannot be excluded that a de Sitter phase, Friedmann phase, de Sitter phase sequence is realized for a set of initial conditions of measure different from zero.

In the case of a power law potential in the presence of a perfect fluid, an oscillating cosmology, in which phases of expansions are alternated to phases of contractions, becomes an attractor for a large set of initial conditions. The period of the oscillations is strictly related to the constant, V_1. The approaching trajectory can be, however, rather complicated and can be approximated with different types of power law behaviors and a de Sitter phase. It is worth stressing that this kind of behavior was not predictable *a priori*, and this shows a limitation of the idea of the design of these models. We can construct a theory whose cosmology shows an exponential behavior, but we cannot easily control the character of the exponential solution.

An interesting phenomenon present in both of the above cases is related to the fact that, since the condensate works as a dust fluid when these cosmologies are filled with dust, one obtains behaviors typical of the $\Omega = 0$ case (modulo the additional complication of the flow, due to the increase of degrees of freedom). This happens because in the case of dust, the theory is not able to 'distinguish' between the perfect fluid and the condensate, so the $\Omega = 0$ phenomenology appears again. This is clear from the structure of the solution associated with $\mathcal{C}$.

All in all, therefore, the presence of matter does have a deep influence on the cosmological models, but in some cases, like the case of the exponential potential, it offers the possibility of a full representation of an inflation plus Λ universes.

The analysis of the above models indicates that the possibility of a non–minimal coupling between a fermion condensate and the geometry can present, in spite of the additional constraint given by the Dirac theory, a phenomenology as rich as the standard scalar field theories. Although still at the toy-model-level of understanding, the theory presented above constitutes an interesting alternative approach to the unification of inflation and dark energy which deserves further study.

To conclude, it is worth stressing that one major issue for these models is to identify a suitable potential for the condensate interaction. This is, however, not new in cosmology and particle physics. Probably the most important example is the theory of inflation in which, in spite of more that thirty years of study, there is no agreement on the form of the potential for the inflation. In this sense, our work aims to highlight an alternative explanation for inflation and/or the cosmic acceleration phenomenon, rather than to prove the actual necessity of such mechanism. Further studies will allow more critical review of these results.

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