Separability of test fields equations on the *C*-metric background

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In the Kerr-Newman spacetime the Teukolsky master equation, governing the fundamental test fields, is of great importance. We derive an analogous master equation for the nonrotating *C*-metric which encompasses a massless Klein-Gordon field, neutrino field, Maxwell field, Rarita-Schwinger field and gravitational perturbations. This equation is shown to be separable in terms of "accelerated spin-weighted spherical harmonics." It is shown that, contrary to ordinary spin-weighted spherical harmonics, the "accelerated" ones are different for different spins. In some cases, the equations for eigenfunctions and eigenvalues are explicitly solved.

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I. INTRODUCTION

Boost-rotation-symmetric spacetimes are a special class of solutions of the vacuum Einstein's field equations with two symmetries which can represent moving objects and contain radiation. Some exact explicit solutions belonging to this class are e.g. the *C*-metric [1] and the solutions found by Bonnor and Swaminarayan [2]. In general these spacetimes are algebraically general, radiative as shown in Ref. [3] and possess a plausible Newtonian limit [4].

Among these solutions the *C*-metric is a special case. It is of Petrov type D—and thus a generalization [5] admitting charge and rotation could have been found, involving "uniformly accelerated black holes."

The method of decoupling and separating variables of the equations governing fundamental fields is, of course, of great importance but it has not been applied to the *C*-metric so far.

In general relativity, the decoupling and separation of variables is often done by employing the Newman-Penrose (NP) formalism. If the NP formalism is applied to the type D spacetimes, the equations for radiative (ingoing and outgoing radiation) components of a (a) massless Klein-Gordon field ($s = 0$), (b) neutrino field ($s = ½$), (c) test Maxwell field ($s = 1$), (d) Rarita-Schwinger field ($s = 3/2$) and (e) linear gravitational perturbations ($s = 2$) can be decoupled [6,7], and in the case of the Kerr-Newman metric they can also be separated. We rewrite these equations in the Geroch-Held-Penrose (GHP) formalism [8] in Sec. IV. In Sec. II, we review some elementary concepts of the Ernst projection formalism which we later connect to the NP formalism.

The background metric—the *C*-metric—is presented in Sec. III. For more details see Refs. [9–11] which encompass a comprehensive historical introduction. The equation analogous to the Teukolsky master equation is obtained in Sec. IV for the charged *C*-metric. We show that the solution of this equation can be found using separation of variables in the canonical coordinates [10]. The angular part of the solution leads to generalized "accelerated spin-weighted spherical harmonics."[1] These are in general Heun [12] functions, but for the extremal case the solutions reduce to rational functions. We find the eigenfunctions and eigenvalues for axially symmetric fields ($m = 0$) in Sec. V and show why for m different from zero this is difficult, solving the same problem for the cosmic string spacetime (the *C*-metric inherently contains a deficit angle) in Sec. X.

The electromagnetic field which was the primary motivation is then analyzed in a more detailed way in Sec. VIII.

Contrary to ordinary spin-weighted spherical harmonics the accelerated ones form a more complicated structure as they are split into different families according to the spin S of the field; this issue is discussed in Sec. IX.

II. ERNST PROJECTION FORMALISM

A well-known fact is that in vacuum spacetimes every Killing vector can serve as a 4-potential of a test electromagnetic field. The straightforward generalization to spacetimes with matter can be done with the help of the Ernst projection formalism.

In the presence of the Killing vector field $\boldsymbol{\xi}$, the Ernst projection formalism [13,14] can be used to solve the full Einstein-Maxwell equations. But the Ernst equations can be used for solving Maxwell equations for a test electromagnetic field (which respects the symmetry induced by the aforementioned Killing vector field, i.e. $\mathcal{L}_{\xi}\boldsymbol{F} = 0$) on a given background.

Let $\boldsymbol{F}$ be an electromagnetic field for which $\mathcal{L}_{\xi}\boldsymbol{F} = 0$; then the potential Ψ with respect to the Killing vector $\boldsymbol{\xi}$ can be introduced[2]

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[1]This terminology can be disputed since these solutions are not in general eigenfunctions of Laplace's operator.

[2]The complex self-dual 2-form is defined as $\boldsymbol{F}^* = \boldsymbol{F} + \frac{i}{2}\star\boldsymbol{F}$.

$$\sqrt{\frac{\kappa}{2}}\,\xi^a F^*_{ab} = \Psi_{,b}, \qquad \Psi_{,a}\xi^a = 0, \qquad F^{*ab}{}_{;b} = 0, \tag{1}$$

and, conversely,

$$\sqrt{\frac{\kappa}{2}}\,F^*_{ab} = 2F^{-1}(\xi_{[a}\Psi_{,b]})^*. \tag{2}$$

The inverse of the norm of the Killing vector (KV) in this equation is the reason that only very special solutions with respect to a general linear combination of timelike and spacelike KVs are, though mathematically possible, physically relevant. The covariant divergence of Eq. (2) gives the equation for Ψ

$$F\left(\frac{\Psi_{,a}}{F}\right)^{;a} + i\omega^a\left(\frac{\Psi_{,a}}{F}\right) = 0 \tag{3}$$

which is a *linear* partial differential equation where

$$F = \xi^a\xi_a, \qquad \omega^a = \varepsilon^{abcd}\xi_b\xi_{c;d}. \tag{4}$$

The equation (3) is a mid-step in the derivation of the Ernst equations (for details presented in a modern fashion see Ref. [8]) where the norm F of the KV $\boldsymbol{\xi}$ is identified with $-F = \frac{1}{2}(\mathcal{E} + \bar{\mathcal{E}}) + \bar{\Phi}\Phi$, where $\mathcal{E}$ is the so-called gravitational Ernst potential whereas Φ is the electromagnetic Ernst potential

$$F\Box\mathcal{E} = (\nabla_a\mathcal{E} + 2\bar{\Phi}\nabla_a\Phi)\nabla^a\mathcal{E}, \tag{5}$$

$$F\Box\Phi = (\nabla_a\mathcal{E} + 2\bar{\Phi}\nabla_a\Phi)\nabla^a\Phi. \tag{6}$$

For a given spacetime the Ernst potentials can be calculated. And these two potentials are solutions of the Maxwell equations for a test field (3) with $\Psi = \Phi$ or $\Psi = \mathcal{E}$.

III. NONROTATING C-METRIC

The nonrotating C-metric, a spacetime representing two[3] in general charged uniformly accelerated black holes [9,11], is given in a modern form [10] with the factorized structure function $G(\xi)$ by

$$\mathbf{d}s^2 = \frac{1}{A^2(x-y)^2}\left[K_\tau^2 G(y)\mathbf{d}\tau^2 - \frac{1}{G(y)}\mathbf{d}y^2 + \frac{1}{G(x)}\mathbf{d}x^2 + K_\varphi^2 G(x)\mathbf{d}\varphi^2\right]. \tag{7}$$

The structure function $G(\xi)$ is given by the fourth-order polynomial

$$G(\xi) = (1-\xi^2)(1+Ar_p\xi)(1+Ar_m\xi), \tag{8}$$

in which the structure of roots is defined by $-\infty < -1/Ar_m < -1/Ar_p < -1$. The parameter A defines the acceleration of the black hole and the parameters r_p, r_m can be related to the mass parameter M and charge[4] parameter q by

$$r_p = M + \sqrt{M^2 - q^2}, \qquad r_m = M - \sqrt{M^2 - q^2}. \tag{9}$$

The metric (7) covers several regions of the spacetime but we are interested in the asymptotically flat region outside the black hole which is covered by the coordinate ranges $\tau \in \mathbb{R}$, $y \in \langle -1/Ar_p, 1\rangle$ (with $y = -1/Ar_p$ being the black hole horizon and $y = -1$ being the acceleration horizon), $x \in \langle 1, -1\rangle$ (with the axis given by $x = \pm 1$) and $x - y < 0$ (the asymptotic infinity is located at $x - y = 0$) and, finally, $\varphi \in \langle 0, 2\pi\rangle$. The strength of the conical singularity which is inevitably present in the C-metric is governed by the parameter K_φ. The constant K_τ can, of course, be absorbed in the definition of τ but we leave it explicitly present.

The null tetrad adapted to the principal null directions[5] reads

$$\boldsymbol{l} = -\frac{A^2(x-y)^2}{\sqrt{2}}\left(\frac{1}{G(y)K_\tau}\frac{\partial}{\partial\boldsymbol{\tau}} - \frac{\partial}{\partial\boldsymbol{y}}\right), \tag{10}$$

$$\boldsymbol{n} = -\frac{G(y)}{\sqrt{2}}\left(\frac{1}{G(y)K_\tau}\frac{\partial}{\partial\boldsymbol{\tau}} - \frac{\partial}{\partial\boldsymbol{y}}\right), \tag{11}$$

$$\boldsymbol{m} = \frac{1}{\sqrt{2}}\frac{A(x-y)}{\sqrt{G(x)}}\left(-G(x)\frac{\partial}{\partial\boldsymbol{x}} + \frac{i}{K_\varphi}\frac{\partial}{\partial\boldsymbol{\varphi}}\right), \tag{12}$$

$$\bar{\boldsymbol{m}} = \frac{1}{\sqrt{2}}\frac{A(x-y)}{\sqrt{G(x)}}\left(-G(x)\frac{\partial}{\partial x} - \frac{i}{K_\varphi}\frac{\partial}{\partial\boldsymbol{\varphi}}\right), \tag{13}$$

and the corresponding nonzero NP coefficients are

$$\pi = \frac{1}{\sqrt{2}}A\sqrt{G(x)}, \tag{14}$$

$$\mu = \frac{1}{\sqrt{2}}\frac{G(y)}{x-y}, \tag{15}$$

$$\tau = -\frac{1}{\sqrt{2}}A\sqrt{G(x)}, \tag{16}$$

[3]After the analytic continuation across the acceleration horizon.

[4]For the sake of completeness, let us mention that the 4-potential of the electromagnetic field is given by $\boldsymbol{A} = K_\tau qy\boldsymbol{d\tau}$.

[5]It is chosen such that in the limit $r_p = r_m = 0$, i.e. in the flat spacetime where $G(\xi) = 1 - \xi^2$, and after a standard coordinate transformation $x = \cos\theta$ the vector $\boldsymbol{m}$ becomes $\partial_{\boldsymbol{\theta}} + \frac{i}{\sin\theta}\partial_{\boldsymbol{\varphi}}$.

$$\varrho = -\frac{1}{\sqrt{2}}A^2(x-y), \tag{17}$$

$$\gamma = -\frac{\sqrt{2}}{4}\left[\frac{G(y)}{(x-y)^2}\right]_{,y}(x-y)^2, \tag{18}$$

$$\beta = -\frac{\sqrt{2}}{4}A(x-y)(\sqrt{G(x)})_{,x}, \tag{19}$$

$$\alpha = \frac{A\sqrt{2}}{4}\left[\frac{\sqrt{G(x)}}{(x-y)^2}\right]_{,x}(x-y)^3. \tag{20}$$

Finally, the only nonzero Weyl NP scalar is

$$\psi_2 = -A^3(x-y)^3\left(\frac{r_p+r_m}{2} + Ar_p r_m(x+y)\right). \tag{21}$$

Let us assume that we have a potential $\Psi^{(\tau)}(y,x,\varphi)$ of Eq. (3) with respect to the boost Killing vector $\boldsymbol{\xi}_{(\tau)}$ [respectively $\Psi^{(\varphi)}(\tau,y,x)$ with respect to the axial Killing vector $\boldsymbol{\xi}_{(\varphi)}$]. The equation (3) for the Ernst electromagnetic potential on the C-metric background is separable in both cases, i.e. let us assume that $\Psi^{(\tau)} = \hat{Y}(y)\hat{X}(x)\hat{P}(\varphi)$ and $\Psi^{(\varphi)} = \tilde{T}(\tau)\tilde{Y}(y)\tilde{X}(x)$. Then the separated equations read

$$G(y)\hat{Y}_{,yy} + \Lambda\hat{Y} = 0, \tag{22}$$

$$[G(x)\hat{X}_{,x}]_{,x} + \left[\Lambda - \frac{m^2}{K_\varphi^2 G(x)}\right]\hat{X} = 0, \tag{23}$$

$$\hat{P}_{,\varphi\varphi} + m^2\hat{P} = 0; \tag{24}$$

and

$$\tilde{T}_{,\tau\tau} + \omega^2\tilde{T} = 0, \tag{25}$$

$$[G(y)\tilde{Y}_{,y}]_{,y} + \left[\Lambda + \frac{\omega^2}{K_\tau^2 G(y)}\right]\tilde{Y} = 0, \tag{26}$$

$$G(x)\tilde{X}_{,xx} + \Lambda\tilde{X} = 0. \tag{27}$$

From the Ernst potential $\Psi^{(\cdot)}$ we can construct the electromagnetic field tensor $\boldsymbol{F}$ by Eq. (2) and calculate the NP scalars of the electromagnetic field corresponding to the potential $\Psi^{(\varphi)}$ (respectively $\Psi^{(\tau)}$). They are given by

$$\Phi_0^{(\varphi)} = \frac{-i\sqrt{2}A^3(x-y)^3}{K_\varphi K_\tau}\frac{(K_\tau G(y)\Psi^{(\varphi)}_{,y} - \Psi^{(\varphi)}_{,\tau})}{G(y)\sqrt{G(x)}}, \tag{28}$$

$$\Phi_1^{(\varphi)} = \frac{i\sqrt{2}A^2(x-y)^2\Psi^{(\varphi)}_{,x}}{K_\varphi}, \tag{29}$$

$$\Phi_2^{(\varphi)} = \frac{i\sqrt{2}A(x-y)}{K_\varphi K_\tau}\frac{(K_\tau G(y)\Psi^{(\varphi)}_{,y} + \Psi^{(\varphi)}_{,\tau})}{\sqrt{G(x)}}, \tag{30}$$

and

$$\Phi_0^{(\tau)} = \frac{-\sqrt{2}iA^3(x-y)^3}{K_\varphi K_\tau}\frac{(K_\varphi G(x)\Psi^{(\tau)}_{,x} - i\Psi^{(\tau)}_{,\varphi})}{G(y)\sqrt{G(x)}}, \tag{31}$$

$$\Phi_1^{(\tau)} = \frac{\sqrt{2}A^2(x-y)^2\Psi^{(\varphi)}_{,y}}{K_\varphi}, \tag{32}$$

$$\Phi_2^{(\tau)} = \frac{i\sqrt{2}A(x-y)}{K_\varphi K_\tau}\frac{(K_\varphi G(x)\Psi^{(\tau)}_{,x} + i\Psi^{(\tau)}_{,\varphi})}{\sqrt{G(x)}}. \tag{33}$$

We observe that there exists a class of solutions for which $\mathcal{L}_{\xi_{(\varphi)}}\boldsymbol{F} = \mathcal{L}_{\xi_{(\tau)}}\boldsymbol{F} = 0$ at the same time and thus we can find both potentials for this field. This gives us a relation between solutions of Eqs. (24) and (27). If $\mathcal{V}$ is the solution of $(G\mathcal{V}_{,z})_{,z} + \lambda\mathcal{V} = 0$ then $\mathcal{U} = G\mathcal{V}_{,z}$ is the solution of the equation $G\mathcal{U}_{,zz} + \lambda\mathcal{U} = 0$. And, conversely, let $\mathcal{U}$ be the solution of $G\mathcal{U}_{,zz} + \lambda\mathcal{U} = 0$ then $\mathcal{V} = \mathcal{U}_{,z}$ is the solution of $(G\mathcal{V}_{,z})_{,z} + \lambda\mathcal{V} = 0$.

More generally, the Ernst potential can be found for a general linear combination of $\boldsymbol{\xi}_{(\tau)}$ and $\boldsymbol{\xi}_{(\varphi)}$ if the Lie derivative of the electromagnetic field along this vector field vanishes, but then the separation of variables leads to static and axially symmetric solutions only.

IV. MASTER EQUATION

Teukolsky [6] provided decoupled equations for NP components of the gravitational perturbation Ψ_0 and Ψ_4, of the test electromagnetic field Φ_0 and Φ_2 and of the neutrino field χ_0 and χ_1 in a general type D spacetime and performed a detailed analysis of these equations on the Kerr-Newman background. The C-metric is also a type D solution and therefore we can perform a similar analysis. The equations[6] for gravitational perturbations read

$$\begin{aligned}&[(D - 3\epsilon + \bar{\epsilon} - 4\varrho - \bar{\varrho})(\Delta - 4\gamma + \mu)\\ &\quad - (\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)(\bar{\delta} + \pi - 4\alpha) - 3\psi_2]\Psi_0 = 0,\end{aligned} \tag{34}$$

$$\begin{aligned}&[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \varrho)\\ &\quad - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\psi_2]\Psi_4 = 0;\end{aligned} \tag{35}$$

the equations for the Rarita-Schwinger equation read [7]

$$\begin{aligned}&[(D - 2\epsilon + \bar{\epsilon} - 3\varrho - \bar{\varrho})(\Delta - 3\gamma + \mu)\\ &\quad - (\delta + \bar{\pi} - \bar{\alpha} - 2\beta - 3\tau)(\bar{\delta} + \pi - 3\alpha) - \psi_2]\Sigma_0 = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}&[(\Delta + 2\gamma - \bar{\gamma} + 3\mu + \bar{\mu})(D + 3\epsilon - \varrho)\\ &\quad - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 2\alpha + 3\pi)(\delta - \tau + 3\beta) - \psi_2]\Sigma_3 = 0;\end{aligned} \tag{37}$$

[6]We consider vacuum solutions only.

the equations for the test electromagnetic field read

$$[(D-\epsilon+\bar{\epsilon}-2\varrho-\bar{\varrho})(\Delta-2\gamma+\mu) - (\delta+\bar{\pi}-\bar{\alpha}-\beta-2\tau)(\bar{\delta}+\pi-2\alpha)]\Phi_0=0, \quad (38)$$

$$[(\Delta+\gamma-\bar{\gamma}+2\mu+\bar{\mu})(D+2\epsilon-\varrho) - (\bar{\delta}-\bar{\tau}+\bar{\beta}+\alpha+2\pi)(\Delta-\tau+2\beta)]\Phi_2=0; \quad (39)$$

and the equations for the neutrino field read

$$[(D+\bar{\epsilon}-\varrho-\bar{\varrho})(\Delta-\gamma+\mu) - (\delta+\bar{\pi}-\bar{\alpha}-\tau)(\bar{\delta}+\pi-\alpha)]\chi_0=0, \quad (40)$$

$$[(\Delta-\bar{\gamma}+\mu+\bar{\mu})(D+\epsilon-\varrho) - (\bar{\delta}-\bar{\tau}+\bar{\beta}+\pi)(\delta-\tau+\beta)]\chi_1=0. \quad (41)$$

Equations (34), (36), (38) and (40) for NP scalars can be rewritten using the GHP formalism in a more compact form:

$$[(þ-\bar{\varrho}-4\varrho)(þ'+\mu) - (ð+\bar{\pi}-4\tau)(ð'+\pi)-3\psi_2]\Psi_0=0 \quad \text{for } s=2, \quad (42)$$

$$[(þ-\bar{\varrho}-3\varrho)(þ'+\mu) - (ð+\bar{\pi}-3\tau)(ð'+\pi)-\psi_2]\Sigma_0=0 \quad \text{for } s=3/2, \quad (43)$$

$$[(þ-\bar{\varrho}-2\varrho)(þ'+\mu) - (ð+\bar{\pi}-2\tau)(ð'+\pi)]\Phi_0=0 \quad \text{for } s=1, \quad (44)$$

$$[(þ-\bar{\varrho}-\varrho)(þ'+\mu) - (ð+\bar{\pi}-\tau)(ð'+\pi)]\chi_0=0 \quad \text{for } s=1/2, \quad (45)$$

from which we infer, for $\Phi\in(\Psi_0,\Sigma_0,\Phi_0,\chi_0)$ and general $s>0$

$$[(þ-\bar{\varrho}-2s\varrho)(þ'+\mu)-(ð+\bar{\pi}-2s\tau)(ð'+\pi) - (2s-1)(s-1)\psi_2]\Phi=0, \quad (46)$$

and the same analysis of Eqs. (35), (37), (39) and (41) results in an equation for $\Phi\in(\Psi_4,\Sigma_3,\Phi_2,\chi_1)$ for general $s<0$

$$[(þ'+\bar{\mu}-2s\mu)(þ-\varrho)-(ð'-\bar{\tau}-2s\pi)(ð-\tau) - (2s+1)(s+1)\psi_2]\Phi=0, \quad (47)$$

which leads to the massless Klein-Gordon ($s=0$) equation in the GHP formalism

$$[(þ-\bar{\varrho})(þ'+\mu)-(ð+\bar{\pi})(ð'+\bar{\pi})-\psi_2]\Phi=0, \quad (48)$$

or, in the NP formalism

$$[(D+\epsilon+\bar{\epsilon}-\bar{\varrho})(\Delta+\mu) - (\delta+\bar{\pi}-\bar{\alpha}+\beta)(\bar{\delta}+\pi)-\psi_2]\Phi=0, \quad (49)$$

which is an assertion we have to prove.

The d'Alembert operator acting on scalar is simply given by

$$\Box=\nabla^a\nabla_a=(-n_aD-l_a\Delta+\bar{m}_a\delta+m_a\bar{\delta}) \times(-n^aD-l^a\Delta+\bar{m}^a\delta+m^a\bar{\delta}). \quad (50)$$

The d'Alembert operator then can be expressed (cf. Ref. [15]) as

$$-\frac{1}{2}\Box=D\Delta-\delta\bar{\delta}+\mu D+(\epsilon+\bar{\epsilon}-\bar{\varrho})\Delta -\varphi\delta+(\bar{\alpha}-\beta-\bar{\varrho})\bar{\delta}, \quad (51)$$

but expanding $[(þ-\bar{\varrho})(þ'+\mu)-(ð+\bar{\pi})(ð'+\pi)]$ [acting on the field of GHP weight (0, 0)] and using the Ricci identity for $D\mu-\delta\varphi$, or expanding $[(þ'+\bar{\mu})(þ-\varrho)-(ð'-\bar{\tau})(ð-\tau)]$ and using the Ricci identity for $\Delta\varrho-\bar{\delta}\tau$ results in

$$\begin{aligned}[(þ-\bar{\varrho})(þ'+\mu)-(ð+\bar{\pi})(ð'+\pi)] &= [(þ'+\bar{\mu})(þ-\varrho)-(ð'-\bar{\tau})(ð-\tau)] \\ &= -\frac{1}{2}\left[\Box-2\psi_2-\frac{4}{3}R-2\sigma\lambda+2\nu\kappa\right].\end{aligned} \quad (52)$$

Thus Eq. (48) is indeed the massless Klein-Gordon equation (for Ricci flat type D spacetimes).

Regarding the NP spin coefficients given in Eq. (20), the equations listed above can be represented as a master equation for $\hat{\Phi}(\tau,y,x,\varphi)$ which is a function listed in Table I

TABLE I. Separable ansatz, spin and GHP weight of field components.

$\hat{\Phi}$	$\varrho^{-1}\Psi_4$	$\varrho^{-1}\Sigma_3^{RS}$	$\varrho^{-1}\Phi_2^{EM}$	$\varrho^{-1}\chi_1$	$\varrho^{-1}\Phi^{KG}$	$\varrho^{-2}\chi_0$	$\varrho^{-3}\Phi_0^{EM}$	$\varrho^{-4}\Sigma_0^{RS}$	$\varrho^{-5}\Psi_0$
s	−2	−3/2	−1	½	0	½	1	3/2	2
GHP weight	(−4, 0)	(−3, 0)	(−2, 0)	(−1, 0)	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)

$$\frac{1}{G(y)}\left[\frac{\hat{\Phi}_{,\tau\tau}}{K_\tau^2}+\frac{sG(y)_{,y}\hat{\Phi}_{,\tau}}{K_\tau}-\frac{(s+1)(2s+1)}{6}G(y)G(y)_{,yy}\hat{\Phi}\right]-\frac{[G(y)^{s+1}\hat{\Phi}_{,y}]_{,y}}{G(y)^s}+\frac{1}{G(x)}\left[\frac{\hat{\Phi}_{,\varphi\varphi}}{K_\varphi^2}-\frac{isG(x)_{,x}\hat{\Phi}_{,\varphi}}{K_\varphi}-\frac{s^2G(x)_{,x}^2\hat{\Phi}}{4}\right]$$
$$+[G(x)\hat{\Phi}_{,x}]_{,x}+\frac{s^2+\frac{1}{2}}{3}G(x)_{,xx}\hat{\Phi}=0. \tag{53}$$

This equation, which is a generalization of the Teukolsky master equation for the C-metric (yet, contrary to the Teukolsky equation itself, only for the nonrotating case), can be separated by the following ansatz:

$$\hat{\Phi}=e^{-i\omega t}e^{im\varphi}\mathcal{Y}(y)\mathcal{X}(x), \tag{54}$$

which leads to the equation for the angular part (56) in the next section and the equation for the radial part (71) in Sec. VII.

Only massless Klein-Gordon equation is separable on the C-metric background and so is the Dirac equation—which in the massless limit is the Weyl neutrino equation and thus is already included above.

We have also discussed only NP scalars of maximal spin weight because only for these the equations are decoupled in type D spacetimes. Of course, to solve completely, for example, the electromagnetic field, we would have to also analyze Φ_1.

V. ACCELERATED SPIN-WEIGHTED SPHERICAL HARMONICS

According to Ref. [16] the spin-weighted spheroidal harmonics are regular solutions to the equation

$$[(1-x^2)S^{(s)}_{(lm),x}]_{,x}+\left[(cx-s)^2-s(s-1)+A^{(s)}_{(lm)}-\frac{(m+sx)^2}{1-x^2}\right]S^{(s)}_{(lm)}=0. \tag{55}$$

The constant c arises in the equation during the separation of variables of various fields on the Kerr-Newman background only if $a\omega\neq 0$. In the limit $c=0$ the separation constant is $A^{(s)}_{(lm)}=l(l+1)-s(s+1)$ and then the standard spin-weighted spherical harmonics are obtained.

The separation of variables of the master equation for the C-metric (53) leads to the equation

$$[G(x)\mathcal{X}^{(s)}_{(lm),x}]_{,x}+\left[\frac{s^2+\frac{1}{2}}{3}G(x)_{,xx}+\Lambda^{(s)}_{(lm)}-\frac{(\hat{m}-\frac{s}{2}G(x)_{,x})^2}{G(x)}\right]\mathcal{X}^{(s)}_{(lm)}=0,\qquad \hat{m}\equiv\frac{m}{K_\varphi}, \tag{56}$$

for the angular part which, according to Sturm-Liouville theory, possesses an infinite number of regular orthogonal solutions. These solutions form a base for L^2 functions on the interval $x\in\langle -1,1\rangle$. This equation is a generalization of Eq. (55) with $c=0$ (we are investigating the nonrotating case, i.e. $a=0$) for accelerated sources.[7]

If the function $G(\xi)$ is even $[G(\xi)=G(-\xi)]$ and if $\mathcal{X}^{(s)}_{(lm)}(x)$ is a solution of Eq. (56) then the solution for $-s$ is given by $\mathcal{X}^{(-s)}_{(lm)}(x)=\mathcal{X}^{(s)}_{(lm)}(-x)$.

With the help of computer algebra systems we are able to find a solution of Eq. (56) for arbitrary s, m, Λ and the general structure function $G(\xi)=(1-\xi^2)(1+Ar_p\xi)(1+Ar_m\xi)$ in terms of a general Heun function. For some special cases we can find the solution explicitly, even in terms of rational functions, and not only formally as a Heun general function.

A. Extremal case

For the extremal case $r_p=r_m$ the solution of Eq. (56) that is regular in the entire interval $x\in\langle -1,1\rangle$ with $m=0$ for general half-integer s has been found in terms of rational functions (in this case, the hypergeometric function reduces to a polynomial)

$$\mathcal{X}^{(s)}_{(l0)}=C^{(s)}_{(l0)}\frac{(1-x^2)^{\frac{1}{2}s}}{(1+Ar_px)^{1+s}}\times{}_2F_1\left({s-l,s+l+1 \atop 1+s};\frac{1+Ar_p}{2}\frac{1+x}{1+Ar_px}\right), \tag{57}$$

with the eigenvalues

$$\Lambda^{(s)}_{(l0)}=(1-A^2r_p^2)\left[l(l+1)+\frac{1}{3}(1-s^2)\right]. \tag{58}$$

Solutions of Eq. (56) with $r_m=r_p$ and $m\neq 0$ are given in terms of a Heun confluent function with unknown eigenvalues. But in the Minkowski limit $r_p\to 0$ and $K_\varphi=1$, these solutions go over to (valid for $m\geq 0$)

$$S^{(s)}_{(lm)}=\left(\frac{1-x}{1+x}\right)^{s/2}(1-x^2)^{m/2}\times{}_2F_1\left({-l+m,l+m+1 \atop m-s+1};\frac{1}{2}(1+x)\right), \tag{59}$$

[7] Notice that this is a general-relativistic effect; in the limit $r_p=r_m=0$ and $K_\varphi=1$ the standard spin-weighted spherical harmonics are recovered.

where l, s, m are from either $\mathbb{N}_0$ or $\mathbb{N}_0 + ½$. Up to normalization, these solutions are standard spin-weighted spherical harmonics, which are given in terms of the Wigner d-functions (see Refs. [17,18])

$$Y^{(s)}_{(lm)}(x,\varphi) = \sqrt{\frac{2l+1}{4\pi}} e^{im\varphi} d^l_{-s,m}(x), \qquad (60)$$

where

$$d^l_{sm}(x) = \sum_{k=\max(0,s-m)}^{\min(l+s,l-s)} (-1)^{k-s+m} \times \frac{\sqrt{(l+s)!(l-s)!(l+m)!(l-m)!}}{k!(l+s-k)!(l-k-m)!(k-s+m)!} \times \left(\frac{\sqrt{1+x}}{2}\right)^{2l-2k+s-m} \left(\frac{\sqrt{1-x}}{2}\right)^{2k-s+m}, \qquad (61)$$

in which we trivially replaced $\cos\frac{\theta}{2} = \sqrt{\frac{1+x}{2}}$ and $\sin\frac{\theta}{2} = \sqrt{\frac{1-x}{2}}$. This gives us a relation (up to the normalization) of the Wigner d-function and hypergeometric function (59).

B. General case

For a general case ($r_p \neq r_m$) the solutions of Eq. (56) (only for $m = 0$) are found in terms of the Heun functions Hf_l:

$$\mathcal{X}^{(s)}_{(l0)} = \frac{1}{1+Ar_m x}\left(\frac{1-x^2}{(1+Ar_p x)(1+Ar_m x)}\right)^{\frac{1}{2}s} \times Hf_l\left(\boldsymbol{a},\boldsymbol{q};1,1-s,1-s,1+s;\frac{1+Ar_m}{2}\frac{1+x}{1+Ar_m x}\right) \qquad (62)$$

where

$$\boldsymbol{a} = \frac{1}{2}\frac{(1-Ar_p)(1+Ar_m)}{A(r_p-r_m)}, \qquad (63)$$

$$\boldsymbol{q} = \frac{1}{6}\frac{3\Lambda^{(s)}_{(l0)} - (s+1)(2s+1)(1-A^2 r_p r_m)}{A(r_p-r_m)} + \frac{s+1}{2}, \qquad (64)$$

with the eigenvalues given by

$$\Lambda^{(s)}_{(l0)} = \left[l(l+1) + \frac{1}{3}(1-s^2)\right]\sqrt{1-A^2 r_m^2}\sqrt{1-A^2 r_p^2}. \qquad (65)$$

VI. SCALAR HARMONICS ON THE "SPHERE" (x,φ)

The conformally rescaled 2-metric of constant τ and y is

$$\mathbf{d}s^2_{(2D)} = \frac{\mathbf{d}x^2}{G(x)} + K_\varphi^2 G(x)\mathbf{d}\varphi^2. \qquad (66)$$

The Laplace operator on this "sphere" reads

$$\Delta_{(2D)}\Phi(x,\varphi) = [G(x)\Phi_{,x}] + \frac{1}{K_\varphi^2}\frac{\Phi_{,\varphi\varphi}}{G(x)}, \qquad (67)$$

so the eigenvalue problem $\Delta_{(2D)}\Phi = -\Lambda_{(lm)}\Phi$ can be separated using the ansatz $\Phi = e^{im\varphi}\tilde{\mathcal{X}}$, where

$$[G(x)\tilde{\mathcal{X}}_{(lm),x}]_{,x} + \left[\Lambda_{(lm)} - \frac{\hat{m}^2}{G(x)}\right]\tilde{\mathcal{X}}_{(lm)} = 0. \qquad (68)$$

Together with the appropriate boundary conditions this is again the formulation of a Sturm-Liouville problem for te eigenfunctions $\tilde{X}_{(lm)}$ and eigenvalues Λ_{lm}.

This equation [which arises from the separation of variables for the electromagnetic Ernst potential (3)] is clearly different from Eq. (56) [which arises during the separation of variables for the master equation (53)] because the latter contains a term proportional to $G(x)_{,xx}$ for any s. Thus, Eq. (68) does not fit in the scheme immediately. Yet, it shows how to construct a basis of spin-1 weighted accelerated spherical harmonics from a scalar basis on this "sphere" $\tilde{Y}_{(lm)} = \tilde{\mathcal{X}}_{(lm)}e^{im\varphi}$; the solutions of Eq. (56) are given by

$$Y^{(-1)}_{(lm)} = \frac{\bar{\eth}\tilde{Y}_{(lm)}}{A(x-y)} = \frac{-K_\varphi G(x)\tilde{Y}_{(lm),x} - i\tilde{Y}_{(lm),\varphi}}{K_\varphi\sqrt{G(x)}}, \quad \left(\text{cf.}\,\bar{\eth} = -\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right); \qquad (69)$$

$$Y^{(1)}_{(lm)} = \frac{\eth\tilde{Y}_{(lm)}}{A(x-y)} = \frac{-K_\varphi G(x)\tilde{Y}_{(lm),x} + i\tilde{Y}_{(lm),\varphi}}{K_\varphi\sqrt{G(x)}}, \quad \left(\text{cf.}\,\eth = -\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right) \qquad (70)$$

which resembles the standard procedure of generating spin-weighted spherical harmonics.

VII. RADIAL FUNCTION

The separation of the master equation (53), furthermore, leads to the equation for the radial function $\mathcal{Y}^{(s)}_{(lm)}$

$$\frac{[G(y)^{s+1}\mathcal{Y}^{(s)}_{(lm),y}]_{,y}}{G(y)^s}+\left[\frac{1}{G(y)}\left(\omega^2-is\omega G(y)_{,y}+\frac{(2s+1)(s+1)}{6}G(y)G(y)_{,yy}\right)+\Lambda^{(s)}_{(lm)}\right]\mathcal{Y}^{(s)}_{(lm)}=0. \tag{71}$$

Naturally there is a "symmetry" between solutions with $\pm s$. Let $\mathcal{Y}^{(|s|)}_{(lm)}$ be the solution of Eq. (71) with spin $|s|$. Then the function

$$\mathcal{Y}^{(-|s|)}_{(ml)}=G(y)^{|s|}\mathcal{Y}^{(|s|)}_{(lm)} \tag{72}$$

is the solution of Eq. (71) with spin $-|s|$ and frequency $-\omega$.

The structure of regular singular points of the ordinary differential equation (71) is the same as the structure of roots of $G(\xi)$ and the infinity:

$$-\frac{1}{Ar_m},\qquad -\frac{1}{Ar_p},\qquad -1,1,\infty. \tag{73}$$

The explicit solutions of Eq. (72) are probably impossible to find but let us analyze the behavior of the solutions at the outer black hole horizon $y=-1/Ar_p$, and the acceleration horizon $y=-1$ for the static case $\omega=0$. Then $y=-1/Ar_p$, $y=-1$ are regular singular points of the equation (72) all of them with characteristic exponents $(0,-s)$ regardless of the $\Lambda^{(s)}_{(lm)}$ for the nondegenerated case $r_p\neq r_m$. In the extremal case[8] the characteristic exponents of the singular point $y=-1/Ar_p$ are $(l-s,-l-s-1)$. In the C-metric the infinity which is interesting is $\mathcal{S}^+$ which is not easy to describe in these coordinates and thus the asymptotic behaviour of fields at $\mathcal{S}$ will be discussed elsewhere.

The theory of ordinary differential equations can provide us with the behavior of two linearly independent solutions in the vicinity of regular singular points. Sort the exponents at the singularity and thus define $R_1=\max\{0,-s\}$ and $R_2=\min\{0,-s\}$ so that $R_1\geq R_2$. Let $\sigma_j=1+\sum_{k=1}^{\infty}a_{jk}(y-y_s)^k$; the first solution around a singular point y_s is then

$$\mathcal{Y}^{(s)A}_{(lm)}=(y-y_s)^{R_1}\sigma_1, \tag{74}$$

and the second one is for $R_1-R_2\in\mathbb{N}_0+½$

$$\mathcal{Y}^{(s)B}_{(lm)}=(y-y_s)^{R_2}\sigma_2, \tag{75}$$

or, for $s\in\mathbb{N}_0$

$$\mathcal{Y}^{(s)B}_{(lm)}=(y-y_s)^{R_2}\sigma_2+c\ln(y-y_s)\mathcal{Y}^{(s)A}_{(lm)}, \tag{76}$$

where the constant c can be zero.

In the static ($\omega=0$), axisymmetric ($m=0$) [i.e. for static (moving along $\boldsymbol{\xi_\tau}$) axisymmetric sources of the test field] and extremal ($r_m=r_p$) case the two linearly independent solutions can be found explicitly in terms of hypergeometric functions, for $s\geq 0$

$$\mathcal{Y}^{(s)r}_{(l0)}=\frac{(1+y)^{l-s}}{(1+Ar_py)^{l+s+1}}\times{}_2F_1\left(\begin{matrix}-l,s-l\\-2l\end{matrix};\frac{2}{1+Ar_p}\frac{1+Ar_py}{1+y}\right), \tag{77}$$

$$\mathcal{Y}^{(s)s}_{(l0)}=\frac{(1+Ar_py)^{l-s}}{(1+y)^{l+s+1}}\times{}_2F_1\left(\begin{matrix}l+1,l+s+1\\2(l+1)\end{matrix};\frac{2}{1+Ar_p}\frac{1+Ar_py}{1+y}\right). \tag{78}$$

The solution $\mathcal{Y}^{(s)r}_{(l0)}$ is a rational function that is divergent at the black hole horizon (i.e. it is the physical outer solution) and the solution $\mathcal{Y}^{(s)s}_{(l0)}$ is divergent at the acceleration horizon (i.e. it is a physical inner solution; the divergence at the acceleration horizon means that no static sources can exist above the acceleration horizon, which is a desirable property of the solution).

VIII. ELECTROMAGNETIC FIELD: A DEEPER ANALYSIS

First of all let us recall that there is an almost forgotten reference in Bičák [19] that for the D type spacetimes Fackerell and Ipser [20] gave a decoupled equation even for the electromagnetic scalar Φ_1

$$[(D+\epsilon+\bar{\epsilon}-\varrho-\bar{\varrho})(\Delta+2\mu)-(\delta+\bar{\pi}-\bar{\alpha}+\beta-\tau)(\bar{\delta}+2\pi)]\Phi_1=0. \tag{79}$$

This equation is separable, using the ansatz

$$\Phi_1=A^2(x-y)^2e^{-i\omega\tau}e^{im\varphi}\mathcal{Y}^{(0)}_{(lm)}(y)\mathcal{X}^{(0)}_{(lm)}(x), \tag{80}$$

and the separation leads to the equations

$$(G(x)\mathcal{X}^{(0)}_{(lm),x})_{,x}+\left[\Lambda^{(0)}_{(lm)}-\frac{m^2}{K_\varphi^2G(x)}\right]\mathcal{X}^{(0)}_{(lm)}=0, \tag{81}$$

$$(G(y)\mathcal{Y}^{(0)}_{(lm),y})_{,y}+\left[\Lambda^{(0)}_{(lm)}+\frac{\omega^2}{K_\tau^2G(y)}\right]\mathcal{Y}^{(0)}_{(lm)}=0, \tag{82}$$

with eigenvalues

[8] And thus only for axially symmetric configurations.

$$\Lambda^{(0)}_{(l0)} = l(l+1)\sqrt{1-A^2 r_m^2}\sqrt{1-A^2 r_p^2}. \tag{83}$$

In the equation (81) we can plainly recognize the equation (68) for eigenfunctions of the Laplace operator on the "sphere" (x, φ).

The self-conjugated form of the electromagnetic field tensor $\boldsymbol{F}^*$ can be reconstructed from the complex NP scalars in the case of the nonrotating C-metric in the tetrad (13).

Let us take a closer look at the static electromagnetic field ($\omega = 0$). The static observer, comoving with the black hole, has 4-velocity

$$\boldsymbol{u} = \frac{A(x-y)}{\sqrt{-G(y)}}\frac{\boldsymbol{\partial}}{\partial\tau}. \tag{84}$$

This observer measures an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$

$$\boldsymbol{M} \equiv \boldsymbol{E} + i\boldsymbol{B} = \boldsymbol{u}\cdot\boldsymbol{F}^*, \tag{85}$$

which we project onto the orthonormal basis $(\hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_\varphi)$

$$\hat{M}_y \equiv \boldsymbol{M}\cdot\hat{\boldsymbol{e}}_y = -\Phi_1, \tag{86}$$

$$\hat{M}_x \equiv \boldsymbol{M}\cdot\hat{\boldsymbol{e}}_x = \frac{1}{2}\frac{G(y)\Phi_0 + A^2(x-y)^2\Phi_2}{A(x-y)\sqrt{-G(y)}}, \tag{87}$$

$$\hat{M}_\varphi \equiv \boldsymbol{M}\cdot\hat{\boldsymbol{e}}_\varphi = \frac{i}{2}\frac{G(y)\Phi_0 - A^2(x-y)^2\Phi_2}{A(x-y)\sqrt{-G(y)}}. \tag{88}$$

The scalar Φ_1 describes the purely radial behavior of the field and so it is not important for the discussion of the regularity conditions of the field in the vicinity of the axis.

Summing up all the known information, i.e. the ansatz from Table I and the "symmetry" in the radial solutions (72), we get

$$\Phi_0 = a^{(1)}_{(lm)}A^3(x-y)^3\mathcal{Y}^{(1)}_{(lm)}\mathcal{X}^{(1)}_{(lm)}, \tag{89}$$

$$\Phi_2 = a^{(-1)}_{(lm)}A(x-y)\mathcal{Y}^{(-1)}_{(lm)}\mathcal{X}^{(-1)}_{(lm)} \tag{90}$$

$$= a^{(-1)}_{(lm)}A(x-y)G(y)\mathcal{Y}^{(1)}_{(lm)}\mathcal{X}^{(-1)}_{(lm)}, \tag{91}$$

and thus, for the orthonormal tetrad components of the electromagnetic field as measured by astatic observer we get

$$M_x = \frac{-1}{2}A^2(x-y)^2\sqrt{-G(y)}\mathcal{Y}^{(1)}_{(lm)} \times (a^{(1)}_{(lm)}\mathcal{X}^{(1)}_{(lm)} + a^{(-1)}_{(lm)}\mathcal{X}^{(-1)}_{(lm)})e^{im\varphi}, \tag{92}$$

$$M_\varphi = \frac{-1}{2}A^2(x-y)^2\sqrt{-G(y)}\mathcal{Y}^{(1)}_{(lm)} \times (a^{(1)}_{(lm)}\mathcal{X}^{(1)}_{(lm)} - a^{(-1)}_{(lm)}\mathcal{X}^{(-1)}_{(lm)})e^{im\varphi}, \tag{93}$$

for which the standard discussion (i.e. as done for a Reissner-Nordström black hole) of the regularity of the vector fields can be done.

In Sec. II we showed how to find two solutions of the test Maxwell equations using the Ernst formalism and provided some symmetries between the equations for $m = 0$. Let us use these results in the extreme and static cases (so explicit solutions of the radial and angular parts exist).

We can calculate the Ernst potentials $\Phi^{(\tau)}$ and $\mathcal{E}^{(\tau)}$ for the nonrotating C-metric. They are

$$\Phi^{(\tau)} = y, \tag{94}$$

$$\mathcal{E}^{(\tau)} = \frac{G(y)}{(x-y)^2} - q^2y^2. \tag{95}$$

The appropriate NP scalars are given by Eqs. (31)–(33).

These solutions are static and axially symmetric and therefore they can be expanded in terms of the basis given by the solution of Eqs. (81) and (82) with $\omega = m = 0$. They can be solved by looking for a solution of $G\mathcal{U}_{,\xi\xi} + \Lambda\mathcal{U}$ as discussed in Sec. II.

Let us investigate the extreme case $q = r_m = r_p$. Then we can find the solutions of Eqs. (81) and (82) in terms of hypergeometric functions

$$\mathcal{X}^{(0)r}_{(l0)} = \mathcal{Y}^{(0)r}_{(l0)} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi)\left(\frac{1+Ar_p\xi}{1+\xi}\right)^{-l} \times {}_2F_1\left(\begin{matrix}-l, 1-l\\ -2l\end{matrix}; \frac{2}{1+Ar_p}\frac{1+Ar_p\xi}{1+\xi}\right)\right], \tag{96}$$

which is a regular solution in the interval $\xi \in \langle -1, 1\rangle$ for $l \geq 1$ (for $l = 0$ the solution is constant) and reduces to rational functions whereas

$$\mathcal{Y}^{(0)s}_{(l0)} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi)\left(\frac{1+Ar_p\xi}{1+\xi}\right)^{l+1} \times {}_2F_1\left(\begin{matrix}2+l, 1+l\\ 2+2l\end{matrix}; \frac{2}{1+Ar_p}\frac{1+Ar_p\xi}{1+\xi}\right)\right], \tag{97}$$

is singular[9] at $\xi = -1$ and, due to the contiguous relations among hypergeometric functions (see Sec. 15.5(ii) of Ref. [21]), can be expressed in terms of ${}_2F_1\left(\begin{smallmatrix}1,1\\2\end{smallmatrix}; z\right)$ which equals $-z^{-1}\ln(1-z)$ (a few formulas can be found in the Appendix).

[9]Singular solutions for the angular part are irrelevant.

The trivial solution of Maxwell's equation, the "Coulomb potential," given by the Ernst potential $\Psi^{(\tau)} = \Phi^{(\tau)} = y$ leads to NP coefficients $\Phi_0 = 0$, $\Phi_1 = -A^2(x-y)^2\sqrt{2}/K_\tau$ and $\Phi_2 = 0$ which compared with Eq. (80) trivially yield the expansion as $\Phi_1 \sim \mathcal{X}^{(0)r}_{(00)}\mathcal{Y}^{(0)r}_{(00)r}$.

The other solution of the test Maxwell equations which is given by $\Psi^{(\tau)} = \mathcal{E}^{(\tau)}$ represents, in the Minkowski limit, a uniform electric field aligned parallel to the symmetry axis. From Eqs. (32) and (80) it follows that

$$\Psi^{(\tau)}_{,y} = \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{G(y)}{(x-y)^2} - r_p^2 y^2\right] = \sum_{l=0}^{\infty} a^{(0)}_{(l0)}\mathcal{X}^{(0)r}_{(l0)}\mathcal{Y}^{(0)s}_{(l0)}. \tag{98}$$

First, we notice, that even the "Coulomb part" is present in the expansion. Second, although it looks nontrivial, it can be explicitly checked that the integral $\int_{-1}^{1}[\frac{G(y)}{(x-y)^2} - r_p^2 y^2]\mathcal{X}^{(0)r}_{(l0)}\mathrm{d}x$ (including logarithmic terms) is proportional to $\mathcal{Y}^{(0)s}_{(l0)}$ after using the contiguous relations for the hypergeometric function.

For this test field the NP scalars diverge at $y = -1$ because the "sources" are located at infinity. Yet, it is merely an artifact of the choice of tetrad; the invariant $\Phi_0\Phi_2 - \Phi_1^2$ does not have any singularities at the acceleration horizon.

Clearly, the integral $\int_{-1}^{1}\Phi_1/(x-y)^2\mathrm{d}x$ can be performed only for $y < -1$ because the range of the coordinates is given by the condition $x - y < 0$ and thus the "sphere" $y = \mathrm{const}$ exists only for $y < -1$.

The series is infinite even in the Minkowski limit which only seemingly contradicts the known result that the homogeneous field is just an $l = 1$ mode. The reason lies in the fact that this system is uniformly accelerated and therefore also even Φ_0 and Φ_2 are nonzero.

The detail analysis of the electromagnetic field will be given in a forthcoming paper.

IX. SPIN WEIGHT RAISING AND LOWERING OPERATORS

From what we have seen so far it is necessary, contrary to the standard spin-weighted harmonics, to distinguish between fields of different spin. For every spin S, there is a sequence[10] of spin-weighted accelerated spherical harmonics $Y^{(s,S)}_{(lm)} = \mathcal{X}^{(s,S)}_{(lm)}e^{im\varphi}$ with spin weight $s \in (-S, -S+1, \ldots, S)$. This sequence can be generated using the universal spin weight raising operator acting on the spin weight component s

$$\begin{aligned} Y^{(s+1,S)}_{(lm)} &= \eth Y^{(s,S)}_{(lm)} \\ &= (G(x))^{\frac{s}{2}}\left(\frac{K_\varphi G(x)\frac{\partial}{\partial x} - i\frac{\partial}{\partial\varphi}}{\sqrt{G(x)}}\right)(Y^{(s,S)}_{(lm)}(G(x))^{-\frac{s}{2}}), \end{aligned} \tag{99}$$

and spin weight lowering operator

$$\begin{aligned} Y^{(s-1,S)}_{(lm)} &= \bar{\eth} Y^{(s,S)}_{(lm)} \\ &= (G(x))^{-\frac{s}{2}}\left(\frac{K_\varphi G(x)\frac{\partial}{\partial x} + i\frac{\partial}{\partial\varphi}}{\sqrt{G(x)}}\right)(Y^{(s,S)}_{(lm)}(G(x))^{\frac{s}{2}}). \end{aligned} \tag{100}$$

The complete scheme can be found in Table II, where the arrows mean lowering and raising the spin weight and the fields are labeled by their spin and spin weight (for example the gravitational perturbation $\Psi_4 \to \Psi^{(-2,2)}$ or $\Sigma_3 \to \Sigma^{(-3/2,3/2)}$ and so on). In the limit $r_p \to 0$ and $K_\varphi \to 1$, the eigenfunctions of operators no longer depend on the spin of the field S but only on the spin weight s of the particular component (as schematically depicted in the last two lines of Table II).

Thus, more properly, the equation (56) for accelerated spin-weighted spherical harmonics should be written as

$$\begin{aligned} &[G(x)\mathcal{X}^{(\pm S,S)}_{(lm),x}]_{,x} + \left[\frac{S^2+\frac{1}{2}}{3}G(x)_{,xx} + \Lambda^{(\pm S,S)}_{(lm)}\right. \\ &\left. - \frac{(\hat{m} - \frac{(\pm S)}{2}G(x)_{,x})^2}{G(x)}\right]\mathcal{X}^{(\pm S,S)}_{(lm)} = 0, \quad \hat{m} \equiv \frac{m}{K_\varphi}; \end{aligned} \tag{101}$$

it holds only for the maximal spin weight $s = \pm S$.

Although the equations for nonextreme components ($|s| \neq S$) are not known for $S > 1$, the following statement can be proved. Let $Y^{(S,S)}_{(lm)}$ be a solution of Eq. (101) for arbitrary S. Then by applying $(2S-1)$ times $\bar{\eth}$ and thus obtaining $Y^{(-S,S)}_{(lm)}$ we get a solution of the eigenvalue problem (101) with $s = -S$ if

$$\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}G(\xi) = 0 \quad \text{for } n \geq 5, \tag{102}$$

which holds even for the charged and rotating C-metric.

X. COSMIC STRING SPACETIME

As soon as the spherical symmetry is abandoned everything becomes more complicated. The cosmic string spacetime can serve as one of the simplest examples. The metric can be obtained as a zero-mass and zero-charge limit of the C-metric; written in quasispherical coordinates it reads

[10] Therefore it would be more appropriate to denote these spin-weighted spherical harmonics with $Y^{(s,S)}_{(lm)}$, or the x component with $\mathcal{X}^{(s,S)}_{(lm)}$; we have not used this notation in the previous text. There it would be just a complication; we omitted the spin S.

TABLE II. Accelerated spin-weighted spherical harmonics schema.

$\overleftarrow{r_p \to 0, K_\varphi \to 1}$	$\Psi^{(-2,2)}$	$\longleftrightarrow$	$\Psi^{(-1,2)}$	$\longleftrightarrow$	$\Psi^{(0,2)}$	$\longleftrightarrow$	$\Psi^{(1,2)}$	$\longleftrightarrow$	$\Psi^{(2,2)}$
		$\Sigma^{(-3/2,3/2)}$	$\longleftrightarrow$	$\Sigma^{(-1/2,3/2)}$	$\longleftrightarrow$	$\Sigma^{(1/2,3/2)}$	$\longleftrightarrow$	$\Sigma^{(3/2,3/2)}$	
			$\Phi^{(-1,1)}$	$\longleftrightarrow$	$\Phi^{(0,1)}$	$\longleftrightarrow$	$\Phi^{(1,1)}$		
				$\chi^{(-1/2,1/2)}$	$\longleftrightarrow$	$\chi^{(1/2,1/2)}$			
					$\Phi^{(0,0)}$				
	$\Phi^{(-2)}$	$\longleftrightarrow$	$\Phi^{(-1)}$	$\longleftrightarrow$	$\Phi^{(0)}$	$\longleftrightarrow$	$\Phi^{(1)}$	$\longleftrightarrow$	$\Phi^{(2)}$
		$\Sigma^{(-3/2)}$	$\longleftrightarrow$	$\Sigma^{(-1/2)}$	$\longleftrightarrow$	$\Sigma^{(1/2)}$	$\longleftrightarrow$	$\Sigma^{(3/2)}$	

$$\mathbf{d}s^2 = -\mathbf{d}t^2 + \mathbf{d}r^2 + r^2(\mathbf{d}\theta^2 + K_\varphi^2 \sin^2\theta \mathbf{d}\varphi^2), \tag{103}$$

and represents flat spacetime with a cosmic string aligned along the z axis.

To analyze fields on this spacetime it evidently makes no sense to use spherical harmonics.

The relevant (angular) part of the metric is

$$\begin{aligned}\mathbf{d}s^2 &= \mathbf{d}\theta^2 + K_\varphi^2 \sin^2\theta \mathbf{d}\varphi^2 \\ &= \frac{\mathbf{d}x^2}{1-x^2} + (1-x^2)K_\varphi^2 \mathbf{d}\varphi^2.\end{aligned} \tag{104}$$

The separation of variables for the equation $\Delta\Phi(x,\varphi) = -\lambda\Phi(x,\varphi)$ leads to the ansatz $\Phi(x,\varphi) = Y^{(s)}_{(lm)}(x,\varphi) = S^{(s)}_{(lm)}(x)e^{im\varphi}$ where

$$\frac{1}{S^{(0)}_{(lm)}}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}S^{(0)}_{(lm)}}{\mathrm{d}x}\right] - \frac{(\frac{m}{K_\varphi})^2}{1-x^2} - \frac{1}{3} + \Lambda^{(s)}_{(lm)} = 0, \tag{105}$$

which can be considered as a limit case of Eq. (56) (with $r_p = r_m = 0$, $s = 0$) and therefore we will solve the more general case with an arbitrary half-integer s:

$$\begin{aligned}&\frac{1}{S^{(s)}_{(lm)}}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}S^{(s)}_{(lm)}}{\mathrm{d}x}\right] - \frac{(\frac{m}{K_\varphi} + sx)^2}{1-x^2} \\ &\quad - \frac{2}{3}s^2 - \frac{1}{3} + \Lambda^{(s)}_{(lm)} = 0.\end{aligned} \tag{106}$$

The general regular solution reads

$$\begin{aligned}S^{(s)}_{(lm)} &= (1-x)^{l+m\frac{1-K_\varphi}{K_\varphi}}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}(\frac{1}{K_\varphi}m - s)} \\ &\quad \times {}_2F_1\left(\begin{matrix}-l+m, -l-s-m\frac{1-K_\varphi}{K_\varphi} \\ 1+\frac{m}{K_\varphi}-s\end{matrix}; -\frac{1+x}{1-x}\right)\end{aligned} \tag{107}$$

or, equivalently,

$$\begin{aligned}S^{(s)}_{(lm)} &= (1-x)^{\frac{m}{K_\varphi}}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}(\frac{m}{K_\varphi} - s)} \\ &\quad \times \mathrm{JacobiP}\left(l-m, \frac{m}{K_\varphi}+s, \frac{m}{K_\varphi}-s, x\right)\end{aligned} \tag{108}$$

with eigenvalues

$$\begin{aligned}\Lambda^{(s)}_{(lm)} &= \left(l + m\frac{1-K_\varphi}{K_\varphi}\right)\left(l+1+m\frac{1-K_\varphi}{K_\varphi}\right) \\ &\quad + \frac{1}{3}(1-s^2).\end{aligned} \tag{109}$$

The solutions are bounded if s, l, m and K_φ fulfill the following conditions: $\Lambda^{(s)}_{(lm)} - \frac{2}{3}s^2 - \frac{1}{3}s > 0$ and $|\frac{m}{K_\varphi}| \geq |s|$, and in other cases $S^{(s)}_{(lm)} = 0$.

In the limit $K_\varphi \to 1$ the standard spin-weighted spherical harmonics[11] are recovered.

We can see that m enters the eigenvalues (109) even in this trivial case, and therefore we have to expect that they enter eigenvalues of the accelerated spherical harmonics in a more complicated way.

XI. CONCLUSIONS

We have derived an analogy of the Teukolsky master equation for the nonrotating C-metric and provided the notion of accelerated spin-weighted spherical harmonics, some of which we have found explicitly. We paid special attention to the electromagnetic field which we solved completely. We showed that for nonaxisymmetric configurations $m \neq 0$ the complete basis is difficult to find.

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[11]Euler's transformations of hypergeometric function, as well as Heun general functions, are used to change the form to a equivalent one (cf. Ref. [21]).

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APPENDIX: THE GAUSS CONTIGUOUS RELATIONS

The Gauss contiguous relations [21] allow us to reduce some of the hypergeometric functions to elementary functions using the fact that

$$L \equiv {}_2F_1\left(\begin{matrix}1,1\\2\end{matrix};z\right) = -\frac{\ln(1-z)}{z}. \tag{A1}$$

For a general hypergeometric function $W_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix};z\right)$ we can define the operators A^+, B^+ and C^+ which raise the parameters a, b and c by

$${}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix};z\right) = A^+(W) = \frac{z}{a}\frac{\mathrm{d}W}{\mathrm{d}z} + W, \tag{A2}$$

$${}_2F_1\left(\begin{matrix}a,b+1\\c\end{matrix};z\right) = B^+(W) = \frac{z}{b}\frac{\mathrm{d}W}{\mathrm{d}z} + W, \tag{A3}$$

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a,b\\c+1\end{matrix};z\right) &= C^+(W)\\ &= \frac{(1-z)c\frac{\mathrm{d}W}{\mathrm{d}z} - c(a+b-c)W}{(c-a)(c-b)}.\end{aligned} \tag{A4}$$

Using these operators the hypergeometric function for a, b, $c \in \mathbb{N}$ and $c \geq 2$ can be expressed in terms of logarithms by

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) = (A^+)^{a-1}(B^+)^{b-1}(C^+)^{c-2}L. \tag{A5}$$

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