WDS'14 Proceedings of Contributed Papers — Physics, 56–60, 2014. *ISBN 978-80-7378-276-4 © MATFYZPRESS*

Preliminary Results on Type I and II Kaluza–Klein Reductions of Vacuum Spacetimes

T. Tintěra
Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic.
Institute of Mathematics AS CR, Prague, Czech Republic.

Abstract. We review the formalism used for the algebraic classification of higher-dimensional spacetimes, together with a few basic results in this field. Then we use this formalism to discuss some problems involving type I and II Kaluza–Klein reductions of vacuum spacetimes and to present a few results.

Introduction

In 2004, a generalization of the Petrov algebraic classification to higher dimensional spacetimes was presented [*Coley et al.*, 2004]. It is based on the idea of counting the so-called multiplicities of null directions aligned with the Weyl tensor. This strategy is sufficiently general to be applied to any tensor, in addition to the Weyl one [*Milson et al.*, 2005]. In the first section, we provide an introduction to the higher dimensional algebraic classification, based on the papers by *Coley et al.* [2004] and *Coley et al.* [2004], and summarize the basic results in this field.

In the second section, we provide an introduction to the method of Kaluza–Klein reduction, essentially based on the review paper by *Pope* [n.d.].

In the last section, we briefly touch upon some problems regarding algebraic types of a spacetime and of its Kaluza–Klein reduction.

Algebraic classification

In this section, based on the results of *Coley et al.* [2004] and *Milson et al.* [2005], we introduce the higher-dimensional algebraic classification. For a more thorough review, see the paper by *Ortaggio et al.* [2013].

Definition. We call a frame

$$\boldsymbol{\ell} \equiv \boldsymbol{e}_{(0)},\ \boldsymbol{n} \equiv \boldsymbol{e}_{(1)},\ \boldsymbol{e}_{(i)}, \qquad i = 2, \ldots, D-1 \tag{1}$$

in the tangent space the *null frame*, if the inverse metric takes the form

$$^{\sharp\sharp}\boldsymbol{g} = \boldsymbol{\ell} \vee \boldsymbol{n} + \sum_{i=2}^{D-1} \boldsymbol{e}_{(i)}\boldsymbol{e}_{(i)}\,. \tag{2}$$

By indices in brackets, we will denote the tensor frame components, as in

$$C_{(0)(1)(i)(j)} = C_{abcd}\,\ell^a n^b e^c_{(i)} e^d_{(j)}\,. \tag{3}$$

In these cases, i, j, … take values $2, \ldots, D-1$ and the Einstein's summation convention is used.

Definition. If some quantity q transforms under boosts

$$\boldsymbol{\ell} \mapsto \lambda \boldsymbol{\ell}\,, \tag{4a}$$
$$\boldsymbol{n} \mapsto \lambda^{-1} \boldsymbol{n}\,, \tag{4b}$$
$$\boldsymbol{e}_{(i)} \mapsto \boldsymbol{e}_{(i)}\,, \qquad i = 2, \ldots, D-1 \tag{4c}$$

according to

$$q \mapsto \lambda^b q\,, \tag{5}$$

where b is an integer, then we say that q has a *boost weight* b.

Proposition. *The* boost weight *of a frame component of some covariant tensor is the number of* 0*'s minus the number of* 1*'s in the component indices.*

Definition. The *boost order* of a covariant tensor with respect to some null frame is the maximum of boost weights of its non-zero frame components.

Proposition. *The boost order of a covariant tensor is invariant under transformations of the null frame that preserve the direction of* $\boldsymbol{\ell}$.

We will thus denote the boost order of a tensor $\boldsymbol{T}$ with respect to the null direction $\boldsymbol{\ell}$ by $\mathrm{bo}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T}$.

Definition. The *maximum boost order* of a covariant tensor is the maximum of boost orders of that tensor taken over all null directions $\boldsymbol{\ell}$:

$$\mathrm{bo}_{\max} \boldsymbol{T} := \max \left\{ \mathrm{bo}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T} \,|\, \boldsymbol{\ell} \text{ is null} \right\} . \tag{6}$$

Observation. *One can show that any nonzero tensor with the Riemann tensor symmetry has* $\mathrm{bo}_{\max}$ *equal to* 2.

Observation. *By exchanging the roles of* $\boldsymbol{\ell}$ *and* $\boldsymbol{n}$*, one can show that:*

$$-\mathrm{bo}_{\max} \boldsymbol{T} \leq \mathrm{bo}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T} \leq \mathrm{bo}_{\max} \boldsymbol{T} . \tag{7}$$

Definition. Given a covariant tensor $\boldsymbol{T}$, we define the *multiplicity* of a null direction $\boldsymbol{\ell}$ as

$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T} := \mathrm{bo}_{\max} \boldsymbol{T} - \mathrm{bo}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T} . \tag{8}$$

Proposition. *From* (7)*, it is evident that:*

$$0 \leq \mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{T} \leq 2\mathrm{bo}_{\max} \boldsymbol{T} . \tag{9}$$

Definition. Choose a point in the manifold.

If the Weyl tensor $\boldsymbol{C}$ is nonvanishing and the manifold admits a null vector field $\boldsymbol{\ell}$ such that locally,

$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} \geq 1 \,, \text{or} \tag{10a}$$
$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} \geq 2 \,, \text{or} \tag{10b}$$
$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} \geq 3 \,, \text{or} \tag{10c}$$
$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} = 4 \,, \tag{10d}$$

then we say that the manifold is of *type I, II, III, or N* [1], respectively, around the chosen point. We then call $\langle \boldsymbol{\ell} \rangle$ a *Weyl aligned null direction (WAND), multiple WAND (mWAND), 3-WAND, or 4-WAND*, respectively.

If the Weyl tensor is nonvanishing and the manifold locally admits two distinct mWANDs, we say that the manifold is of *type D* around the chosen point.

If for all null vector fields $\boldsymbol{\ell}$,

$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} = 0 \tag{11}$$

around the chosen point, then we say that the manifold is there of *type G*.

If the Weyl tensor vanishes around the chosen point, we say that the manifold is of *type O* there. In this case we say that for any null ℓ,

$$\mathrm{mul}_{\langle \boldsymbol{\ell} \rangle}\, \boldsymbol{C} = 4 \,, \tag{12}$$

In four dimensions, the sum of multiplicities of all distinct WANDs is 4 [see, e.g., *Stephani et al.*, 2003]. There are therefore exactly 5 possibilities of combinations of multiplicities at each point of the manifold, corresponding to the types I, II, D, III and N. These types correspond to the Petrov classification (for introduction to the Petrov classification, see e.g. [*Stephani et al.*, 2003, chap. 4]).

On the other hand, in higher dimensions, further combinations could be achieved. There are however some general results, restricting the allowed combinations [*Milson et al.*, 2005], e.g.,

- If the manifold is of type III, then it admits exactly one mWAND.
- If the manifold is of type N, then it admits exactly one WAND (which, by definition, is a 4-WAND).

Later we will need the notion of the so-called optical matrix and non-geodesity.

[1] Note that a type N manifold is also of type III, and so on.

Definition. For a given null frame, we define:

$$\rho_{(i)(j)} := \ell_{(i);(j)} \equiv \ell_{a;b}\, e^a_{(i)} e^b_{(j)} \qquad \textit{(optical matrix)}, \tag{13a}$$

$$\kappa^a := \ell^a{}_{;b}\ell^b \qquad \textit{(non-geodesity of } \boldsymbol{\ell}\textit{)}. \tag{13b}$$

Proposition. *$\boldsymbol{\ell}$ is geodetic (not necessarily affinely geodetic), iff $\kappa^{(i)} = 0$.*

From the Ricci identity $\ell_{(i);(0)(j)} - \ell_{(i);(j)(0)} = R_{(0)(i)(0)(j)}$, we get for a geodetic $\boldsymbol{\ell}$ and a parallelly transported null frame the transport equation for the optical matrix (*Sachs equation*):

$$\rho_{(i)(j);a}\ell^a = -\rho_{(i)(k)}\rho^{(k)}{}_{(j)} - R_{(0)(i)(0)(j)}\,. \tag{14}$$

For the precise derivation and more discussion, see [*Pravda et al.*, 2004; *Ortaggio et al.*, 2007].

Kaluza–Klein reduction

In this section, based on the review by *Pope* [n.d.], we provide a basic introduction to the method of Kaluza–Klein reduction.

Let us consider a $(D{+}1)$-dimensional manifold admitting a Killing vector field and choose coordinates adapted to this Killing vector field, with z being the Killing coordinate. This choice of coordinates defines a set of natural embeddings, parametrized by z, of the D-dimensional manifold $z = \text{const.}$ into the $(D+1)$-dimensional one. The chosen coordinates are then coordinates adapted to this embedding.

The $(D+1)$-dimensional metric $\hat{g}_{AB}$ can be decomposed into a D-dimensional metric, vector field and scalar field in many ways. Among them, there is one that leads to very simple field equations; it has the following form [*Pope*, n.d.]:

$$\hat{g}_{ab} = \mathrm{e}^{2\alpha\phi} g_{ab} + \mathrm{e}^{2\beta\phi}\mathcal{A}_a\mathcal{A}_b\,, \tag{15a}$$

$$\hat{g}_{az} = \mathrm{e}^{2\beta\phi}\mathcal{A}_a\,, \tag{15b}$$

$$\hat{g}_{zz} = \mathrm{e}^{2\beta\phi}\,, \tag{15c}$$

or equivalently

$$\hat{\boldsymbol{g}} = \mathrm{e}^{2\alpha\phi}\boldsymbol{g} + \mathrm{e}^{2\beta\phi}\left(\mathbf{d}z + \boldsymbol{\mathcal{A}}\right)^2, \tag{16}$$

where α and β are some nonvanishing constants (for fixed D) and where the respective D-dimensional fields are pullbacks by the embeddings mentioned above of the fields $\boldsymbol{g}$, $\boldsymbol{\mathcal{A}}$ and ϕ (where $\boldsymbol{g}$ and $\boldsymbol{\mathcal{A}}$ are orthogonal to $\boldsymbol{\partial}_z$ and, together with ϕ, independent of z), whilst $\hat{g}_{ab}$ and $\hat{g}_{az}$ are orthogonal projections by the vector field $\boldsymbol{\partial}_z$ of $\hat{g}_{AB}$ and $\hat{g}_{Az}$, respectively.

Assuming the original spacetime is vacuum, we can write the $(D{+}1)$-dimensional Einstein equations as:

$$\begin{aligned} R_{ab} = {} & \big(\beta + (D-2)\,\alpha\big)\,\phi_{;ab} + \big(\beta^2 - 2\alpha\beta - (D-2)\,\alpha^2\big)\,\phi_{;a}\phi_{;b} + \\ & + \tfrac{\mathrm{e}^{(2\beta-2\alpha)\phi}}{2}\big(\mathcal{F}^2_{ab} + \tfrac{\alpha}{2\beta}\mathcal{F}^2 g_{ab}\big)\,, \end{aligned} \tag{17a}$$

$$\Big(\mathrm{e}^{((D-4)\alpha+3\beta)\phi} g^{bc}\mathcal{F}_{ab}\Big)_{;c} = 0\,, \tag{17b}$$

$$\mathrm{e}^{(2\beta-2\alpha)\,\phi}\mathcal{F}^2 = 4\beta g^{ab}\big(\phi_{;ab} + (\beta + (D-2)\,\alpha)\phi_{;a}\phi_{;b}\big)\,, \tag{17c}$$

where we have introduced the analog of the Maxwell tensor:

$$\mathcal{F}_{ab} := \mathcal{A}_{b;a} - \mathcal{A}_{a;b}\,, \tag{18a}$$

$$\mathcal{F}^2_{ab} := g^{cd}\mathcal{F}_{ac}\mathcal{F}_{bd}\,, \tag{18b}$$

$$\mathcal{F}^2 := g^{ab}\mathcal{F}^2_{ab}\,. \tag{18c}$$

Choosing

$$\beta + (D-2)\,\alpha = 0\,, \tag{19}$$

the reduced field equations (17) could be further simplified.[2]

[2]The assumption (19) is needed if we wish to interpret ϕ as the scalar in the Einstein–Maxwell–scalar system. For this reason, it is assumed throughout the paper by *Pope* [n.d.] as well.

The form of the Kaluza–Klein ansatz (15) is preserved by any z-independent coordinate transformation

$$\delta x^A = \delta x^A(x^a)\,. \tag{20}$$

In this case, ϕ transforms as a D-dimensional scalar:

$$\delta\phi = -\mathcal{L}_{\delta\tilde{x}}\,\phi\,, \tag{21a}$$

g_{ab} transforms as a D-dimensional symmetric second-order tensor:

$$\delta g_{ab} = -\mathcal{L}_{\delta\tilde{x}}\,g_{ab} \tag{21b}$$

and $\mathcal{A}_a$ transforms as a D-dimensional U(1) gauge field:

$$\delta\mathcal{A}_a = -\mathcal{L}_{\delta\tilde{x}}\,\mathcal{A}_a - \delta x^z{}_{,a}\,. \tag{21c}$$

Here, the Lie derivatives are taken along the orthogonal projections by $\boldsymbol{\partial}_z$ of the vector field δx^A. It is thus not a surprise that the D-dimensional field equations also have the U(1) gauge symmetry, and the electromagnetic potential can be eliminated from them in favor of the Maxwell tensor.

Type I and II Kaluza–Klein reductions

Given a null frame (1) in the reduced manifold, it is natural (in the sense that it simplifies computations) to define the null frame in the original manifold the following way [*Pope*, n.d.]:

$$\hat{\boldsymbol{e}}_{[a]} = \mathrm{e}^{-\alpha\phi}\,(\boldsymbol{e}_{(a)} - \mathcal{A}_{(a)}\boldsymbol{\partial}z)\,, \qquad a = 0,\ldots,D-1\,, \tag{22a}$$

$$\hat{\boldsymbol{e}}_{[z]} = \mathrm{e}^{-\beta\phi}\,\boldsymbol{\partial}z\,, \tag{22b}$$

where $\boldsymbol{e}_{(a)}$ is meant as the pushforward of the null frame (1) by the natural embedding mentioned in the previous section. The notation using the tensor frame components in this frame will be governed by conventions similar to those for the reduced frame (1).

The frame (22) is indeed a null frame:

$$^{\sharp\sharp}\hat{\boldsymbol{g}} = \hat{\boldsymbol{\ell}} \vee \hat{\boldsymbol{n}} + \sum_{i=2}^{D-1} \hat{\boldsymbol{e}}_{[i]}\hat{\boldsymbol{e}}_{[i]} + \hat{\boldsymbol{e}}_{[z]}\hat{\boldsymbol{e}}_{[z]}\,. \tag{23}$$

Its dual is

$$\hat{\boldsymbol{\varepsilon}}^{[a]} = \mathrm{e}^{\alpha\phi}\,\boldsymbol{\varepsilon}^{(a)}\,, \qquad a = 0,\ldots,D-1\,, \tag{24a}$$

$$\hat{\boldsymbol{\varepsilon}}^{[z]} = \mathrm{e}^{\beta\phi}\,(\mathbf{d}z + \boldsymbol{\mathcal{A}})\,, \tag{24b}$$

where $\boldsymbol{\varepsilon}^{(a)}$ is the dual of $\boldsymbol{e}_{(a)}$ in the subspace orthogonal to $\boldsymbol{\partial}_z$.

Assuming the original spacetime is vacuum, we would now like to determine the conditions necessary and sufficient for $\hat{\boldsymbol{\ell}}$ to be a WAND with multiplicity at least m, provided that $\boldsymbol{\ell}$ is a WAND with multiplicity at least m of the reduced manifold.[3] After expressing the $(D+1)$-dimensional Riemann tensor (which is equal to the Weyl tensor) in terms of the D-dimensional Weyl tensor and requiring the corresponding components to vanish, we arrive at the following results for Kaluza–Klein reductions of types I and II:

Proposition 1. *Let $D \geq 4$. Let $\boldsymbol{\ell}$ be a WAND of a D-dimensional spacetime. Suppose that this spacetime is a Kaluza–Klein reduction of a $(D+1)$-dimensional vacuum spacetime. Then $\hat{\ell}$ is a WAND of this $(D+1)$-dimensional spacetime, iff the following holds locally:*

$$\left(\mathrm{e}^{(\beta-2\alpha)\phi}\right)_{;(0)(0)} = 0\,, \tag{25a}$$

$$\mathcal{F}_{(0)(i)} = 0\,, \tag{25b}$$

$$\mathcal{F}_{(i)(j)}\kappa^{(j)} = 0\,, \tag{25c}$$

$$\mathcal{F}_{(0)(1)}\kappa^{(i)} = 0\,. \tag{25d}$$

[3] It should be noted that one can also ask whether $\hat{\boldsymbol{g}}$ is more special than $\boldsymbol{g}$. For a subset of the cases studied here, this question has been addressed by *Ortaggio et al.* [2011].

Proposition 2. *Let $D \geq 4$. Let $\boldsymbol{\ell}$ be a mWAND of a D-dimensional spacetime. Suppose that this spacetime is a Kaluza–Klein reduction of a $(D+1)$-dimensional vacuum spacetime. Then $\hat{\ell}$ is a mWAND of this $(D+1)$-dimensional spacetime, iff the following holds locally:*

$$\left(\mathrm{e}^{(\beta-2\alpha)\phi}\right)_{;(0)(i)} = 0\,, \tag{26a}$$

$$\left(\mathrm{e}^{(4\beta-4\alpha)\phi}\mathcal{F}_{(i)(j)}\right)_{;(0)} = 0\,, \tag{26b}$$

$$\left(\mathcal{F}_{(i)(k)} + \mathcal{F}_{(0)(1)}g_{(i)(k)}\right)\rho^{(k)}{}_{(j)} + (i \leftrightarrow j) = -4\alpha\mathcal{F}_{(0)(1)}g_{(i)(j)}\,, \tag{26c}$$

together with the conditions (25).

Remark. The condition (25b) is equivalent to saying that

$$\mathrm{mul}_{\langle\boldsymbol{\ell}\rangle}\,\boldsymbol{\mathcal{F}} \geq 1\,. \tag{27}$$

The conditions (25a), (26a) is equivalent to saying that

$$\mathrm{mul}_{\langle\boldsymbol{\ell}\rangle}\left(\mathrm{e}^{(\beta-2\alpha)\phi}\right)_{;ab} \geq 2\,. \tag{28}$$

Remark. The special case of $\boldsymbol{\mathcal{A}} = 0$ and $\phi = \text{const.}$ was studied earlier by *Ortaggio et al.* [2011] as a subset of metrics with a warped extra dimension. For this case, the results correspond with those covered by propositions 1 and 2.

Acknowledgment. I would like to thank my supervisor Mgr. Vojtěch Pravda, Ph.D., for suggesting this topic and for his constructive notes. I am grateful for support from the Czech Science Foundation No. 14-37086G (Albert Einstein Center for gravitation and astrophysics). I also thank the referees for careful reading and their valuable suggestions.

References

Coley A., R. Milson, V. Pravda, A. Pravdová, Classification of the Weyl tensor in higher dimensions, *Class. Quant. Grav.*, **21**(7), L35–L41, 2004.

Milson R., A. Coley, V. Pravda, A. Pravdová, Alignment and algebraically special tensors in Lorentzian geometry, *Int. J. Geom. Methods M.*, **02**(01), 41–61, 2005.

Ortaggio M., V. Pravda, A. Pravdová, Ricci identities in higher dimensions, *Class. Quant. Grav.*, **24**(6), 1657–1664, 2007.

Ortaggio M., V. Pravda, A. Pravdová, On higher dimensional Einstein spacetimes with a warped extra dimension, *Class. Quant. Grav.*, **28**(10), 105006–17, 2011.

Ortaggio M., V. Pravda, A. Pravdová, Algebraic classification of higher dimensional spacetimes based on null alignment, *Class. Quant. Grav.*, **30**(1), 013001–57, 2013.

Pope C., *Kaluza–Klein Theory*,
retrieved June 11, 2014, from `http://people.physics.tamu.edu/pope/ihplec.pdf`.

Pravda V., A. Pravdová, A. Coley, R. Milson, Bianchi identities in higher dimensions, *Class. Quant. Grav.*, **21**(12), 2873–2897, 2004.

Stephani H., et al., *Exact Solutions of Einstein's Field Equations*, Cambridge: Cambridge University Press, 2003.