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Source integrals for multipole moments in static and axially symmetric spacetimes

Norman Gürlebeck*

ZARM, University of Bremen, Am Fallturm, 28359 Bremen, Germany and Institute of Theoretical Physics, Charles University, V Holešovičkách 2, 180 00 Praha 8, Czech Republic

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In this article, we derive source integrals, i.e., quasilocal expressions, for multipole moments in axially symmetric and static spacetimes. Usually, these multipole moments are read off the asymptotics of the metric close to spatial infinity. Whereas for the evaluation of the here derived source integrals the geometry has to be known in the region containing all sources, i.e., matter as well as singularities. The source integrals can be written either as volume integrals over such a region or as integrals over the surface of that region. Eventually, these source integrals allow assigning to any spacetime regions its contribution to the total multipole moments of the spacetime. Finally, we give an exemplary application that outlines the usefulness and applicability of the source integrals in, e.g., (non)existence proofs.

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I. INTRODUCTION

In general relativity, several definitions of multipole moments were proposed in the past. Since this theory is nonlinear, it is, however, by no means obvious that a meaningful definition can be found. Thus, it is not surprising that in early works multipoles were considered only in approximations to general relativity that lead to linear field equations and allow a classical treatment. The most definitions in this direction were covered in Thorne's review [1].

From the 1960s on, new definitions in the full theory of isolated sources[1] emerged. These definitions of multipole moments can roughly be divided into two classes. In the first, the metric (or a quantity derived from it) is expanded at spacelike or null-like infinity. We will call these *asymptotic definitions* or *asymptotic multipole moments*. Amongst these are the definitions of Bondi, Metzner, Sachs, and van der Burgh (BMSB) [2,3], Geroch and Hansen (GH) [4,5], Simon and Beig[2] [6], Janis, Newman, and Unti (JNU) [7,8], Thorne[3] [1], the Arnowitt-Deser-Misner (ADM) approach [10], and the Komar integrals [11]; for reviews see [1,12]. Their scope of applicability varies greatly. Whereas the GH multipole moments are defined only in stationary spacetimes, the BMSB, JNU, Thorne, and ADM definitions hold in a more general setting. The Komar expression for the mass (angular momentum), on the other hand, requires stationarity (stationarity and axial symmetry). Higher order multipoles are not defined in the Komar approach. Despite their conceptual differences, Gürsel showed in [13] the equivalence of the GH and Thorne's multipole moments, in case the requirements of both definitions are met. Additionally, the mass and the angular momentum in the GH, Thorne, ADM, and Komar approach can be shown to agree.

Multipoles that are determined by the metric in a compact region fall in the second class. Dixon's multipoles in [14] are of this kind. They are given in the form of *source integrals*. However, it is not yet known how they are related with the asymptotic multipole moments. A main application of these multipoles is in the theory of the motion of test bodies with internal structure. But for test bodies such a relation between Dixon's multipole moments and the asymptotic multipole moments of the background spacetime cannot exist. Furthermore, Dixon's definition is in general not applicable if caustics of geodesics appear inside the source, i.e., if the gravitational field is strong compared to a characteristic radius of the source.

Ashtekar *et al.* defined in [15] multipole moments of isolated horizons that are source integrals as well and require only the knowledge of the interior geometry of the horizon. In [15], it was also shown that the so defined multipoles of the Kerr black hole deviate from the GH multipoles. This effect becomes more pronounced the greater the rotation parameter. In both cases [14,15], it is interesting to find the relation of these multipole moments to asymptotically defined GH multipole moments and the interpretation of possible deviations. Also a comparison with source integrals and multipole definitions in approximations to general relativity like those recommended by the International Astronomical Union (IAU) (see [16–18]) or those discussed in [19] might prove insightful.

Other definitions, in particular for the quasilocal mass and the quasilocal angular momentum, can be found in [20]. Here we aim at source integrals for *all* multipole moments in static spacetimes for arbitrary sources; i.e., we

*norman.guerlebeck@zarm.uni-bremen.de

[1]This means that all sources of the gravitational field (matter and singularities) are located in a sphere of finite radius and the spacetime is assumed to be asymptotically flat. A precise meaning is given in Sec. II C.

[2]This approach reproduces the GH multipole moments.

[3]For a more recent approach following Thorne, see [9].

want to express asymptotic multipole moments by surface (volume) integrals, where the surface envelopes (the volume covers) all the sources of the gravitational field. That such source integrals can be found is not trivial. This is due to the nonlinearity of the Einstein equations yielding a gravitational field, which acts again as a source. To overcome the principle difficulties, we will focus here on static and axially symmetric spacetimes. In this case, the vacuum Einstein equations can be cast in an essentially linear form. Additionally, they allow the introduction of a linear system [21–23]. The latter might seem superfluous, if the former holds. However, we derive our source integrals relying solely on the existence of the linear system and the applicability of the inverse scattering technique. This will allow us in future work to apply the same formalism to stationary and axially symmetric isolated systems. This is especially relevant for the description of relativistic stars. We summarize possible applications in this direction below. A generalization to spacetimes with an Einstein-Maxwell field exterior to the sources seems also feasible. In all these cases, multipole moments of the GH type exist, too.

Source integrals will prove useful in many respects, in particular in the search of global solutions of Einstein equations describing figures of equilibrium or relativistic stars; for recent efforts see, e.g., [24–27] and references therein. For example, an exterior solution can be constructed for a known interior solution by employing the source integrals to calculate its multipole moments. From these, the exterior solution can be completely determined. This yields an exterior solution, which does not necessarily match to the interior solution. However, if it does not, this construction shows that there is *no* asymptotically flat solution, which can be matched to the given interior. Conversely, possible matter sources can be analyzed for given exterior solutions. The latter approach is also of astrophysical interest, since often only the asymptotics of the gravitational field and the asymptotic multipole moments are accessible to experiments. Source integrals can be applied to restrict the equation of state of a rotating perfect fluid from observed multipole moments.

Furthermore, source integrals can be used to compare numerical solutions, analytical solutions, and analytical approximations by calculating their multipole moments; see, for example, [28]. This would also solve the difficulties of extracting the multipole moments of a given numerically determined metric as described in [29]. On the other hand, such an approach can also be used to approximate the vacuum exterior of a given numerical solution by an analytical one, which exhibits the correct multipole moments up to a prescribed order. The difference from the earlier work by [30] is that source integrals determine the multipole moments using the matter region only, which captures the internal structure of the relativistic object and is usually determined with high accuracy. Additionally, source integrals provide the means to test the accuracy of numerical methods, which are used to determine relativistic stars (cf. [26,31,32]) by calculating the multipole moments in two independent ways: First, using the asymptotics and, second, using the source integrals. This will also give a physical interpretation to possible deviations: Up to which multipole moment is the numerical solution viable.

The paper is organized as follows: In Sec. II, we introduce the different concepts used later, i.e., the GH and the Weyl multipole moments as well as the inverse scattering technique. Section III is devoted to the derivation of the source integrals and includes the main results. In Secs. IV and V, we discuss some properties of the source integrals and their application.

II. PRELIMINARIES

In this section, we will repeat the notions required in the present paper. Note that we use geometric units, in which $G = c = 1$, where c is the velocity of light and G Newton's gravitational constant. The metric has the signature $(-1, 1, 1, 1)$. Greek indices run from 0 to 3, lowercase Latin indices run from 1 to 3 and uppercase Latin indices from 1 to 2.

A. The line element and the field equation

We consider static and axially symmetric spacetimes admitting a timelike Killing vector ξ^α and a spacelike Killing vector η^α, which commutes with ξ^α, which has closed orbits, and which vanishes at the symmetry axis. If the orbits of the so-defined isometry group admit orthogonal 2-surfaces,[4] which is the case for vacuum, for static perfect fluids, or for static electromagnetic fields (see, e.g., [34]), then the metric can be written in the Weyl form,

$$ds^2 = \mathrm{e}^{2k-2U}(d\rho^2 + d\zeta^2) + W^2\mathrm{e}^{-2U}d\varphi^2 - \mathrm{e}^{2U}dt^2, \quad (1)$$

where the functions U, k, and W depend only on ρ and ζ. Note that the metric functions U and W can be expressed by the Killing vectors,

$$\mathrm{e}^{2U} = -\xi_\alpha\xi^\alpha, \qquad W^2 = -\eta_\alpha\eta^\alpha\xi_\beta\xi^\beta. \quad (2)$$

The Einstein equations simplify for the metric (1); cf. [35]. Since we do not specify the matter here, we give only a complete set of combinations of the nontrivial components of the Ricci tensor,

[4]That this is not the general case can be seen from the conformally flat, static, and axially symmetric spacetimes in [33]. The field equations are not imposed there, but they can be used to define a (rather unphysical) stress energy tensor.

$$0 = e^{2k-4U} W R_{tt} - W\Delta^{(2)} U - U_{,\rho} W_{,\rho} - U_{,\zeta} W_{,\zeta},$$
$$0 = W(R_{\zeta\zeta} - R_{\rho\rho}) - W_{,\rho\rho} + W_{,\zeta\zeta} - 2(k_{,\zeta} W_{,\zeta} - k_{,\rho} W_{,\rho} + W(U_{,\rho}^2 - U_{,\zeta}^2)),$$
$$0 = e^{2k-4U} W^2 R_{tt} - e^{2k} R_{\varphi\varphi} - W\Delta^{(2)} W,$$
$$0 = W R_{\rho\zeta} - W_{,\zeta} k_{,\rho} - W_{,\rho} k_{,\zeta} + 2W U_{,\zeta} U_{,\rho} + W_{,\rho\zeta},$$
$$0 = (R_{\rho\rho} + R_{\zeta\zeta}) - e^{2k-4U} R_{tt} - \frac{e^{2k}}{W^2} R_{\varphi\varphi} + 2\Delta^{(2)} k, \qquad (3a)$$

where $\Delta^{(n)} = (\frac{\partial^2}{\partial\rho^2} + \frac{n-2}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2})$. The third equation implies that we can introduce canonical Weyl coordinates $(\tilde{\rho}, \tilde{\zeta})$ with $W = \tilde{\rho}$ via a conformal transformation in vacuum or in matter regions, where $\Delta^{(2)} W = 0$ holds. This includes the important case of a perfect fluid with vanishing pressure, i.e., dust. After this coordinate system is chosen in the vacuum region, we drop the tilde again. The remaining coordinate freedom is a shift of the origin along the symmetry axis, which is characterized by $\rho = 0$. Note that the metric functions are assumed to be sufficiently smooth so that the field equations are well defined.

Equations (3a) simplify in vacuum to the well-known equations

$$\Delta^{(3)} U = 0,$$
$$k_{,\zeta} = 2\rho U_{,\rho} U_{,\zeta},$$
$$k_{,\rho} = \rho((U_{,\rho})^2 - (U_{,\zeta})^2). \qquad (3b)$$

The last two equations determine k via a line integration once U is known. This k automatically satisfies the last equation in (3a). Hence, only a Laplace equation for U remains to be solved. Therefore, the Newtonian theory and general relativity can be treated on the same formal footing. The disadvantage of using the canonical Weyl coordinates is that they cannot necessarily be introduced in the interior of the matter, where we have to use other (noncanonical) Weyl coordinates.

B. The sources

We assume isolated sources of the gravitational field, which are in compliance with Eq. (1). Hence, there exists a 2-sphere with radius $\mathcal{R}_0$ ($\rho = \mathcal{R}_0 \sin\theta, \zeta = \mathcal{R}_0 \cos\theta$), $\mathcal{S}_0^{(2)}$, enclosing all sources; cf. Fig. 1. The upper index in brackets is necessary because of the axial symmetry. It indicates the dimensionality of the respective set. It is used to distinguish between, say $\mathcal{S}_0^{(3)}$ and $\mathcal{S}_0^{(2)}$, where $\mathcal{S}_0^{(3)}$ is the direct product of $\mathcal{S}_0^{(2)}$ with the orbits of η^α.

If there are black holes with horizons $\mathcal{H}_i$ present in the spacetime, we assume that they admit a neighborhood, which is free of any matter. This is satisfied for static black holes in static spacetimes if the energy conditions are met; see, e.g., [36]. Hence, we can define closed surfaces $\mathcal{S}_{S,i}^{(3)}$ inside of $\mathcal{S}_0^{(3)}$ enclosing only the black hole with $\mathcal{H}_i$. Similarly, we can introduce surfaces $\mathcal{S}_{S,i}^{(3)}$ enclosing other kinds of singularities if they have a vacuum neighborhood. Then, we define $\mathcal{V}_0^{(3)}$ as the spacelike region bounded by $\mathcal{S}_0^{(3)}$ and $\mathcal{S}_{S,i}^{(3)}$. The vacuum region exterior to $\mathcal{S}_0^{(3)}$ is denoted by $\mathcal{E}^{(3)}$ and extends to infinity. The regions $\mathcal{V}_i^{(3)}$ with the surfaces $\mathcal{S}_i^{(3)}$ are those where the stress energy tensor is nonvanishing. They are not necessarily connected.

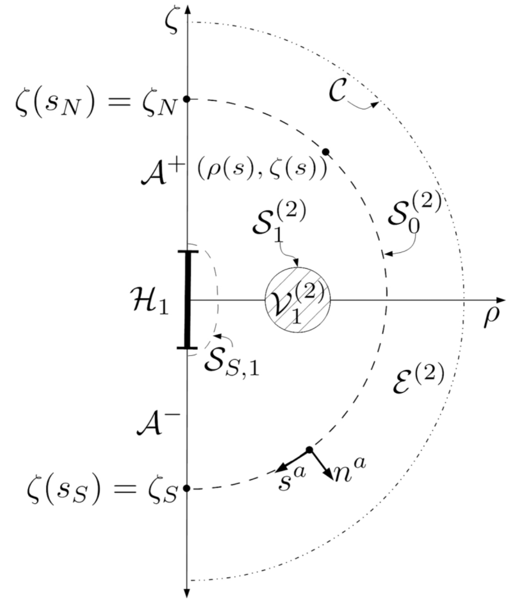

FIG. 1. The different surfaces $\mathcal{H}_i$, $\mathcal{S}_i^{(2)}$, $\mathcal{S}_{S,i}^{(2)}$ and volumes $\mathcal{V}_i^{(2)}$ are depicted for a black hole surrounded by a thick ring. The curves $\mathcal{A}^\pm$ and $\mathcal{C}$ are relevant in Sec. II D.

Moreover, we assume that there are no matter layers at the surfaces $\mathcal{S}_i^{(3)}$ for technical reasons. If surface layers are present, $\mathcal{V}_i^{(3)}$ has to be chosen slightly bigger such that it covers also some part of the vacuum. Then the first and the second junction conditions (cf. [37]) are satisfied at its surface.

C. Asymptotic multipole moments

Isolated gravitating systems are described by an asymptotically flat spacetime. The precise meaning of this is given below. It is used to define Geroch's multipole moments. Afterwards, Weyl's multipole moments are introduced, and they are compared with Geroch's multipole moments. Let us denote our static spacetime by $(M, g_{\alpha\beta})$ with the metric $g_{\alpha\beta}$ and by $M^{(3)}$ a hypersurface that is orthogonal to ξ^α endowed with the induced metric $g_{\alpha\beta}^{(3)} = -\xi^\gamma\xi_\gamma g_{\alpha\beta} + \xi_\alpha\xi_\beta$. We use lowercase Latin indices for tensors on $M^{(3)}$. The asymptotic flatness of $M^{(3)}$ is sufficient for the definition of Geroch's multipole moments; see [4].

Definition 1 $(M^{(3)}, g_{ab}^{(3)})$ is asymptotically flat iff there exists a point Λ, a manifold $\tilde{M}^{(3)}$, and a conformal factor $\Omega \in C^2(\tilde{M}^{(3)})$ such that

(1) $\tilde{M}^{(3)} = M^{(3)} \cup \Lambda$,

(2) $\tilde{g}_{ab}^{(3)} = \Omega^2 g_{ab}^{(3)}$ is a smooth metric of $\tilde{M}^{(3)}$,

(3) $\Omega = \tilde{D}_a\Omega = 0$ and $\tilde{D}_a\tilde{D}_b\Omega = 2\tilde{g}^{(3)}_{ab}$ at Λ, where $\tilde{D}_a$ is the covariant derivative in $\tilde{M}^{(3)}$ associated with the metric $\tilde{g}^{(3)}_{ab}$.

Let us define $\tilde{\psi}$,

$$\tilde{\psi} = \frac{1-(-\xi^\alpha\xi_\alpha)^{\frac{1}{2}}}{\Omega^{\frac{1}{2}}}, \tag{4}$$

which is also a scalar in $\tilde{M}^{(3)}$. Introducing the Ricci tensor $\tilde{R}^{(3)}_{ab}$ built from the metric $\tilde{g}^{(3)}_{ab}$, tensors $P_{a_1\ldots}$ can be defined recursively,

$$P = \tilde{\psi},$$
$$P_{a_1\ldots a_r} = C\left[P_{a_2\ldots a_r||a_1} - \frac{(n-1)(2n-3)}{2}\tilde{R}^{(3)}_{a_1a_2}P_{a_3\ldots a_r}\right], \tag{5}$$

where $C[A_{a_1\ldots a_r}]$ denotes the symmetric and trace-free part of $A_{a_1\ldots a_r}$. The covariant derivative with respect to $\tilde{g}^{(3)}_{ab}$ is indicated by $||$. Geroch's multipole moments are defined as the tensors $P_{a_1\ldots a_r}$ evaluated at Λ,

$$M_{a_1\ldots a_r} = P_{a_1\ldots a_r}|_\Lambda. \tag{6}$$

This expansion at infinity is the reason for the name *asymptotic multipole moments*. The degree of freedom in the choice of the conformal factor reflects the choice of an origin, with respect to which the multipole moments are taken; see [4]. This definition was generalized to the stationary case in [5], to the electrostatic case in [38], and to the electrostationary case in [39]. Its merit is that both multipole conjectures of Geroch can be proved. The first states roughly that two spacetimes with the same asymptotic multipole moments are isometric in a neighborhood of Λ. Thus, a spacetime is characterized by its multipole moments. The proof of the second conjecture established that a spacetime can be found for any given set of asymptotic multipole moments, which satisfy a certain convergence condition; cf. [40–44].

For axially symmetric spacetimes the multipole structure simplifies to

$$m_r = \frac{1}{n!}P_{a_1\ldots a_r}\tilde{z}^{a_1}\cdots\tilde{z}^{a_r}\Big|_\Lambda, \tag{7}$$

where $\tilde{z}^a$ is the unit vector pointing in the direction of the symmetry axis and the scalars m_r define the multipole moments completely. Hence, we will refer to them as multipole moments, as well.

Although the definition due to Geroch is conceptually pleasing, it is not always the most practical approach. However, Fodor *et al.* showed in [45] that an expansion of the potential U [cf. Eq. (1)] along the axis of symmetry in powers of $\frac{1}{|\zeta|}$ is sufficient to determine the m_r. The expansion coefficients $U^{(r)}$, which will be called Weyl's multipole moments, are defined via

$$U(\rho=0,\zeta) = \sum_{r=1}^{\infty}\frac{U^{(r)}}{|\zeta|^{r+1}}. \tag{8}$$

Note that this expansion is well defined, since U is harmonic in the vacuum region close to infinity [cf. Eqs. (3b)]. The relation $m_r(U^{(j)})$ can be obtained in principle to any order. Thus, we limit ourselves here to the calculation of the $U^{(r)}$. Up to now, however, only the $m_0,\ldots,m_{11}$ were explicitly expressed using the $U^{(r)}$; see [45,46]. We give here the first four for illustration,

$$m_0 = -U^{(0)}, \qquad m_1 = -U^{(1)},$$
$$m_2 = \frac{1}{3}U^{(0)3} - U^{(2)}, \qquad m_3 = U^{(0)2}U^{(1)} - U^{(3)}. \tag{9}$$

Equations (8) and (9) show that the mass dipole moment, $U^{(1)}$, can be transformed away if the origin of ζ is chosen appropriately. For a general discussion, further references, and expressions of the center of mass in static spacetimes, see [47]. In [48], a method to obtain the m_r was proposed, which could help to overcome the nonexplicit structure of $m_r(U^{(j)})$. Also the results in [43,49] are useful in this respect.

D. The linear problem of the Laplace equation

Last, we give a short review of the linear problem associated with the Laplace equation. Although the equations involved are fairly simple, we employ this technique here because it is readily generalizable to the nonlinear stationary case.

In the more general case of stationarity and axially symmetry, the Einstein equation simplifies to the Ernst equation (see [50]). The solution of this equation can be achieved by the inverse scattering technique (see, e.g., [21–23] and for a more recent account [51]). In the static case, the linear problem associated with the field equations (3b) reads[5]

$$\sigma_{,z} = (1+\lambda)U_{,z}\sigma, \qquad \sigma_{,\bar{z}} = \left(1+\frac{1}{\lambda}\right)U_{,\bar{z}}\sigma, \tag{10}$$

where $z = \rho + \mathrm{i}\zeta$. The spectral parameter $\lambda = \sqrt{\frac{K-\mathrm{i}\bar{z}}{K+\mathrm{i}z}}$ depends on a constant $K \in \mathbb{C}$. A bar denotes complex conjugation. The function σ depends on $z, \bar{z}$, and λ. The integrability condition of Eqs. (10) yields the first equation in (3b).

[5] The formulas are easily inferred from [51] by setting $g_{t\varphi} = 0$.

Next we repeat some known properties of σ, which we need in the next section, without proof. For details we refer the reader to [51]. The four curves $\mathcal{A}^{\pm}$, $\mathcal{S}_0^{(2)}$, and $\mathcal{C}$ in the (ρ,ζ) plane are of particular interest and are described momentarily; cf. Fig. 1 as well. The axis of symmetry is divided by $\mathcal{V}_0^{(2)}$ into upper and lower parts $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$, respectively. $\mathcal{C}$ describes a half circle with a sufficiently large radius connecting $\mathcal{A}^{+}$ with $\mathcal{A}^{-}$.

Along $\mathcal{A}^{\pm}$ and $\mathcal{C}$, Eq. (10) can be integrated. For a suitable choice of the constant of integration, this gives

$$\begin{aligned}(0,\zeta)\in\mathcal{A}^{+}\colon\ &\sigma(\lambda=+1,\rho=0,\zeta)=F(K)\mathrm{e}^{2U(\rho=0,\zeta)},\\ &\sigma(\lambda=-1,\rho=0,\zeta)=1,\\ (0,\zeta)\in\mathcal{A}^{-}\colon\ &\sigma(\lambda=+1,\rho=0,\zeta)=\mathrm{e}^{2U(\rho=0,\zeta)},\\ &\sigma(\lambda=-1,\rho=0,\zeta)=F(K).\end{aligned}\tag{11}$$

The function $F\colon\ \mathbb{C}\to\mathbb{C}$ is given for $K\in\mathbb{R}$ by

$$F(K)=\begin{cases}\mathrm{e}^{-2U(\rho=0,\zeta=K)} & (0,K)\in\mathcal{A}^{+},\\ \mathrm{e}^{2U(\rho=0,\zeta=K)} & (0,K)\in\mathcal{A}^{-}.\end{cases}\tag{12}$$

The integration along $\mathcal{S}_0^{(2)}$ is the crucial part for our considerations in the next section.

III. SOURCE INTEGRALS

Let us assume that the line element is written in canonical Weyl coordinates in $\mathcal{E}^{(3)}$; cf. Sec. II A. The scalars U and W [cf. (2)] are supposed to be continuously differentiable in $\mathcal{E}^{(2/3)}\cup\mathcal{S}_0^{(2/3)}$. Let us introduce the vectors[6] $(s^A)=(s^\rho,s^\zeta)=(\frac{\mathrm{d}\rho}{\mathrm{d}s},\frac{\mathrm{d}\zeta}{\mathrm{d}s})$ and $(n^A)=(n^\rho,n^\zeta)=(-\frac{\mathrm{d}\zeta}{\mathrm{d}s},\frac{\mathrm{d}\rho}{\mathrm{d}s})$ that are tangential and normal to the curve $\mathcal{S}_0^{(2)}\colon s\in[s_N,s_S]\to(\rho(s),\zeta(s))$. The parameter values $s_{N/S}$ denote the "north/south" pole, i.e., $(\rho=0,\zeta=\zeta_{N/S})$; cf. Fig. 1. Analogously, the indices N/S indicate that a function is evaluated at the respective pole. Note that s^A and n^A are not necessarily normalized allowing an arbitrary parametrization of $\mathcal{S}_0^{(2)}$. If the vectors are normalized with respect to the induced metric on $\mathcal{S}_0^{(2)}$, we distinguish them by a hat. Moreover, the projection of the derivative of a function, say U, in the direction of $s^A, n^A, \hat{s}^A$, and $\hat{n}^A$ is denoted by $U_{,s}, U_{,n}, U_{,\hat{s}}$, and $U_{,\hat{n}}$, respectively. With this notation, we can consider the linear problem (10) along $\mathcal{S}_0^{(2)}$,

$$\sigma_{,s}=\left[U_{,s}+\frac{1}{2}\left(\left(\frac{1}{\lambda}+\lambda\right)U_{,s}+\mathrm{i}\left(\frac{1}{\lambda}-\lambda\right)U_{,n}\right)\right]\sigma.\tag{13}$$

[6]Capital Latin indices are used for objects that are projected orthogonal to the orbits of η^α and ξ^α.

Equation (13) constitutes an ordinary differential equation of first order with the boundary conditions as given in (11) assuming $(0,K)\in\mathcal{A}^{+}\cup\mathcal{A}^{-}$. As such it is an overdetermined system and the boundary data have to satisfy a compatibility condition, which corresponds to the integrability of Eqs. (10). Nonetheless, Eq. (13) is readily integrated and the compatibility condition can be read off explicitly,

$$U(0,K)=\frac{U_N-U_S}{2}+\frac{1}{4}\int_{s_N}^{s_S}(N_+U_{,s}+N_-U_{,n})\mathrm{d}s,\tag{14}$$

where we introduced the abbreviations $N_+=\lambda^{-1}+\lambda$ and $N_-=\mathrm{i}(\lambda^{-1}-\lambda)$. Equation (14) determines the axis values of U from the Dirichlet data and the Neumann data of U along $\mathcal{S}_0^{(2)}$.

Weyl's multipole moments follow from an expansion of Eq. (14) with respect to K^{-1}, which is possible because λ is holomorphic for sufficiently large $|K|$. Let us denote by $f^{(r)}$ the expansion coefficient to order $|K|^{-r-1}$ of a function $f(K)$, which is constant at infinity,

$$f(K)=\sum_{r=-1}^{\infty}f^{(r)}|K|^{-r-1}.\tag{15}$$

The coefficients $N_\pm^{(r)}$ depend still on (ρ,ζ), and they satisfy the equations

$$\begin{aligned}N_{+,\rho}^{(r)}-N_{-,\zeta}^{(r)}&=0,\\ N_{+,\zeta}^{(r)}+N_{-,\rho}^{(r)}-\frac{1}{\rho}N_{-}^{(r)}&=0.\end{aligned}\tag{16}$$

Equations (16) follow directly from the form of the spectral parameter λ; cf. the text after Eq. (10). Furthermore, $N_\pm^{(r)}$ and their radial derivatives evaluate at the axis to

$$\begin{aligned}N_{-}^{(r)}(\rho=0,\zeta)&=0\qquad\forall r\geq-1,\\ N_{+}^{(-1)}(\rho=0,\zeta)&=2,\\ N_{+}^{(r)}(\rho=0,\zeta)&=0\qquad\forall r\geq0,\\ N_{-,\rho}^{(-1)}(\rho=0,\zeta)&=0,\\ N_{-,\rho}^{(r)}(\rho=0,\zeta)&=-2\zeta^r\qquad\forall r\geq0.\end{aligned}\tag{17}$$

Together with Eq. (17), the zeroth order of Eq. (14) implies $U^{(-1)}=0$. This reflects the assumption of asymptotic flatness employed in the derivation of (11). Solving Eqs. (16) with the initial values (17), a lengthy calculation gives us the $N_\pm^{(r)}$ for $r\geq0$ everywhere,

$$N_-^{(r)}(x,y)=\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\frac{2(-1)^{k+1}r!x^{2k+1}y^{r-2k}}{4^k(k!)^2(r-2k)!},$$
$$N_+^{(r)}(x,y)=\sum_{k=0}^{\lfloor\frac{r-1}{2}\rfloor}\frac{2(-1)^{k+1}r!x^{2k+2}y^{r-2k-1}}{4^k(k!)^2(r-2k-1)!(2k+2)}. \tag{18}$$

Here, (x,y) replaces (ρ,ζ) to avoid later confusion. Note that $N_\pm^{(r)}=O(x)$ for all $r\geq 0$.

For orders $r\geq 0$, Eq. (14) yields together with Eq. (18) the following line integrals defining Weyl's multipole moments:

$$U^{(r)}=\frac{1}{4}\int_{\mathcal{S}_0^{(2)}}(N_+^{(r)}U_{,\hat{s}}+N_-^{(r)}U_{,\hat{n}})\mathrm{d}\mathcal{S}_0^{(2)}, \tag{19}$$

where $\mathrm{d}\mathcal{S}_0^{(2)}$ denotes the proper distance along $\mathcal{S}_0^{(2)}$. We define $\mathrm{d}\mathcal{S}_i^{(2/3)}$, $\mathrm{d}\mathcal{S}_{S,i}^{(2/3)}$, $\mathrm{d}\mathcal{V}_0^{(2/3)}$, and $\mathrm{d}\mathcal{V}_i^{(2/3)}$ analogously.

Equations (19) are already quasilocal expressions, since they determine the multipole moments from the metric given in a compact region. Subsequently, we rewrite these multipole moments as volume integrals, and we generalize them to arbitrary spacetime regions. Thereby, the term *source integral* is justified. The main obstacle to overcome is that Weyl's multipole moments are given in Eq. (19) using canonical Weyl coordinates. Hence, neither the coordinate invariance of these expressions is transparent nor is it obvious how to continue ρ and ζ into $\mathcal{V}_0^{(2)}$. Thus, we express first Eq. (19) covariantly.

U can be expressed by the norm of the timelike Killing vector; cf. Eq. (2). Moreover, ρ equals W in the vacuum region $\mathcal{E}^{(3)}$ and, thus, can be expressed by the product of the norms of the Killing vectors; see Eq. (2). Hence, a continuation of these two functions to $\mathcal{V}_0^{(3)}$ is straightforward. It only remains the coordinate ζ, for which we derive a covariant expression and an unambiguous continuation subsequently.

Let us introduce the 1-form

$$Z_\alpha=\epsilon_{\alpha\beta\gamma\delta}W^{,\beta}W^{-1}\eta^\gamma\xi^\delta, \tag{20}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ denotes the volume form of the static spacetime. A simple calculation shows that Z_α is closed in $\mathcal{E}^{(3)}$ and hypersurface orthogonal in the entire spacetimes. Thus, we can introduce a scalar potential Z with $Z_{,\alpha}=XZ_\alpha$, where $X=1$ in $\mathcal{E}^{(3)}$.[7] In canonical Weyl coordinates and in $\mathcal{E}^{(3)}$, the potential Z has the form $Z=\zeta+\zeta_0$. Because we did not fix the origin of our Weyl coordinates, e.g., the value of ζ_N, we can set the constant $\zeta_0=0$ without loss of generality. This integration constant describes the arbitrary choice of the origin with respect to which the multipole moments are measured. This allows us to change to the center of mass frame, where the mass dipole moment vanishes. Now, we can extend ζ to $\mathcal{V}_0^{(3)}$ using Z, which is defined everywhere and coincides with ζ in $\mathcal{E}^{(3)}$. Note that $W_{,\alpha}$ and $Z_{,\alpha}$ are orthogonal everywhere and have the same norm in $\mathcal{E}^{(3)}$. The line integral (19) becomes inherently coordinate independent, if one reads $N_\pm^{(r)}$ not as functions of $(\rho(s),\zeta(s))$ but rather as a function of (W,Z), which in turn depend on s. This dependence, however, will not be shown explicitly in what follows.

In $\mathcal{E}^{(3)}$, the functions W and Z satisfy the Cauchy-Riemann equations as a consequence of the field equations, which read along $\mathcal{S}_0^{(2)}$ as

$$W_{,s}=Z_{,n},\qquad W_{,n}=-Z_{,s}. \tag{21}$$

After an integration by parts, we rewrite the line integral Eq. (19) as a surface integral using the axial symmetry and Eqs. (21),

$$U^{(r)}=\int_{\mathcal{S}_0^{(3)}}\eta_a^{(r)}\hat{n}^a\mathrm{d}\mathcal{S}_0^{(3)},$$
$$\eta_a^{(r)}=\frac{1}{8\pi}\frac{\mathrm{e}^U}{W}(N_-^{(r)}U_{,a}-N_{+,W}^{(r)}Z_{,a}U+N_{+,Z}^{(r)}W_{,a}U). \tag{22}$$

Since we ruled out surface distributions, Israel's junction conditions imply that Eqs. (22) can be understood as integrals over the surfaces $\mathcal{S}_0^{(3)}$ as seen from the exterior and the interior.

Using Stokes' theorem, we rewrite the surface integral in (22) as volume integrals over $\mathcal{V}_0^{(3)}$,

$$U^{(r)}=\int_{\mathcal{V}_0^{(3)}}\mu^{(r)}\mathrm{d}\mathcal{V}_0^{(3)}+\sum_i\int_{\mathcal{S}_{S,i}^{(3)}}\eta_a^{(r)}\hat{n}^a\mathrm{d}\mathcal{S}_{S,i}^{(3)},$$
$$\mu^{(r)}=\eta^{(r)a}{}_{:a}. \tag{23}$$

A colon denotes a covariant derivative with respect to the 3-metric $g_{ab}^{(3)}$ of $M^{(3)}$; cf. Sec. II C. Here we use the extension of $\eta_a^{(r)}$ described after Eqs. (19) and (20). Moreover, the surface of $\mathcal{V}_0^{(3)}$ is the union of $\mathcal{S}_0^{(3)}$ and all $\mathcal{S}_{S,i}^{(3)}$; cf. Fig. 1. Thus, the integrals over $\mathcal{S}_{S,i}^{(3)}$, the surfaces enclosing singularities, have to be subtracted from the volume integral over $\mathcal{V}_0^{(3)}$. We concentrate here on the volume integral and simplify $\mu^{(r)}$ subsequently. The calculation of the surface terms has to be carried out independently.

To simplify $\mu^{(r)}$, we use general Weyl coordinates and the field equations (3a) for arbitrary matter. Additionally, we employ the identities

$$W_{,a}Z^a=W_{,a}Z^{,a}=Z^\alpha_{;\alpha}=W^{,\alpha}W_{,\alpha}-Z^\alpha Z_\alpha=0 \tag{24}$$

[7]Wherever $\Delta^{(2)}W=0$ holds, $X=1$ can be chosen.

and Eqs. (16). Then standard calculations yield

$$\mu^{(r)} = \frac{1}{8\pi\sqrt{g^{(3)}}}\sum_a\left(\sqrt{g^{(3)}}\frac{\mathrm{e}^U}{W}g^{\rho\rho}(N_-^{(r)}U_{,a} - N_{+,W}^{(r)}Z_{,a}U + N_{+,Z}^{(r)}W_{,a}U)\right)_{,a}$$
$$= \frac{1}{8\pi\sqrt{g^{(3)}}}\sum_{a\in\{\rho,\zeta\}}((N_{-,W}^{(r)} + N_{+,Z}^{(r)})W_{,a}U_{,a} + (N_{-,Z}^{(r)} - N_{+,W}^{(r)})Z_{,a}U_{,a} + N_-^{(r)}U_{,aa} - N_{+,W}^{(r)}Z_{,aa}U$$
$$+ N_{+,Z}^{(r)}W_{,aa}U + N_{+,WZ}^{(r)}(W_{,a}W_{,a} - Z_{,a}Z_{,a})U), \tag{25}$$

where $g^{(3)}$ is the determinant of the metric $g_{ab}^{(3)}$ in Weyl coordinates. Analogously, we define the determinant $g^{(4)}$ of the metric $g_{\alpha\beta}$. They read

$$g^{(3)} = \mathrm{e}^{4k-6U}W^2, \qquad g^{(4)} = -\mathrm{e}^{4k-4U}W^2. \tag{26}$$

The second term in the sum in the second line of Eq. (25) vanishes because of (16). The first simplifies to

$$\frac{1}{8\pi\sqrt{g^{(3)}}}\sum_{a\in\{\rho,\zeta\}}\frac{N_-^{(r)}}{W}W_{,a}U_{,a}, \tag{27}$$

which in turn gives together with the third term and the first equation in (3a)

$$\frac{\mathrm{e}^{-U}}{8\pi W}N_-^{(r)}R_{tt} = -\frac{\mathrm{e}^U}{8\pi W}N_-^{(r)}R_{\alpha\beta}\frac{\xi^\alpha\xi^\beta}{\xi^\gamma\xi_\gamma}. \tag{28}$$

Using the third field equation in (3a), the fifth term in the second line of (25) can be rewritten as

$$\frac{N_{+,Z}^{(r)}U}{8\pi\sqrt{g^{(3)}}}\sum_{a\in\{\rho,\zeta\}}W_{,aa} = \frac{N_{+,Z}^{(r)}U}{8\pi\sqrt{g^{(3)}}}\Delta^{(2)}W$$
$$= -\frac{\mathrm{e}^U}{8\pi}N_{+,Z}^{(r)}UR_{\alpha\beta}\left(\frac{\xi^\alpha\xi^\beta}{\xi_\gamma\xi^\gamma} + \frac{\eta^\alpha\eta^\beta}{\eta_\gamma\eta^\gamma}\right). \tag{29}$$

A semicolon denotes the covariant derivative with respect to $g_{\alpha\beta}$. The last equality follows from Eq. (3a). Analogously, we have

$$-\frac{N_{+,W}^{(r)}U}{8\pi\sqrt{g^{(3)}}}\sum_{a\in\{\rho,\zeta\}}Z_{,aa} = -\frac{\mathrm{e}^U}{8\pi}N_{+,W}^{(r)}U\left(\frac{Z^{;\alpha}}{W}\right)_{;\alpha}$$
$$= -\frac{\mathrm{e}^U}{8\pi W}N_{+,W}^{(r)}UX_{,\alpha}Z^\alpha. \tag{30}$$

The last equality uses the definition of the potential Z of Z_α; see the discussion after Eq. (20). Also the last two terms in Eq. (25) can be expressed in terms of X as

$$\frac{1}{8\pi\sqrt{g^{(3)}}}N_{+,WZ}^{(r)}U\sum_{a\in\{\rho,\zeta\}}(W_{,a}W_{,a} - Z_{,a}Z_{,a})$$
$$= \frac{\mathrm{e}^U}{8\pi W}N_{+,WZ}^{(r)}UW_{,\alpha}W^{;\alpha}(1 - X^2). \tag{31}$$

Thus, we arrive at the covariant form

$$\mu^{(r)} = \frac{\mathrm{e}^U}{8\pi W}\left(-R_{\alpha\beta}\left[WUN_{+,Z}^{(r)}\left(\frac{\eta^\alpha\eta^\beta}{\eta^\gamma\eta_\gamma} + \frac{\xi^\alpha\xi^\beta}{\xi^\gamma\xi_\gamma}\right) + N_-^{(r)}\frac{\xi^\alpha\xi^\beta}{\xi^\gamma\xi_\gamma}\right]\right.$$
$$\left.- N_{+,W}^{(r)}UX_{,\alpha}Z^\alpha + N_{+,WZ}^{(r)}UW_{,\alpha}W^{;\alpha}(1 - X^2)\right). \tag{32}$$

In Eq. (32), it is apparent that $\mu^{(r)}$ vanishes in vacuum, where $R_{\alpha\beta} = 0$ and $X = 1$. Hence, the integrals have to be evaluated only in the matter regions $\mathcal{V}_i^{(3)}$,

$$U^{(r)} = \sum_i\int_{\mathcal{V}_i^{(3)}}\mu^{(r)}\mathrm{d}\mathcal{V}_i^{(3)} + \sum_i\int_{\mathcal{S}_{S,i}^{(3)}}\eta_a^{(r)}\hat{n}^a\mathrm{d}\mathcal{S}_{S,i}^{(3)}. \tag{33}$$

This form justifies the term *source integrals*: Only the regions with sources contribute to the asymptotic Weyl multipole moments, and they all can be linearly superposed to yield the total asymptotic Weyl multipole moments. However, the transformation from Weyl's multipole moments to Geroch's is nonlinear except for the mass and the mass dipole; cf. Eq. (9). Thus, there is no *linear* superposition of the multipole contributions of the individual sources to the total Geroch multipole moments, and a mixing of the contributions of the individual sources takes place, which can be disentangled using the above source integrals. For example, Weyl's quadrupole moment of a spacetime containing two sources with the source quadrupole moments $U_i^{(2)}$ is $U_{\mathrm{tot}}^{(2)} = U_1^{(2)} + U_2^{(2)}$. Let us define now Geroch's mass and quadrupole moment for individual sources following Eq. (9) by

$$m_{0,i} = -U_i^{(0)}, \qquad m_{2,i} = \frac{1}{3}U_i^{(0)2} - U_i^{(2)}. \tag{34}$$

Then Geroch's (total) quadrupole moment is given by

$$m_{2,\mathrm{tot}} = m_{2,1} + m_{2,2} + 2m_{0,1}m_{0,2}. \tag{35}$$

Thus, an additional term mixing the masses appears. This can be generalized to higher orders beyond the quadrupole moment and arbitrary sources: The total Geroch multipole moment $m^{(r)}_{\rm tot}$ will always be given as a sum of the source multipole moments of order r of all individual sources and additional terms mixing the source multipole moments $m^{(k)}_i$ of lower order $k < r$.

Using Eq. (33), we can assign multipole moments to *any* spacetime region $\mathcal{V}^{(3)}$ as integrals over the $\mu^{(r)}$ and, if singularities are present, over the $\eta^{(r)}_a$. We only have to substitute the corresponding regions in Eq. (33) with $\mathcal{V}^{(3)}$ and $\mathcal{S}^{(3)}$. If the metric and its derivatives are not known explicitly in $\mathcal{S}^{(3)}$ but on its surface $\mathcal{S}^{(3)}$, a form of the source integral analogous to Eq. (22) can be used to determine the multipole moments. For this, the aforementioned calculations have to be retraced to arrive at the surface integral form,

$$U^{(r)} = \int\limits_{\mathcal{S}^{(3)}} \eta^{(r)}_a \hat{n}^a \mathrm{d}\mathcal{S}^{(3)}. \tag{36}$$

If $\mathcal{S}^{(3)}$ is bounded by vacuum from one side, we can again use Eq. (21) to write Eq. (36) in a line integral form,

$$U^{(r)} = \frac{1}{4}\int\limits_{\mathcal{S}^{(2)}} (N^{(r)}_{+} U_{,\hat{s}} + N^{(r)}_{-} U_{,\hat{n}})\mathrm{d}\mathcal{S}^{(2)}. \tag{37}$$

Which of the derived forms of the source integrals (19), (22), (23), (33), (36), (37) together with (32) and (33) are employed is a matter of the concrete applications.

So far, we used the Einstein equations only in $\mathcal{E}^{(3)}$. Thus, as long as the used quantities are well defined, the above derivation is also valid for line elements of the form (1). There is a wide variety of alternative theories of gravity equivalent to general relativity in vacuum, e.g., the Eddington-inspired Born-Infeld theory and subclasses of theories admitting a nonminimal coupling; see, e.g., [52–54]. However, note that we can express the Ricci tensor by the stress-energy tensor in Einstein's theory of gravity.

IV. PROPERTIES OF THE SOURCE INTEGRALS

In this section, we discuss the properties of the source integrals in more detail.

A. The Newtonian limit

First we recover the well-known Newtonian multipole definitions taking the Newtonian limit of the source integrals given in Eq. (33). Expanding the metric functions in powers of c^{-1} and keeping only terms of the Newtonian order yields

$$W=\rho+O(c^{-1}),\quad Z=\zeta+O(c^{-1}),\quad X=1+O(c^{-1}),$$
$$U=U_N c^{-2}+O(c^{-3}),\quad k=1+O(c^{-1}), \tag{38}$$

where U_N denotes the Newtonian gravitational potential generated by a mass density μ_N. Hence, the scalars W and Z, which play an important role in the derivation detailed in the last section, are the cylindrical coordinates in the Newtonian limit. This implies that all but one term in $\mu^{(r)}$ vanish; see Eq. (25) and the discussion afterwards. Then the source integrals, excluding singular sources, read

$$U^{(r)}_N = \frac{1}{8\pi}\int\limits_{\mathcal{V}} \frac{N^{(r)}_{-}}{\rho}\Delta^{(3)} U_N \mathrm{d}\mathcal{V} = \frac{1}{2}\int\limits_{\mathcal{V}} \frac{N^{(r)}_{-}}{\rho}\mu_N \mathrm{d}\mathcal{V}. \tag{39}$$

A comparison with the well-known formulas of Newtonian theory suggests that

$$N^{(k)}_{-}(\rho,\zeta) = -2\rho r^k P_k(\cos\theta), \qquad \forall k \geq 0, \tag{40}$$

where $\rho = r\sin\theta$ and $\zeta = r\cos\theta$. This can be verified explicitly using Eq. (18) and Legendre's differential equation.

Thus, the source integrals derived in the last section have the correct Newtonian limit. In a future work, we will go beyond the leading order. A first post-Newtonian approximation will enable us to compare the source integrals with those used by the IAU; see [16–18].

B. A special case

The Einstein equations are nonlinear and, hence, we could not expect a result like Eq. (39), which depends only on the mass density. We rather find source integrals containing terms that are not expressed explicitly by the Ricci tensor or equivalently by the stress-energy tensor; cf. the last two terms in Eq. (32). However, these terms vanish in vacuum. They also vanish in geometries or matter distributions, for which we can choose $W = \rho$ in the interior of the matter as for dust, i.e., where $\Delta W^{(2)} = 0$ is satisfied. In those cases, the density for the source integrals simplifies to

$$\mu^{(r)} = -\mathrm{e}^U N^{(r)}_{-} R_{\alpha\beta}\frac{\xi^\alpha\xi^\beta}{\xi^\gamma\xi_\gamma}. \tag{41}$$

Inserting the Einstein equations, $\mu^{(r)}$ is expressed entirely by the stress-energy tensor.

C. The connection to the Komar mass

In general, the first Weyl multipole moment $U^{(0)}$ coincides with the negative Geroch mass m_0 and, hence, must coincide with the well-known Komar mass. Equations (18) imply $N^{(0)}_{+} = 0$ and $N^{(0)}_{-} = -2W$. Thus, the first multipole moment in Eq. (33) simplifies to

$$m_0 = \frac{1}{4\pi}\sum_i \int_{\mathcal{V}_i^{(3)}} \frac{R_{\alpha\beta}\xi^\alpha\xi^\beta}{\sqrt{-\xi^\gamma\xi_\gamma}} \mathrm{d}\mathcal{V}_i^{(3)} + \frac{1}{4\pi}\sum_i \int_{\mathcal{S}_{S,i}^{(3)}} \mathrm{e}^U U_{,\hat{n}} \mathrm{d}\mathcal{S}_{S,i}^{(3)}.$$

This is, of course, exactly Komar's integral of the mass in static spacetimes. The contributions of singularities can also be cast in the standard form,

$$M_{\mathcal{S}_{S,i}^{(3)}} = \frac{1}{4\pi}\int_{\mathcal{S}_{S,i}^{(3)}} \mathrm{e}^U U_{,\hat{n}} \mathrm{d}\mathcal{S}_{S,i}^{(3)} = \frac{1}{8\pi}\int_{\mathcal{S}_{S,i}^{(3)}} \epsilon_{\alpha\beta\gamma\delta}\xi^{\alpha;\beta}. \tag{42}$$

D. Alternative derivation

Let us also mention an alternative derivation of the source integrals using Green's theorem. Suppose U is a solution to the Laplace equation in $\mathcal{E}^{(3)}$ (cf. Fig. 1), then it is given by virtue of Green's theorem by

$$U(x) = \frac{1}{4\pi}\int_{\mathcal{S}_0^{(3)}} \left(U(y)\frac{\partial G(x,y)}{\partial y^a} - G(x,y)\frac{\partial U(y)}{\partial y^a} \right) \hat{n}^a \mathrm{d}\mathcal{S}_{0,y}^{(3)}, \tag{43}$$

where $G(x,y)$ denotes an arbitrary Green's function, $\mathrm{d}\mathcal{S}_{0,y}^{(3)}$ a surface element of the boundary $\mathcal{S}_0^{(3)}$, and $x \in \mathcal{E}^{(3)}$. $\hat{n}^a$ is the unit normal to $\mathcal{S}_0^{(3)}$ at y pointing into $\mathcal{E}^{(3)}$.

Staring with Eq. (43), we arrive at Eq. (14) in three steps: First, we restrict x to the symmetry axis and use the axial symmetry to rewrite the right hand side of Eq. (43) as a line integral. Second, the choice $G(x,y) = -\frac{1}{4\pi|x-y|}$ allows integration by parts for the second term, which brings the line integral in the form of Eq. (14). Last, a lengthy but straightforward calculation shows that the functions in front of the U and the $U_{,n}$ coincide with the $N_\pm$ and the approach of Sec. III can be followed.

Thus, the multipole moments still contain crucial information of Green's theorem (43) with the difference that the latter is not accessible for stationary, axially symmetric, and isolated sources in general relativity. We chose the approach using the linear system, since it is easily generalizable also to that setting.

V. EXAMPLES

We will first verify the source integrals in their surface integral form and in their volume integral form using the axially symmetric Chazy-Curzon solution and the interior Schwarzschild solution describing a spherically symmetric star with homogeneous mass density. Afterwards, we apply the source integrals to recover a well-known nonexistence result for static, axially symmetric, and isolated dust configurations.

A. The Chazy-Curzon solution

The Chazy-Curzon solution is given in the form (1), if we choose

$$U = \frac{C}{r}, \qquad W = \rho. \tag{44}$$

Here, we used polar coordinates as in Eq. (40). For an interpretation of this solution and the constant C, see [55]. The solution is singular for $\rho = \zeta = 0$ but otherwise it is smooth. The Weyl moments are given by

$$U^{(0)} = C, \qquad U^{(k)} = 0 \quad \forall k \geq 1. \tag{45}$$

Following Eq. (9), the Geroch moments read

$$m_0 = -C, \qquad m_1 = 0, \qquad m_2 = \frac{1}{3}C^2, \ldots. \tag{46}$$

Thus, the Geroch quadrupole moment is not vanishing, and the solution is not spherically symmetric.

To calculate the multipole moments using source integrals, we choose as an integration surface a sphere $\mathcal{S}_R$ of radius $r = R > 0$ centered around $r = 0$, which contains the sources and which is located in the vacuum region. Hence, we can choose canonical Weyl coordinates, i.e., $W = \rho$ and $Z = \zeta$. Then the surface density $\eta_a^{(i)}\hat{n}^a$ in Eq. (22) can be written as

$$\eta_a^{(i)}\hat{n}^a = \frac{1}{8\pi}\mathrm{e}^{2U-k}(N_-^{(i)}U_{,r} + UN_-^{(i)} - UN_{-,r}^{(i)})\Big|_{r=R}, \tag{47}$$

where we used Eqs. (16). Taking Eqs. (40) and (44) into account, Weyl's multipole moments are given by

$$U^{(k)} = \frac{k+1}{2}CR^k \int_0^\pi \sin\theta P_k(\cos\theta) d\theta. \tag{48}$$

The orthogonality of the Legendre polynomials implies Weyl's multipole moments (45).

B. Spherically symmetric star

A spherically symmetric perfect fluid star of radius R_s with a homogenous mass density is given in Schwarzschild coordinates (r_s, θ_s) by

$$
\begin{aligned}
ds^2 &= -e^{2U}dt^2 + \frac{1}{\Delta(r_s)^2}dr_s^2 + r_s^2(d\theta_s^2 + \sin\theta_s^2 d\varphi^2),\\
e^U &= \begin{cases} \frac{3}{2}\Delta(R_s) - \frac{1}{2}(1-\frac{2Mr_s^2}{R_s^3})^{\frac{1}{2}} & \text{for } r_s \le R_s,\\ \Delta(r_s) & \text{for } r > R_s,\end{cases}\\
\Delta(r_s) &= \left(1 - \frac{2m(r_s)}{r_s}\right)^{\frac{1}{2}},\\
m(r_s) &= \begin{cases} M\frac{r_s^3}{R_s^3} & \text{for } r_s \le R_s,\\ M & \text{for } r_s > R_s,\end{cases}\\
p(r_s) &= \frac{3M}{4\pi R_s^3}\frac{\Delta(R_s) - \Delta(r_s)}{\Delta(r_s) - 3\Delta(R_s)}. \qquad (49)
\end{aligned}
$$

We choose a homogeneous mass density solely for brevity and concreteness. The steps presented shortly can also be carried out for more general Tolman-Oppenheimer-Volkoff stars.

The canonical Weyl coordinates read in the vacuum region

$$\rho = r_s\Delta(r_s)\sin\theta, \qquad \zeta = (r_s - M)\cos\theta. \qquad (50)$$

An expansion of U along the axis of symmetry in ζ^{-1} defines Weyl's multipole moments. They also follow from Geroch's multipole moments using Eq. (9). The latter vanish all except for the mass due to symmetry. Thus, the Weyl multipole moments are

$$U^{(0)} = -M, \qquad U^{(1)} = 0, \qquad U^{(2)} = -\frac{1}{3}M^3, \ldots. \qquad (51)$$

In fact, all odd Weyl multipole moments vanish. Since the even ones do not vanish, the Weyl multipole structure is more difficult than in the case of Geroch's multipole moments. This contrasts the situation for the Chazy-Curzon solution, and it relates to the fact that the canonical Weyl coordinates are not very well adapted to spherical symmetry.

For the evaluation of the source integrals using the density $\mu^{(r)}$ of Eq. (32), W, some components of the Ricci tensor, Z, and X have to be calculated. The former are

$$
\begin{aligned}
W &= \Delta(r_s)r_s\sin\theta_s,\\
R_{tt} &= -\Delta(r_s)^2\frac{m_{,r_sr_s}}{r_s}, \qquad R_{\varphi\varphi} = 2m_{,r_s}\sin^2\theta_s, \qquad (52)
\end{aligned}
$$

and the latter follow from the fact that X is defined as the integrating factor of Z_α, i.e.,

$$
\begin{aligned}
Z_\alpha &= e^U X[\Delta(r_s)^{-1}\cos\theta_s dr_s - \Delta(r_s)(r_s + r_s^2U_{,r_s})\sin\theta_s d\theta_s]\\
&= Z_{,r_s}dr + Z_{,\theta_s}d\theta_s. \qquad (53)
\end{aligned}
$$

This equation is easily solved yielding

$$
\begin{aligned}
Z &= X(r_s)e^U\Delta(r_s)(r_s + r_s^2U_{,r_s})\cos\theta_s,\\
X(r_s) &= \exp\left(16\pi\int_{r_s}^{R_s}\frac{r^2p(r)}{r - m(r) + 4\pi r^3p(r)}dr\right), \qquad (54)
\end{aligned}
$$

where the integration constant was chosen such that $X(R_s) = 1$ and $Z(R_s) = \zeta$. In fact, the assumption $X = X(r)$ allows one to determine the integrating factor also for Tolman-Oppenheimer-Volkoff stars with inhomogeneous density.

Inserting Eqs. (49), (52), and (54) into Eq. (25) yields the density for the source integrals Eq. (33). It can easily be checked that this density is even (odd) in θ_s for even (odd) r. Thus, all odd Weyl moments vanish. The density for the mass monopole reads

$$\mu^{(0)} = -\frac{3M}{4\pi R_s^3}\Delta(r_s), \qquad (55)$$

which gives after integration $U^{(0)} = -M$. The density $\mu^{(2)}$ is a rather lengthy expression, which, however, gives after a numerical integration the expected result of Eq. (51) to working precision.

C. An application of the source integrals to dust

Although the main goal of this paper is to present the derivation and definition of the source integrals, we give a short application here. We show that static, axially symmetric, and isolated dust configurations do not exist in general relativity. This is an old result[8] but can easily be recovered using source integrals. This demonstrates also how the quasilocal expressions (36) can be employed in general.

The energy-momentum tensor for static and axially symmetric dust configurations reads in Weyl coordinates

$$T_{\alpha\beta} = \mu e^{2U}\delta^t_\alpha\delta^t_\beta. \qquad (56)$$

The Bianchi identity implies $U_{,\alpha} = 0$ in the interior of the dust and, thus, at its surface. In fact, this yields together with the quasilocal surface integrals for the Weyl moments Eq. (22)

$$U^{(r)} = 0. \qquad (57)$$

Hence, the dust distribution has no mass or any other multipole moment, which implies flat space in the vacuum region. This is clearly a contradiction to a dust source with positive mass density. Note that the source integrals enabled us here to determine the metric close to spatial infinity from the metric and its derivatives at the boundary of the matter region. That this can be achieved is astonishing for a nonlinear theory and is the merit of the source integrals.

[8]For more general nonexistence results for dust, see also [56–59].

VI. CONCLUSIONS

We have derived in this article source integrals or quasilocal expressions for Weyl's multipole moments and, thus, for Geroch's multipole moments for axially symmetric and static sources. These source integrals—written as surface integrals or volume integrals—determine for any spacetime region its contribution to the total asymptotic multipole moments. *A priori*, one could not expect to find any kind of source integrals at all, because of the nonlinear nature of the Einstein equations. That this is, nonetheless, possible in the here considered setting, is not due to the staticity and axial symmetry and the peculiarly simple form of the field equations. But rather a linear system must be available offering a notion of integrability of the Einstein equation. Thus, it appears feasible to find source integrals for stationary and axially symmetric isolated systems, which describe (electro)vacuum close to spatial infinity. These generalizations will be investigated in future work.

We want to study also how the source integrals are connected to the already known source integrals for isolated horizons [15] and to the multipole moments and their source integrals recommended by the IAU (see [16–18], or those discussed in [19]). In [15], it was shown that a horizon is uniquely characterized by a certain set of source multipole moments. However, these specific source integrals do not reproduce the Geroch-Hansen multipole moments in the case of a Kerr black hole. In our approach, the agreement of the source integrals and the asymptotically defined Weyl or Geroch multipole moments is given by construction. Therefore, these source integrals might prove useful for identifying the contributions to the multipole moments, which yield the discrepancies between the isolated horizon multipole moments and those of Geroch-Hansen.

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