Spinning particles in vacuum spacetimes of different curvature types: Natural reference tetrads and massless particles

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In a previous paper, we considered the motion of massive spinning test particles in the "pole-dipole" approximation, as described by the Mathisson-Papapetrou-Dixon (MPD) equations and examined its properties in dependence on the spin supplementary condition. We decomposed the equations in the orthonormal tetrad based on the timelike vector fixing the spin condition and on the corresponding spin, while representing the curvature in terms of the Weyl scalars obtained in the Newman-Penrose null tetrad naturally associated with the orthonormal one; the projections thus obtained did not contain the Weyl scalars Ψ_0 and Ψ_4. In the present paper, we choose the interpretation tetrad in a different way, attaching it to the tangent u^μ of the worldline representing the history of the spinning body. Actually *two* tetrads are suggested, both given "intrinsically" by the problem and each of them incompatible with one specific spin condition. The decomposition of the MPD equation, again supplemented by writing its right-hand side in terms of the Weyl scalars, is slightly less efficient than in the massive case, because u^μ cannot be freely chosen (in contrast to V^μ) and so the u^μ-based tetrad is less flexible. In the second part of this paper, a similar analysis is performed for massless spinning particles; in particular, a certain intrinsic interpretation tetrad is again found. The respective decomposition of the MPD equation of motion is considerably simpler than in the massive case, containing only Ψ_1 and Ψ_2 scalars and not the cosmological constant. An option to span the spin-bivector eigenplane, besides the worldline null tangent, by a main principal null direction of the Weyl tensor can lead to an even simpler result.

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I. INTRODUCTION

In Ref. [1] (henceforth referred to as paper I), we studied the problem of motion of a massive spinning test particle ("pole-dipole" body) as described by the Mathisson-Papapetrou-Dixon (MPD) equations

$$\dot{p}^\mu = -\frac{1}{2} R^\mu{}_{\nu\kappa\lambda} u^\nu S^{\kappa\lambda}, \tag{1}$$

$$\dot{S}^{\alpha\beta} = p^\alpha u^\beta - u^\alpha p^\beta, \tag{2}$$

where p^μ and u^μ denote the total momentum and four-velocity of the particle, $S^{\mu\nu}$ is the particle-spin bivector, and the dot denotes the absolute derivative along u^μ. We restricted to *vacuum* space-times and focused on the dependence of the exercise on the spin supplementary condition $S^{\mu\nu}V_\nu = 0$, necessary to fix ambiguity in the MPD equations, and on the interpretation of the spin-curvature interaction in terms of the Weyl scalars. Starting from the projection of the equations into a suitable orthonormal tetrad, we chose the latter's time vector to coincide with the "reference observer" V^μ specifying the spin condition and one of the spatial legs to be given by the spin vector s^μ connected with $S^{\mu\nu}$ by $s^\mu := -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}V_\nu S_{\rho\sigma} = -{}^*S^{\mu\nu}V_\nu$. Rewriting the force term representing spin-curvature interaction in terms of the scalars $\Psi_{0\text{–}4}$, obtained by projecting the Weyl tensor onto the associated Newman-Penrose (NP) complex null tetrad, we found that the MPD-equation orthonormal-basis projections do not contain scalars Ψ_0 and Ψ_4. We then suggested a possible way to choose the remaining two spatial basis vectors "intrinsically," that is, along directions provided by the geometry of the problem itself; this choice is applicable when $u^\mu \nparallel p^\mu$ (an alternative tetrad, usable in this situation—but not together with the Mathisson-Pirani condition, on the contrary—is added in the present paper, Sec. III B 1).

In order to find how the problem looks in space-times of some particular curvature type, we aligned the first vector k^μ of the NP tetrad with the highest-multiplicity principal null direction (PND) of the Weyl tensor by a suitable choice of V^μ, reproducing at the same time a given spin, either described by s^μ or $S^{\mu\nu}$ according to the MPD equations. More specifically, the plan goes like this: have a generic space-time (thus *some* k^μ and other PNDs) and a generic particle (with some spin vector s^μ or spin tensor $S^{\mu\nu}$ at a given point). Aligning the first real vector of the NP tetrad with k^μ, its second real vector l^μ can always be chosen so as to satisfy the relation $s^\mu = \frac{s}{\sqrt{2}}(k^\mu - l^\mu)$,[1] or, respectively, as an eigenvector of $S^{\mu\nu}$ independent of k^μ; finally V^μ is

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[1]More precisely, it is only not possible if $k_\mu s^\mu = 0$.

retrodefined by $V^\mu = \frac{1}{\sqrt{2}}(k^\mu + l^\mu)$. Projecting the MPD equation of motion into the orthonormal tetrad involving *these* V^μ and s^μ/s as the zeroth and first vectors (and completed by some orthonormal $e^\mu_{\hat{2}}$ and $e^\mu_{\hat{3}}$), one obtains Eqs. (66)–(69) of paper I,

$$-V_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2\, u^{\hat{1}} - s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1)u^{\hat{2}} - s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1)u^{\hat{3}}, \tag{3}$$

$$e^{\hat{1}}_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2\, u^{\hat{0}} - s(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)u^{\hat{2}} - s(\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)u^{\hat{3}}, \tag{4}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = +s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{3}} - s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1)u^{\hat{0}} + s(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)u^{\hat{1}}, \tag{5}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = -s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{2}} - s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1)u^{\hat{0}} + s(\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)u^{\hat{1}}, \tag{6}$$

where $u^{\hat{\alpha}}$ denote the tetrad components of four-velocity.

It is known that—with the exception of Petrov type III—it is possible to rotate the null tetrad so it becomes "transverse" in the sense that the corresponding Ψ_1 and Ψ_3 projections vanish (instead of the usual elimination of Ψ_0 and Ψ_4). If such a rotation of the tetrad were feasible (*in addition* to the above), the spinning-particle motion would be fully determined by Ψ_2 and by the cosmological constant (because Ψ_0 and Ψ_4 are not involved from the beginning). Unfortunately, this could only be achieved by chance, because the necessary rotation involves all the NP vectors (in dependence on Weyl scalars in the original NP tetrad); in particular, it fixes the (k^μ, l^μ) plane, so l^μ *cannot* be chosen to lie in the (k^μ, s^μ) plane at the same time.

In the last part of paper I, we discussed the implications of the spin conditions mostly considered in the literature, mainly advocating the condition $\dot{V}^\mu = 0$ which leads to $u^\mu \| p^\mu$ and generalizing it, and finally checked several particular types of motion.

In the present paper, let us proceed in a similar way but choosing a different orthonormal tetrad, namely, the one tied to u^μ as the time vector. In Sec. II, we suggest—as a counterpart of the intrinsic tetrad based on V^μ considered in paper I—a u^μ-based tetrad which follows naturally from geometry of the problem. If trying to adapt the interpretation tetrad to the Weyl-tensor PNDs, one is either led to the situation when $S^{\mu\nu}u_\nu = 0$, so the Mathisson-Pirani condition holds (thus returning to the respective section of paper I), or one has to release the "natural" association of the orthonormal tetrad with the NP tetrad, namely, to compute the Weyl scalars in a NP tetrad which is *not* naturally associated with the orthonormal tetrad into which the MPD equations have been projected. Both possibilities are worked out, with the type-N space-time mentioned as an example. Implications of specific spin supplementary conditions are considered in Sec. III, pointing out, in particular, that for $u^\mu \| p^\mu$ a different tetrad has to be devised since the original one degenerates (similarly as its counterpart employed in paper I).

In the second part (Sec. IV), we turn to spinning particles with zero rest mass. Starting from a summary of what is known from the literature, we study the geometry of the massless problem in a similar way as its "massive" counterpart before. In particular, we again propose a certain natural NP tetrad and the associated orthonormal frame, which follow from the geometry of the problem itself, and inquire about the properties of the MPD equation of motion when projected there. Also the properties of the orthonormal frame are examined, including the circumstance $p^\mu \| k^\mu$ when the frame is not available (and $\dot{p}^\mu$ is itself proportional to the worldline null tangent k^μ).

First, however, let us remind the reader that the space-time is supposed to be vacuum, possibly involving a nonzero cosmological constant Λ; the metric signature is $(-+++)$; and geometrized units are used in which $c = 1$, $G = 1$. Greek indices run 0–3, latin indices 1–3, and the summation convention is followed. The dot denotes an absolute derivative with respect to the particle's proper time τ, the asterisk denotes the Hodge dual, and the overbar indicates complex conjugation. The Riemann tensor is defined by $V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} = R^\mu{}_{\nu\kappa\lambda}V_\mu$ and the Levi-Civita tensor as

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,[\mu\nu\rho\sigma], \qquad \epsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}\,[\mu\nu\rho\sigma], \tag{7}$$

where g is the determinant of the covariant metric and $[\mu\nu\rho\sigma]$ is the permutation symbol fixed by $[0123] \coloneqq 1$. Please see (e.g.) paper I for an introductory summary on the spinning-particle problem, including basic as well as recent references.

II. VACUUM MPD EQUATIONS IN A TETRAD TIED TO u^μ

The reference observer V^μ, in terms of which the spin supplementary condition is written ($S^{\mu\sigma}V_\sigma = 0$), can be chosen freely, so it is generically possible to attach it to a *given* NP tetrad by taking $V^\mu \coloneqq \frac{1}{\sqrt{2}}(k^\mu + l^\mu)$. This is not in general possible with u^μ, because this has to be obtained from p^μ which in turn is determined by the MPD equations, so none of these two vectors can be *chosen*. Hence the procedure will have to be different, namely, based on the given u^μ and k^μ.

We will again start from the MPD equation of motion, rewritten in terms of spin vector s^μ in the form (39) of paper I,

$$\dot{p}^\mu = {}^*R^\mu{}_{\nu\alpha\beta}u^\nu s^\alpha V^\beta = \left({}^*C^\mu{}_{\nu\alpha\beta} + \frac{\Lambda}{3}\epsilon^\mu{}_{\nu\alpha\beta}\right)u^\nu s^\alpha V^\beta, \qquad (8)$$

where we have used the vacuum relation between the Riemann-tensor and Weyl-tensor left duals ${}^*R^\mu{}_{\nu\alpha\beta}$ and ${}^*C^\mu{}_{\nu\alpha\beta}$ (in a vacuum they equal the right duals). Two advantages of having u^μ as the time vector of the tetrad are obvious: first, similarly as V^μ (and s^μ, which we used in paper I), the four-velocity u^μ appears on the right-hand side among the vectors on which the dual Riemann is projected, and second, the whole $\dot{p}^\mu$ is from the beginning orthogonal to u^μ, so its "zeroth" component in such a tetrad vanishes automatically. (Note that none of these properties holds for the third major "time" vector of the exercise, p^μ.) Now, however, the following question arises: which spatial vectors should one add to u^μ, in order to complete the basis? Generally, there are two possibilities: either to take some vectors provided intrinsically by the p^μ, u^μ, s^μ, V^μ geometry (possibly also including derivatives of these vectors) or to try to somehow connect the spatial basis directly to the curvature structure, while staying in a space orthogonal to u^μ.

The first, intrinsic possibility can be proposed in analogy with paper I. Actually, denoting

$$\gamma \coloneqq -u_\mu V^\mu (>0), \;\; \mu \coloneqq -p_\mu V^\mu (>0),$$

we chose there the basis

$$V^\mu, s^\mu, \mu u^\mu - \gamma p^\mu, \;\; (s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu \qquad (9)$$

(or rather its normalized version), made of the eigenvectors V^μ and s^μ of the spin bivector $S_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}V^\mu s^\nu$ and of the eigenvectors $(\mu u^\mu - \gamma p^\mu)$ and $(s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu$ of its dual ${}^*S^{\mu\nu} = s^\mu V^\nu - V^\mu s^\nu$. As a counterpart of this basis, we suggested the quadruple made of u^μ and spatial vectors

$$p^\mu - m u^\mu = -\dot{S}^{\mu\alpha}u_\alpha, \qquad (10)$$

$$\gamma s^\mu + s_\nu u^\nu V^\mu = -{}^*S^{\mu\alpha}u_\alpha, \qquad (11)$$

$$\epsilon^{\mu\iota\kappa\lambda}u_\iota(\gamma s_\kappa + s_\nu u^\nu V_\kappa)p_\lambda = -{}^*\dot{S}^{\mu\lambda *}S_{\lambda\nu}u^\nu, \qquad (12)$$

i.e., of the eigenvectors u^μ and $(p^\mu - m u^\mu)$ ("hidden momentum") of the bivector ${}^*\dot{S}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}p^\alpha u^\beta$ and of the eigenvectors $(\gamma s^\mu + s_\nu u^\nu V^\mu)$ and $\epsilon^{\mu\iota\kappa\lambda}u_\iota(\gamma s_\kappa + s_\nu u^\nu V_\kappa)p_\lambda$ of the bivector $\dot{S}^{\alpha\beta} = p^\alpha u^\beta - u^\alpha p^\beta$. In the above, m is the particle mass with respect to u^μ, given by $m \coloneqq -u_\mu p^\mu (>0)$.

Note that the last of the tetrad vectors can also be written in a different way: regarding the formula [see, e.g., Ref. [2], Eq. (7.15)]

$${}^*F^{\mu\lambda *}H_{\lambda\nu} = H^{\mu\lambda}F_{\lambda\nu} + \frac{1}{2}\delta^\mu_\nu F^{\alpha\beta}H_{\alpha\beta},$$

valid for any two bivectors $F_{\mu\nu}$ and $H_{\mu\nu}$, we can rewrite

$$\begin{aligned}
&\epsilon^{\mu\iota\kappa\lambda}u_\iota(\gamma s_\kappa + s_\nu u^\nu V_\kappa)p_\lambda \\
&= -{}^*\dot{S}^{\mu\lambda *}S_{\lambda\nu}u^\nu = -S^{\mu\lambda}\dot{S}_{\lambda\nu}u^\nu - \frac{1}{2}u^\mu \dot{S}^{\alpha\beta}S_{\alpha\beta} \\
&= -S^{\mu\lambda}\dot{S}_{\lambda\nu}u^\nu - u^\mu \epsilon^{\alpha\beta\gamma\delta}p_\alpha u_\beta V_\gamma s_\delta \\
&= -S^{\mu\lambda}\dot{S}_{\lambda\nu}u^\nu - u^\mu s\dot{s} = -(\delta^\mu_\alpha + u^\mu u_\alpha)S^{\alpha\lambda}\dot{S}_{\lambda\nu}u^\nu \\
&= (\delta^\mu_\alpha + u^\mu u_\alpha)\epsilon^{\alpha\lambda\gamma\delta}V_\gamma s_\delta(p_\lambda - m u_\lambda),
\end{aligned} \qquad (13)$$

where we have used just basic forms of all the bivectors and relation (33) from paper I, i.e.,

$$s\dot{s} \equiv s_\mu \dot{s}^\mu = \frac{1}{2}S^{\alpha\beta}\dot{S}_{\alpha\beta} = S^{\alpha\beta}p_\alpha u_\beta = \epsilon^{\mu\nu\alpha\beta}s_\mu V_\nu u_\alpha p_\beta. \qquad (14)$$

Also, instead of the tetrad vectors u^μ and $(p^\mu - m u^\mu)$, it would be possible to use in the basis, for example, p^μ and $(m p^\mu - \mathcal{M}^2 u^\mu)$ (the latter being given by the component of u^μ orthogonal to p^μ).

In order to make the tetrad orthonormal, one needs magnitudes of the spatial vectors:

$$(p_\mu - m u_\mu)(p^\mu - m u^\mu) = m^2 - \mathcal{M}^2, \qquad (15)$$

$$(\gamma s_\mu + s_\nu u^\nu V_\mu)(\gamma s^\mu + s_\nu u^\nu V^\mu) = \gamma^2 s^2 - (s_\nu u^\nu)^2, \qquad (16)$$

$$\begin{aligned}
&\epsilon^{\mu\iota\kappa\lambda}u_\iota(\gamma s_\kappa + s_\nu u^\nu V_\kappa)p_\lambda \epsilon_{\mu\rho\sigma\tau}u^\rho(\gamma s^\sigma + s_\beta u^\beta V^\sigma)p^\tau \\
&= {}^*\dot{S}^{\mu\lambda *}S_{\lambda\nu}u^\nu {}^*\dot{S}_{\mu\kappa}{}^*S^{\kappa\sigma}u_\sigma \\
&= -\frac{1}{2}\dot{S}^{\alpha\beta}\dot{S}_{\alpha\beta}{}^*S_{\lambda\nu}u^\nu {}^*S^{\lambda\sigma}u_\sigma \\
&= (m^2 - \mathcal{M}^2)(\gamma s_\mu + s_\nu u^\nu V_\mu)(\gamma s^\mu + s_\nu u^\nu V^\mu) \\
&= (m^2 - \mathcal{M}^2)[\gamma^2 s^2 - (s_\nu u^\nu)^2].
\end{aligned} \qquad (17)$$

Finally, regarding that the tetrad used in paper I was numbered as

$$e^\mu_{\hat{0}} \coloneqq V^\mu, \qquad (18)$$

$$e^\mu_{\hat{1}} \coloneqq \frac{s^\mu}{s}, \qquad (19)$$

$$e^\mu_{\hat{2}} \coloneqq \frac{\mu u^\mu - \gamma p^\mu}{\sqrt{\gamma^2(m^2 - \mathcal{M}^2) - (\gamma m - \mu)^2}}, \qquad (20)$$

$$e^\mu_{\hat{3}} \coloneqq \frac{\epsilon^{\mu\iota\kappa\lambda}V_\iota s_\kappa(\mu u_\lambda - \gamma p_\lambda)}{s\sqrt{\gamma^2(m^2 - \mathcal{M}^2) - (\gamma m - \mu)^2}}, \qquad (21)$$

let us do it similarly here,

$$e^{\mu}_{(0)} := u^{\mu}, \tag{22}$$

$$e^{\mu}_{(1)} := \frac{\gamma s^{\mu} + s_{\nu}u^{\nu}V^{\mu}}{\sqrt{\gamma^2 s^2 - (s_{\sigma}u^{\sigma})^2}}, \tag{23}$$

$$e^{\mu}_{(2)} := \frac{p^{\mu} - m u^{\mu}}{\sqrt{m^2 - \mathcal{M}^2}}, \tag{24}$$

$$e^{\mu}_{(3)} := \frac{\epsilon^{\mu\iota\kappa\lambda}u_{\iota}(\gamma s_{\kappa} + s_{\nu}u^{\nu}V_{\kappa})p_{\lambda}}{\sqrt{m^2 - \mathcal{M}^2}\sqrt{\gamma^2 s^2 - (s_{\sigma}u^{\sigma})^2}} \tag{25}$$

(we distinguish the two tetrads by the different markings of their vector-numbering indices).

Clearly neither of the tetrads can be erected if $u^{\mu} \| p^{\mu}$ (see Sec. III B below).

A. Basic observations

One of the vectors we have proposed for the u^{μ}-based tetrad, $(\gamma s^{\mu} + s_{\nu}u^{\nu}V^{\mu})$, is a combination of V^{μ} and s^{μ}, so it belongs to the eigenplane of $S^{\mu\nu}$. If we select this plane to coincide with that spanned by the PND k^{μ} and a suitably chosen l^{μ}, the vector $(\gamma s^{\mu} + s_{\nu}u^{\nu}V^{\mu})$ will be linked with the curvature structure. This is actually the *best* that can be done in this respect; in particular, one cannot include in the basis *two* independent vectors lying in the k^{μ}, l^{μ} plane, because it is impossible to make *both* of them orthogonal to u^{μ}. Therefore, the above set of vectors seems to be a reasonable proposal from which to build a u^{μ}-directed basis, which at the same time is attached to the curvature structure as closely as generically possible. (So far, however, the space-time is left completely general, and also the tetrad is not necessarily linked to the Weyl-tensor PNDs.)

Introducing the tetrad (22)–(25), we can first write (8) as

$$\dot{p}^{\mu} = \frac{1}{\gamma}\left({}^{*}C^{\mu}{}_{\nu\alpha\beta} + \frac{\Lambda}{3}\epsilon^{\mu}{}_{\nu\alpha\beta}\right)u^{\nu}(\gamma s^{\alpha} + s_{\iota}u^{\iota}V^{\alpha})V^{\beta} \tag{26}$$

$$= \frac{\sqrt{\gamma^2 s^2 - (s_{(0)})^2}}{\gamma}\left({}^{*}C^{\mu}{}_{(0)(1)(\delta)} + \frac{\Lambda}{3}\epsilon^{\mu}{}_{(0)(1)(\delta)}\right)V^{(\delta)}, \tag{27}$$

where the relevant components of $V^{(\delta)}$ read

$$V^{(0)} := e^{(0)}_{\mu}V^{\mu} = -u_{\mu}V^{\mu} \equiv \gamma, \tag{28}$$

$$V^{(2)} := e^{(2)}_{\mu}V^{\mu} = \frac{\gamma m - \mu}{\sqrt{m^2 - \mathcal{M}^2}}, \tag{29}$$

$$V^{(3)} := e^{(3)}_{\mu}V^{\mu} = \frac{\gamma\epsilon^{\mu\iota\kappa\lambda}V_{\mu}u_{\iota}s_{\kappa}p_{\lambda}}{\sqrt{m^2 - \mathcal{M}^2}\sqrt{\gamma^2 s^2 - (s_{\sigma}u^{\sigma})^2}} = \frac{\gamma s_{\mu}\dot{s}^{\mu}}{\sqrt{m^2 - \mathcal{M}^2}\sqrt{\gamma^2 s^2 - (s_{\sigma}u^{\sigma})^2}} \tag{30}$$

[Eq. (14) has been used]. It is clear that the cosmological constant does not occur in the $e^{(1)}_{\mu}\dot{p}^{\mu}$ component, i.e., in the projection on $(\gamma s^{\mu} + s_{\nu}u^{\nu}V^{\mu})$. Since the latter plays the role of spin in (26), this implies the same property we observed on V^{μ}-tetrad decomposition in paper I: Λ only influences motion in directions perpendicular to the spin.

When projecting $\dot{p}^{\mu}$ to the "parenthesis" tetrad, one also notices that due to the orthogonality $u_{\mu}\dot{p}^{\mu} = 0$ the "second" component yields just

$$e^{(2)}_{\mu}\dot{p}^{\mu} = \frac{p_{\mu}\dot{p}^{\mu}}{\sqrt{m^2 - \mathcal{M}^2}} = \frac{-\mathcal{M}\dot{\mathcal{M}}}{\sqrt{m^2 - \mathcal{M}^2}},$$

where the mass $\mathcal{M}$ is given by $\mathcal{M}^2 := -p_{\mu}p^{\mu} (> 0)$.

Let us also add some obvious identities useful when transforming between the "hatted" and the parenthesized tetrads:

$$\gamma \equiv -u_{\mu}V^{\mu} \equiv u^{\hat{0}} \equiv V^{(0)},$$
$$\mu \equiv -p_{\mu}V^{\mu} \equiv p^{\hat{0}},$$
$$m \equiv -u_{\mu}p^{\mu} \equiv p^{(0)},$$
$$s_{\mu}u^{\mu} \equiv s u^{\hat{1}} \equiv -s^{(0)}.$$

B. Decomposition in a curvature-adjusted tetrad. Which one?

Employing Appendix A of paper I, where orthonormal components of the Weyl tensor (and consequently those of its dual) are expressed in terms of the Ψ_0–Ψ_4 scalars, it is now easy to write down the decomposition of the MPD equation of motion (8):

$$\frac{1}{\sigma}e^{(1)}_{\mu}\dot{p}^{\mu} = -2\,\mathrm{Im}\Psi_2\,V^{(0)} - (\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)V^{(2)} - (\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)V^{(3)}, \tag{31}$$

$$\frac{1}{\sigma}e^{(2)}_{\mu}\dot{p}^{\mu} \equiv \frac{-\mathcal{M}\dot{\mathcal{M}}}{\sigma\sqrt{m^2 - \mathcal{M}^2}} = -(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1)V^{(0)} + \frac{1}{2}(\mathrm{Im}\Psi_0 - \mathrm{Im}\Psi_4)V^{(2)} + \left[\mathrm{Re}\Psi_2 - \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4) + \frac{\Lambda}{3}\right]V^{(3)}, \tag{32}$$

$$\frac{1}{\sigma}e_\mu^{(3)}\dot{p}^\mu = -(\mathrm{Re}\Psi_3+\mathrm{Re}\Psi_1)V^{(0)} - \left[\mathrm{Re}\Psi_2+\frac{1}{2}(\mathrm{Re}\Psi_0+\mathrm{Re}\Psi_4)+\frac{\Lambda}{3}\right]V^{(2)} - \frac{1}{2}(\mathrm{Im}\Psi_0-\mathrm{Im}\Psi_4)V^{(3)}, \tag{33}$$

where we abbreviated $\sigma := \frac{\sqrt{\gamma^2 s^2-(s_\sigma u^\sigma)^2}}{\gamma}$. Apparently the result is similar to the decomposition with respect to the V^μ-based tetrad, given in Eqs. (3)–(6), with one important difference: the components obtained in paper I do not contain Ψ_0 and Ψ_4, whereas now these scalars *are* present. On the other hand, the present approach has one big advantage: at any point, the reference observer V^μ can be chosen arbitrarily (in contrast to u^μ), so one can in fact eliminate much of the above formulas.

Let us remind the reader that the complex Ψ scalars featuring in Eqs. (31)–(33) represent projections of the Weyl tensor onto the NP tetrad $(k^\mu, l^\mu, m^\mu, \bar{m}^\mu)$ naturally associated with its orthonormal counterpart (22)–(25), namely, connected with the latter by

$$k^\mu := \frac{1}{\sqrt{2}}(u^\mu + e^\mu_{(1)}), \qquad l^\mu := \frac{1}{\sqrt{2}}(u^\mu - e^\mu_{(1)}),$$
$$m^\mu := \frac{1}{\sqrt{2}}(e^\mu_{(2)} + \mathrm{i}e^\mu_{(3)}), \qquad \bar{m}^\mu := \frac{1}{\sqrt{2}}(e^\mu_{(2)} - \mathrm{i}e^\mu_{(3)}).$$

One might also express the projections of the MPD equation onto the (22)–(25) tetrad in terms of Weyl scalars obtained in some *different* NP tetrad, not associated with the given orthonormal tetrad, but then Eqs. (31)–(33) would look differently.

Consider now shortly our plan, i.e., tuning the tetrad to a given space-time curvature, similarly as in paper I. It will certainly be advantageous to identify the first vector k^μ of the NP tetrad with the Weyl-tensor PND of the highest multiplicity again. Should now the plane determined by u^μ and k^μ be made an eigenplane of the spin bivector $S^{\mu\nu}$, one would have to resort to only one viable spin condition, with $V^\mu \equiv u^\mu$. This would, however, mean returning to paper I, Sec. V.A, on MPD equations supplemented by the Mathisson-Pirani condition. Actually, setting $V^\mu = u^\mu$, one has $s_\sigma u^\sigma = 0$, $\sigma = s$, $V^{(0)} = 1$, and $V^{(i)} = 0$, reducing Eqs. (31)–(33) to

$$e_\mu^{(1)}\dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2,$$
$$e_\mu^{(2)}\dot{p}^\mu = -s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1),$$
$$e_\mu^{(3)}\dot{p}^\mu = -s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1),$$

which are just Eqs. (97)–(99) of paper I.

If one insisted on the tight connection between the tetrad and the curvature structure, and at once on a sufficiently generic view (not pushing one into $V^\mu = u^\mu$), there is an alternative—with u^μ used as the time vector of the orthonormal tetrad in which the MPD equations are decomposed, yet with the reference observer V^μ left free (for later adaptation of the NP tetrad to a given algebraic type). If adopting such a compromise, it is necessary to release the tight (natural) connection between the NP tetrad and the orthonormal one. Specifically, one could consider instead the NP tetrad naturally associated with the *same* orthonormal tetrad as in paper I, i.e., with (18)–(21). Expressing such an alternative in other words, one could keep the NP tetrad (thus the Weyl scalars) from paper I but decompose the MPD equations in the orthonormal tetrad (22)–(25) instead of (18)–(21). Rather than deriving such "hybrid" relations by transformation of the Weyl scalars, it is simpler to start from Eqs. (3)–(6) and compose their new components according to the transformation of the orthonormal basis. One finds easily that

$$e_\mu^{(1)}\dot{p}^\mu = \frac{u^{\hat{0}}e^{\hat{1}}_\mu + u^{\hat{1}}V_\mu}{\sqrt{(u^{\hat{0}})^2-(u^{\hat{1}})^2}}\dot{p}^\mu, \tag{34}$$

$$e_\mu^{(2)}\dot{p}^\mu = -\sqrt{1-\frac{(\gamma m-\mu)^2}{\gamma^2(m^2-\mathcal{M}^2)}}\,e^{\hat{2}}_\mu\dot{p}^\mu. \tag{35}$$

To also find the hatted decomposition of $e^\mu_{(3)}$, we recall $e^{(3)}_\mu V^\mu$ given in (30) and calculate the remaining components,

$$e^{(3)}_\alpha e^\alpha_{\hat{1}} = -\frac{u^{\hat{1}}e^{\hat{1}}_\alpha \dot{s}^\alpha}{\sqrt{m^2-\mathcal{M}^2}\sqrt{(u^{\hat{0}})^2-(u^{\hat{1}})^2}},$$
$$e^{(3)}_\alpha e^\alpha_{\hat{2}} = 0,$$
$$e^{(3)}_\alpha e^\alpha_{\hat{3}} = -\frac{1}{\sqrt{(u^{\hat{0}})^2-(u^{\hat{1}})^2}\sqrt{1-\frac{(\gamma m-\mu)^2}{\gamma^2(m^2-\mathcal{M}^2)}}},$$

which can then be inserted into

$$e^{(3)}_\mu\dot{p}^\mu = (e^{(3)}_\alpha e^\alpha_{\hat{0}})e^{\hat{0}}_\mu\dot{p}^\mu + (e^{(3)}_\alpha e^\alpha_{\hat{1}})e^{\hat{1}}_\mu\dot{p}^\mu + (e^{(3)}_\alpha e^\alpha_{\hat{3}})e^{\hat{3}}_\mu\dot{p}^\mu.$$

Since the decomposition (3)–(6) from paper I is expressed in terms of the hatted four-velocity components, it is useful to add, as a counterpart of (28)–(30), that

$$u^{\hat{0}} \equiv V^{(0)} \equiv \gamma, \tag{36}$$

$$u^{\hat{1}} = -s^{(0)}/s, \tag{37}$$

$$u^{\hat{2}} = \frac{\gamma m-\mu}{\sqrt{\gamma^2(m^2-\mathcal{M}^2)-(\gamma m-\mu)^2}}, \tag{38}$$

$$u^{\hat{3}} = \frac{\gamma s_\mu \dot{s}^\mu}{s\sqrt{\gamma^2(m^2-\mathcal{M}^2)-(\gamma m-\mu)^2}}. \tag{39}$$

The last two components are proportional to $V^{(2)}$ and $V^{(3)}$; see (29) and (30), respectively.

C. Algebraically special space-times: Type-N example

It is only meaningful to discuss the particular curvature types if one accepts the above compromise view, i.e., decomposes the MPD equations into the u^μ-based orthonormal tetrad, but keeps the NP tetrad (in which Ψ scalars are computed) unrelated, and thus free for adaptation to the curvature structure as in paper I. We saw above that one pays for this freedom by longer expressions for the MPD-equation projections. On the other hand, these equations "inherit" from those obtained in paper I the lack of the Ψ_0 and Ψ_4 scalars.

For the most special Petrov type N, by using Eqs. (82)–(83) of paper I, i.e.,

$$-V_\mu\dot{p}^\mu = 0,\quad e^{\hat{1}}_\mu\dot{p}^\mu = 0,\quad e^{\hat{2}}_\mu\dot{p}^\mu = -\frac{\Lambda}{3}su^{\hat{3}},\quad e^{\hat{3}}_\mu\dot{p}^\mu = \frac{\Lambda}{3}su^{\hat{2}},$$

and (39) from above, we obtain

$$e^{(1)}_\mu\dot{p}^\mu = 0, \tag{40}$$

$$e^{(2)}_\mu\dot{p}^\mu = \frac{\Lambda}{3}su^{\hat{3}}\sqrt{1-\frac{(\gamma m-\mu)^2}{\gamma^2(m^2-\mathcal{M}^2)}} = \frac{\Lambda}{3}\frac{s_\mu\dot{s}^\mu}{\sqrt{m^2-\mathcal{M}^2}}, \tag{41}$$

$$\begin{aligned} e^{(3)}_\mu\dot{p}^\mu &= \frac{-\frac{\Lambda}{3}su^{\hat{2}}}{\sqrt{(u^{\hat{0}})^2-(u^{\hat{1}})^2}\sqrt{1-\frac{(\gamma m-\mu)^2}{\gamma^2(m^2-\mathcal{M}^2)}}} \\ &= \frac{-\frac{\Lambda}{3}s^2\gamma(\gamma m-\mu)\sqrt{m^2-\mathcal{M}^2}}{\sqrt{\gamma^2s^2-(s^{(0)})^2}\,[\gamma^2(m^2-\mathcal{M}^2)-(\gamma m-\mu)^2]}. \end{aligned} \tag{42}$$

In the intrinsic tetrad, we found, in Eq. (84) of paper I, that $\mathcal{M}\dot{\mathcal{M}} = -(\Lambda/3)s_\mu\dot{s}^\mu$, so we can also write the second equation as

$$e^{(2)}_\mu\dot{p}^\mu = \frac{-\mathcal{M}\dot{\mathcal{M}}}{\sqrt{m^2-\mathcal{M}^2}}. \tag{43}$$

The decomposition forms following for other Petrov types can also be obtained straightforwardly, and we will not discuss them.

III. SPECIFIC SPIN CONDITIONS

Let us briefly consider how the exercise looks when supplemented by the main spin conditions. We will, however, not include the Mathisson-Pirani spin condition, $V^\mu \equiv u^\mu$, anymore, because this simply reduces the problem to the form already treated in Sec. V.A of paper I.

A. Tulczyjew spin condition, $V^\mu \equiv p^\mu/\mathcal{M}$

We know from paper I (Sec. V.B) that the Tulczyjew condition implies $\gamma = m/\mathcal{M}$, $\mu = \mathcal{M}$, $s_\mu p^\mu = 0 = s_\mu u^\mu$, $\dot{\mathcal{M}} = 0$, $\dot{s} = 0$, and $\sigma = s$, so we have

$$V^{(0)} \equiv \gamma = \frac{m}{\mathcal{M}},\qquad V^{(2)} = \frac{\sqrt{m^2-\mathcal{M}^2}}{\mathcal{M}},\qquad V^{(3)} = 0, \tag{44}$$

$$e^{(1)}_\mu\dot{p}^\mu = \frac{s_\mu\dot{p}^\mu}{s} = e^{\hat{1}}_\mu\dot{p}^\mu, \tag{45}$$

$$e^{(2)}_\mu\dot{p}^\mu = 0 = e^{\hat{2}}_\mu\dot{p}^\mu, \tag{46}$$

$$\begin{aligned} e^{(3)}_\mu\dot{p}^\mu &= \frac{\epsilon_{\mu\iota\kappa\lambda}\dot{p}^\mu p^\iota s^\kappa u^\lambda}{s\sqrt{m^2-\mathcal{M}^2}} = \frac{\mathcal{M}S_{\mu\lambda}\dot{p}^\mu u^\lambda}{s\sqrt{m^2-\mathcal{M}^2}} \\ &= -\frac{\mathcal{M}\dot{S}_{\mu\lambda}p^\mu u^\lambda}{s\sqrt{m^2-\mathcal{M}^2}} = \frac{\mathcal{M}}{s}\sqrt{m^2-\mathcal{M}^2} \\ &= e^{\hat{3}}_\mu\dot{p}^\mu, \end{aligned} \tag{47}$$

which reduces Eqs. (31)–(33) to

$$\begin{aligned} \frac{\mathcal{M}}{s^2}s_\mu\dot{p}^\mu &= -2m\,\mathrm{Im}\Psi_2 \\ &\quad - \sqrt{m^2-\mathcal{M}^2}\,(\mathrm{Im}\Psi_3+\mathrm{Im}\Psi_1), \end{aligned} \tag{48}$$

$$\begin{aligned} 0 &= -2m(\mathrm{Im}\Psi_3-\mathrm{Im}\Psi_1) \\ &\quad + \sqrt{m^2-\mathcal{M}^2}\,(\mathrm{Im}\Psi_0-\mathrm{Im}\Psi_4), \end{aligned} \tag{49}$$

$$\begin{aligned} \frac{\mathcal{M}^2}{s^2} &= -\frac{m(\mathrm{Re}\Psi_3+\mathrm{Re}\Psi_1)}{\sqrt{m^2-\mathcal{M}^2}} \\ &\quad - \mathrm{Re}\Psi_2 - \frac{1}{2}(\mathrm{Re}\Psi_0+\mathrm{Re}\Psi_4) - \frac{\Lambda}{3}. \end{aligned} \tag{50}$$

So the projections of $\dot{p}^\mu$ into the u^μ-based tetrad (parenthesized) equal those into the V^μ-based tetrad (hatted), and they appear somewhat simpler when written in terms of the Ψ scalars computed in the null tetrad associated with the V^μ-based orthonormal tetrad; this was done in paper I, Eqs. (121)–(124):

$$\begin{aligned} 0 &= s(\mathrm{Im}\Psi_1-\mathrm{Im}\Psi_3), \\ e^{\hat{1}}_\mu\dot{p}^\mu &= -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2 - s(\mathrm{Im}\Psi_1+\mathrm{Im}\Psi_3)u^{\hat{2}}, \\ 0 &= s(\mathrm{Im}\Psi_1-\mathrm{Im}\Psi_3), \\ \mathcal{M}^2 &= s^2\left(\frac{\Lambda}{3}-2\mathrm{Re}\Psi_2\right) - \frac{ms^2(\mathrm{Re}\Psi_1+\mathrm{Re}\Psi_3)}{\sqrt{m^2-\mathcal{M}^2}}. \end{aligned}$$

The reason for the difference is that when the above expression is written in terms of the Weyl scalars computed in the null tetrad associated with the u^μ-based orthonormal tetrad, it contains, in addition to Ψ_1, Ψ_2, and Ψ_3, also Ψ_0 and Ψ_4.

B. Condition $u^\mu \| p^\mu$: An alternative tetrad

If $u^\mu \| p^\mu$, then $p^\mu = m u^\mu$, $m = \mathcal{M}$, $\dot{m} = \dot{\mathcal{M}} = 0$, $\dot{p}^\mu = m\dot{u}^\mu$, $\dot{S}^{\mu\nu} = 0$, ${}^*\dot{S}^{\mu\nu} = 0$, $\dot{s} = 0$, and $\mu = \gamma m$. The intrinsic tetrad tied to u^μ, Eqs. (22)–(25), cannot be used, because its last two vectors degenerate (the hidden momentum vanishes).[2] One can, however, find a different orthonormal tetrad, usable even when $p^\mu = m u^\mu$—for example, one can choose, besides (22) and (23), a vector orthogonal to u^μ, V^μ as well as s^μ, i.e.,

$$\epsilon^{\mu\nu\kappa\lambda} u_\nu V_\kappa s_\lambda = S^{\mu\nu} u_\nu,$$

and add the last one orthogonal to all u^μ, $e^\mu_{(1)}$, and $S^{\mu\nu}u_\nu$ (we will number the two vectors in a reverse order):

$$\begin{aligned} e^\mu_{(2)} &:= \frac{\epsilon^{\mu\alpha\beta\gamma} u_\alpha (\gamma s_\beta + s_\rho u^\rho V_\beta) \epsilon_{\gamma\nu\kappa\lambda} u^\nu V^\kappa s^\lambda}{\sqrt{\gamma^2 s^2 - (s_\sigma u^\sigma)^2}\sqrt{(\gamma^2-1)s^2 - (s_\sigma u^\sigma)^2}} \\ &= \frac{(\delta^\mu_\nu + u^\mu u_\nu)(\gamma s^2 V^\nu + s_\rho u^\rho s^\nu)}{\sqrt{\gamma^2 s^2 - (s_\sigma u^\sigma)^2}\sqrt{(\gamma^2-1)s^2 - (s_\sigma u^\sigma)^2}}, \end{aligned} \tag{51}$$

$$e^\mu_{(3)} := \frac{S^{\mu\nu} u_\nu}{\sqrt{S^{\alpha\rho} u_\rho S_{\alpha\sigma} u^\sigma}} = \frac{\epsilon^{\mu\nu\kappa\lambda} u_\nu V_\kappa s_\lambda}{\sqrt{(\gamma^2-1)s^2 - (s_\sigma u^\sigma)^2}}. \tag{52}$$

Obviously, these vectors are not defined if the Mathisson-Pirani spin condition $S^{\mu\nu}u_\nu = 0$ is applied.

The new basis vectors (51) and (52) provide—independently of the spin condition—projections

$$V^{(2)} = \gamma \frac{\sqrt{(\gamma^2-1)s^2 - (s_\rho u^\rho)^2}}{\sqrt{\gamma^2 s^2 - (s_\sigma u^\sigma)^2}}, \qquad V^{(3)} = 0, \tag{53}$$

so Eqs. (31)–(33) assume a similar form as (48)–(50), just with $e^{(2)}_\mu \dot{p}^\mu$ no longer equal to $\frac{-\mathcal{M}\dot{\mathcal{M}}}{\sqrt{m^2 - \mathcal{M}^2}}$ and with slightly more complicated σ, $V^{(2)}$, $e^{(1)}_\mu \dot{p}^\mu$, and $e^{(3)}_\mu \dot{p}^\mu$. Note that, if employed together with the *Tulczyjew* condition, this alternative tetrad yields *exactly the same* projections of V^μ and $\dot{p}^\mu$ as the original tetrad, that is (44)–(47).

In Sec. V.C.1 of paper I, we showed that the freedom which the condition $u^\mu \| p^\mu$ leaves to the choice of V^μ can be used to select the latter in such a manner that the corresponding spin s^μ is orthogonal to u^μ (thus also to p^μ) and remains so along the whole trajectory. Specifically, this requires selecting $V^\mu = u^\mu$ at some initial point and then prescribing evolutions

$$\dot{V}^\mu = \frac{\alpha}{\mu m^2} s^\mu, \qquad \dot{s}^\mu = \frac{\alpha s^2}{\mu m^2} V^\mu, \tag{54}$$

with α given by

$$\alpha = \frac{\mathcal{M}^2}{s^2} \dot{p}^\mu s_\mu = \frac{\mathcal{M}^2}{s} e^{\hat{1}}_\mu \dot{p}^\mu.$$

Ensuring the above setting, one gets, at a generic point, the "alternative" tetrad[3]

$$\begin{aligned} e^\mu_{(0)} &= u^\mu, & e^\mu_{(1)} &= \frac{s^\mu}{s}, \\ e^\mu_{(2)} &= \frac{V^\mu - \gamma u^\mu}{\sqrt{\gamma^2-1}}, & e^\mu_{(3)} &= \frac{S^{\mu\nu} u_\nu}{s\sqrt{\gamma^2-1}} \end{aligned} \tag{55}$$

and hence projections

$$V^{(0)} = \gamma, \quad V^{(1)} = 0, \quad V^{(2)} = \sqrt{\gamma^2-1}, \quad V^{(3)} = 0. \tag{56}$$

Consequently, Eqs. (31)–(33) reduce to

$$\frac{\alpha}{m^2} = -2\gamma\,\mathrm{Im}\Psi_2 - \sqrt{\gamma^2-1}\,(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1), \tag{57}$$

$$\frac{V_\mu \dot{p}^\mu}{s(\gamma^2-1)} = \frac{\gamma(\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3)}{\sqrt{\gamma^2-1}} + \frac{1}{2}(\mathrm{Im}\Psi_0 - \mathrm{Im}\Psi_4), \tag{58}$$

$$\begin{aligned} \frac{S_{\mu\nu} u^\nu \dot{p}^\mu}{s^2(\gamma^2-1)} &= -\frac{\gamma(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1)}{\sqrt{\gamma^2-1}} \\ &\quad - \mathrm{Re}\Psi_2 - \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4) - \frac{\Lambda}{3}. \end{aligned} \tag{59}$$

This form is slightly more complicated than the Tulczyjew-condition counterpart (48)–(50). Note that one cannot obtain the latter, or any other more special form, by resorting to $V^\mu \sim p^\mu$ and so, because by prescribing the initial value (u^μ) and evolution of V^μ, the reference observer was fixed and cannot be adjusted any more (it cannot be set proportional to p^μ or u^μ, in particular).

1. Remark: Alternative to the intrinsic tetrad of paper I

If $p^\mu = m u^\mu$, the tetrad (18)–(21) employed in paper I is clearly meaningless as well. Let us suggest its substitute even usable when $p^\mu = m u^\mu$, thus supplementing paper I where we did not go into this detail. One of the vectors orthogonal to both V^μ and s^μ can obviously be chosen like above in the u^μ-based tetrad case, namely, according to (52), and the last vector can be found in analogy with (51), i.e., as the one orthogonal to all V^μ, s^μ, and (52):

[2]The tetrad (18)–(21) suggested in paper I degenerates then in the same manner.

[3]Let us stress that $s_\mu u^\mu = 0$ does not in general mean $S_{\mu\nu}u^\mu = 0$ (i.e., the Mathisson-Pirani condition); the spin bivector still has V^μ and s^μ as its eigenvectors, while u^ν need not belong to the eigenplane.

$$e^{\mu}_{(3)} := \frac{-S^{\mu\nu}u_\nu}{\sqrt{S^{\alpha\rho}u_\rho S_{\alpha\sigma}u^\sigma}} = \frac{\epsilon^{\mu\nu\kappa\lambda}V_\nu u_\kappa s_\lambda}{\sqrt{(\gamma^2-1)s^2-(s_\sigma u^\sigma)^2}}, \quad (60)$$

$$e^{\mu}_{(2)} := \frac{\epsilon^{\mu\alpha\beta\gamma}V_\alpha s_\beta \epsilon_{\gamma\nu\kappa\lambda}V^\nu u^\kappa s^\lambda}{s\sqrt{(\gamma^2-1)s^2-(s_\sigma u^\sigma)^2}} = \frac{(\delta^\mu_\nu + V^\mu V_\nu - e^\mu_{\hat 1} e^{\hat 1}_\nu)u^\nu}{\sqrt{(\gamma^2-1)s^2-(s_\sigma u^\sigma)^2}}, \quad (61)$$

where we remind the reader that $e^{\mu}_{\hat 1} \equiv s^\mu/s$. Therefore, the $e^{\mu}_{(2)}$ vector is represented by the component of u^μ orthogonal to both V^μ and s^μ. Again, this tetrad is not available if the Mathisson-Pirani condition holds, $S^{\mu\nu}u_\nu = 0$.

IV. MASSLESS PARTICLES

In paper I as well as so far here, we have been considering particles with nonzero rest mass. Let us now reserve some space to localized *massless* particles. It was shown by Refs. [3,4] that the "massless" situation, represented by a traceless energy-momentum tensor, implies[4]

$$m := -p_\mu k^\mu = 0 \, (= \text{const along } k^\mu); \quad (62)$$

the same was also obtained by Ref. [5] from conformal invariance of the action functional. Other results were that, if $S_{\mu\nu}$ is spacelike, $S_{\mu\nu}S^{\mu\nu} =: 2s^2 > 0$,[5] then

$$S_{\mu\nu}k^\nu = 0, \qquad k_\mu k^\mu = 0, \qquad \dot k^\mu \sim k^\mu, \quad (63)$$

so—as already suggested by Ref. [6]—the Mathisson-Pirani condition automatically holds, and the particle follows a null geodesic.

The MPD equations themselves (1), (2) remain the same,

$$\dot p^\mu = -\frac{1}{2}R^\mu{}_{\nu\kappa\lambda}k^\nu S^{\kappa\lambda}, \qquad \dot S^{\alpha\beta} = p^\alpha k^\beta - k^\alpha p^\beta, \quad (64)$$

yet one can only rarely take over results from the massive case (paper I) simply by putting $m = 0$; namely, the assumption $u_\mu u^\mu = -1$ (and $V_\mu V^\mu = -1$) was used there frequently, whereas now $V^\mu \to u^\mu \to k^\mu$ turns out to be lightlike. From the second MPD equation, one sees immediately that the scalar s called *helicity* is constant along k^μ,

$$2s\dot s = S_{\mu\nu}\dot S^{\mu\nu} = 0,$$

and that $\dot S^{\alpha\beta}$ is null since $\dot S^{\alpha\beta}\dot S_{\alpha\beta} = 0$, with k^μ being a common eigenvector of $S^{\alpha\beta}$ and $\dot S^{\alpha\beta}$.

[4] In the massless case, let us use k^μ instead of u^μ for the tangent of the representative worldline k^μ, while keeping the dot for covariant differentiation along that worldline, i.e., $\dot X := X_{;\iota}k^\iota$.

[5] Otherwise it could hardly be understood as describing the rotational angular momentum.

Let us stop at k^μ for a while: here it represents the worldline tangent, while in paper I we denoted by k^μ the first vector of the NP interpretation tetrad. However, the tetrad was chosen so that k^μ (as well as its second vector l^μ) lay in the eigenplane of $S_{\mu\nu}$, which is just consistent with the present notation since the spin condition $S^{\mu\nu}k_\nu = 0$ now necessarily holds, so k^μ is naturally taken as the main vector of the interpretation tetrad.

Multiplication of the second of the MPD equations (64) by p_β and by $\dot p_\beta$ yields

$$\mathcal{M}^2 k^\alpha = -\dot S^{\alpha\beta}p_\beta, \quad (65)$$

$$\mathcal{M}\dot{\mathcal{M}}k^\alpha = (p_\mu \dot p^\mu)k^\alpha = -\dot S^{\alpha\beta}\dot p_\beta, \quad (66)$$

from which one sees that

$$\dot S^{\alpha\beta}\dot p_\beta = \ddot S^{\alpha\beta}p_\beta, \qquad \dot{\mathcal{M}}\dot S^{\alpha\beta}p_\beta = \mathcal{M}\dot S^{\alpha\beta}\dot p_\beta.$$

Above, we have introduced $p_\mu p^\mu =: \mathcal{M}^2$ as in the massive case, but with a different (*plus*) sign—we will see below that p^μ is spacelike now.

Another difference from the massive case is that the spin vector defined analogously as there, by projection of the spin-bivector dual onto $V^\mu \to k^\mu$, is also null ($s_\mu s^\mu = 0$) and proportional to k^μ,

$$s^\mu := -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}k_\nu S_{\rho\sigma} = -{}^*S^{\mu\nu}k_\nu = sk^\mu. \quad (67)$$

The null character of s^μ is seen immediately: $s_\mu s^\mu$ only contains terms involving $S_{\rho\sigma}k^\sigma = 0$ or $k_\nu k^\nu = 0$. The second claim, $s^\mu = sk^\mu$, follows from the fact that two real null vectors are orthogonal if and only if they are proportional to each other. The above result also implies that s^μ parallel transports along k^μ; specifically, if k^μ is affinely parametrized ($\dot k^\mu = 0$), then

$$\dot s^\mu = \dot s k^\mu + s\dot k^\mu = 0. \quad (68)$$

Once one knows that the particle moves on a geodesic and that its spin is proportional to the latter's tangent k^μ, one might have little reason to continue the study, because the momentum p^μ is a "strange thing" (spacelike) anyway, so there is actually no demand to interpret its evolution $\dot p^\mu$. However, we show below that even in the massless case there naturally follows a (timelike) "reference observer" and an associated (spacelike) spin vector (whether the former is taken as primary or the latter), i.e., quantities which have the same meaning as in the massive-particle case and which are worth further consideration. We first realize that the null version of the Mathisson-Pirani condition leaves more freedom to the spin bivector than the timelike version and then fix the remaining freedom by determining the remaining independent dimension of the spin-bivector eigenplane. In doing so, we naturally introduce the reference observer V^μ and the corresponding spin S^μ and also note that one can in fact take advantage of this freedom and adjust the spin eigenplane so as to contain a desired direction (independent of k^μ), in particular, the main PND of the host space-time.

A. Null spin condition: $S^{\mu\nu}k_\nu = 0$, $k_\mu k^\mu = 0$

As stressed by Ref. [7] in their treatment of massless spinning particles, the null version of the spin condition is less restricting than the "full" timelike case. Generally speaking, the vanishing of the projection of an object onto a *null* direction k^μ does not exclude that the object has a component proportional to k^μ. In the case of our bivector $S^{\mu\nu}$, the "timelike" condition $S^{\mu\nu}V_\nu = 0$, considered in paper I, strictly determined its eigenplane and blade; in particular, it implied that the bivector must read

$$S_{\mu\nu} = \epsilon_{\mu\nu\kappa\lambda}V^\kappa s^\lambda = -s\,\epsilon_{\mu\nu\kappa\lambda}k^\kappa l^\lambda = \mathrm{i}s\, m_\mu \wedge \bar{m}_\nu,$$

where the real null vectors k^μ and l^μ were related to V^μ and s^μ by

$$k^\mu = \frac{1}{\sqrt{2}}\left(V^\mu + \frac{s^\mu}{s}\right), \qquad l^\mu = \frac{1}{\sqrt{2}}\left(V^\mu - \frac{s^\mu}{s}\right),$$

and m^μ and $\bar{m}^\mu$ are complex null vectors (mutual complex conjugates) orthogonal to both k^μ and l^μ and normalized to $m_\mu \bar{m}^\mu = 1$. In contrast, the condition $S^{\mu\nu}k_\nu = 0$ admits a more general form,

$$S_{\mu\nu} = \mathrm{i}s\, m_\mu \wedge \bar{m}_\nu + k_\mu \wedge (L m_\nu + \bar{L}\bar{m}_\nu), \tag{69}$$

where m^μ and $\bar{m}^\mu$ are some complex null vectors orthogonal to k^μ and normalized to $m_\mu \bar{m}^\mu = 1$ and L denotes an (arbitrary) magnitude of "the other" independent spin component. Speaking more generally, the spin vector (67) follows uniquely from a *known* bivector, but the converse is not true; the bivector is not fully determined by the spin vector.

However, a simple non-null bivector has the whole *plane* of eigendirections (with zero eigenvalue), so there exists (or one can choose) a second null direction l^μ, independent of k^μ, which is also "annihilated," $S^{\mu\nu}l_\nu = 0$. Provided it is normalized so that $k_\mu l^\mu = -1$, the conditions $S^{\mu\nu}k_\nu = 0$ and $S^{\mu\nu}l_\nu = 0$ require $L = -\mathrm{i}s\bar{m}_\nu l^\nu$ (ergo $\bar{L} = \mathrm{i}s m_\nu l^\nu$). For the eigendirections k^μ and l^μ known/chosen, the bivector is already determined uniquely (and it is possible to choose m^μ and $\bar{m}^\mu$ perpendicular to both, making $L = 0$). Clearly, if there is some privileged null direction in space-time (call it l^μ), one can take advantage of the freedom still remaining in the spin bivector subjected to the null condition $S^{\mu\nu}k_\nu = 0$ and require that it also satisfy $S^{\mu\nu}l_\nu = 0$, thus inclining the bivector's eigenplane in the desired way.

B. Spin-bivector eigenplane

Having introduced l^μ as the second independent eigenvector of the spin bivector, we can multiply by l_β the second equation of (64) to get

$$p^\alpha = -p^\beta l_\beta k^\alpha + p_\perp^\alpha \tag{70}$$

as a counterpart of equation $\gamma p^\alpha = \mu u^\alpha + S^{\alpha\beta}\dot{V}_\beta$ which was numbered (21) in paper I. We have introduced

$$S^{\alpha\beta}\dot{l}_\beta = -\dot{S}^{\alpha\beta}l_\beta = p^\alpha + k^\alpha l_\beta p^\beta = (\delta^\alpha_\beta + k^\alpha l_\beta + l^\alpha k_\beta)p^\beta =: p_\perp^\alpha \tag{71}$$

as the part of p^μ orthogonal to the plane (k^μ, l^μ); it is a counterpart of the hidden momentum

$$p^\alpha_{\mathrm{hidden}} := (\delta^\alpha_\beta + u^\alpha u_\beta)p^\beta = p^\alpha - m u^\alpha = -\dot{S}^{\alpha\beta}u_\beta$$

from the massive case.

As already suggested above, we will use in the next section the NP tetrad based on independent real null vectors k^μ and l^μ which are both annihilated by $S_{\mu\nu}$ and which are normalized as $k_\mu l^\mu = -1$. Being null, l^μ certainly satisfies $\dot{l}^\mu l_\mu = 0$, and, if the particle's geodesic is affinely parametrized ($\dot{k}^\mu = 0$), $\dot{l}^\mu k_\mu = 0$ as well, but l^μ need not be parallel along k^μ (i.e., $\dot{l}^\mu \neq 0$ in general). Actually, with helicity s known, one can "reconstruct" the spin bivector (and its dual) by

$$S_{\alpha\beta} = -s\,\epsilon_{\alpha\beta\gamma\delta}k^\gamma l^\delta, \qquad {}^*S^{\mu\nu} = s(k^\mu l^\nu - l^\mu k^\nu). \tag{72}$$

Multiplying the derivative

$$\dot{S}_{\alpha\beta} = -s\,\epsilon_{\alpha\beta\gamma\delta}k^\gamma \dot{l}^\delta \tag{73}$$

by $\epsilon^{\mu\nu\alpha\beta}l_\nu$, we have

$$s^2\dot{l}^\mu = -S^{\mu\alpha}p_\alpha \Rightarrow s^2\dot{l}^\mu\dot{l}_\mu = \dot{S}^{\mu\alpha}l_\mu p_\alpha = \mathcal{M}^2. \tag{74}$$

We have again used $p^\alpha p_\alpha =: \mathcal{M}^2$, so with the sign different from the massive case. Namely, $\dot{l}^\mu$ is clearly orthogonal to both k^μ and l^μ which span the eigenplane of $S^{\mu\nu}$, and this eigenplane is timelike by assumption, so $\dot{l}^\mu$ has to be spacelike; hence, $\mathcal{M}^2 > 0$. Besides, $\dot{l}^\mu$ is also seen to be orthogonal to p^μ; the reason cannot (in general) be that p^μ also belongs to the eigenplane of $S^{\mu\nu}$, because this would mean $\dot{l}^\mu = 0$, so $\dot{S}^{\mu\nu} = 0$ and, consequently, $p^\mu \| k^\mu$, which is not in general consistent with the MPD equation for $\dot{p}^\mu$ (cf. Ref. [4], Sec. V, and Sec. IV G 1 below). Therefore, in the generic situation the vectors k^μ, l^μ, and p^μ are independent.

Note that one learns from (72) that k^μ is also annihilated by

$$^*\dot{S}^{\mu\nu} = s(k^\mu \dot{l}^\nu - \dot{l}^\mu k^\nu), \tag{75}$$

so it is *the* common null eigenvector of $\dot{S}^{\mu\nu}$ and $^*\dot{S}^{\mu\nu}$. This confirms that $\dot{S}^{\mu\nu}$ is null and thus $^*\dot{S}^{\mu\nu}\dot{S}_{\alpha\nu} = 0$ like in the massive case, similarly as $^*S^{\mu\nu}S_{\alpha\nu} = 0$.

C. Summary of eigenvectors of the spin bivectors

It is very easy now to summarize the independent eigenvectors of all the bivectors involved. The eigenplane of $S^{\mu\nu}$ is timelike, and it is spanned by k^μ and l^μ, while the eigenplane of $\dot{S}^{\mu\nu}$ is null and spanned by k^μ and $\dot{l}^\mu$ (the two eigenplanes intersect "along" k^μ). The eigenvectors of $^*\dot{S}^{\mu\nu}$ are k^μ and p^μ, their plane being null (because $^*\dot{S}^{\mu\nu}$ is null, as inherited from $\dot{S}^{\mu\nu}$). The last bivector, $^*S^{\mu\nu}$, is the only one which does not annihilate k^μ, but clearly this is true for $\dot{l}^\mu$, while its second eigenvector can be found among projections of $S^{\mu\nu}$; in particular, $S^{\mu\nu}\dot{l}_\nu \equiv p^\mu_\perp$ is certainly independent of $\dot{l}^\mu$. Both $\dot{l}^\mu$ and $p^\mu_\perp$ are spacelike as well as the eigenplane spanned by them (this is confirmed by the timelike character of the dual spin bivector, $^*S^{\mu\nu\,*}S_{\mu\nu} = -2s^2$).

Therefore, the massless case differs from the massive one in the null character of $\dot{S}^{\mu\nu}$ and $^*\dot{S}^{\mu\nu}$ (for a massive particle, $\dot{S}^{\mu\nu}$ is timelike, and $^*\dot{S}^{\mu\nu}$ is (thus) spacelike).

D. Natural tetrad

In Sec. III.D of paper I, we suggested a natural orthonormal tetrad which is provided intrinsically, by geometry of the spinning-particle problem itself. In case of the Mathisson-Pirani supplementary condition, it was given by u^μ, s^μ (the eigenvectors of the spin bivector), the hidden momentum $(p^\mu - mu^\mu)$, and the vector product of the three. The vectors

$$k^\mu, l^\mu; \qquad p^\mu_\perp, \dot{l}^\mu \tag{76}$$

we listed in the previous subsection can be used as such a natural tetrad here in the massless case. Actually, k^μ and l^μ span the (timelike) eigenplane of $S^{\mu\nu}$, and $p^\mu_\perp$ with $\dot{l}^\mu$ span the spacelike plane orthogonal to it, being orthogonal to each other as well. The first two, null vectors are normalized by $k_\mu l^\mu = -1$, and the second, spacelike couple is seen immediately to have norms given by

$$p^\mu_\perp p^\perp_\mu = p^\mu p_\mu = \mathcal{M}^2, \qquad \dot{l}^\mu \dot{l}_\mu = \frac{\mathcal{M}^2}{s^2}.$$

Needless to say, the spacelike basis vectors

$$e^\mu_{(2)} := \frac{p^\mu_\perp}{\mathcal{M}}, \qquad e^\mu_{(3)} := \frac{s\dot{l}^\mu}{\mathcal{M}} \tag{77}$$

can be transformed into null ones by

$$m^\mu := \frac{1}{\sqrt{2}}(e^\mu_{(2)} + \mathrm{i}e^\mu_{(3)}), \qquad \bar{m}^\mu := \frac{1}{\sqrt{2}}(e^\mu_{(2)} - \mathrm{i}e^\mu_{(3)}), \tag{78}$$

to complete the NP null tetrad to $(k^\mu, l^\mu, m^\mu, \bar{m}^\mu)$.

E. Vacuum MPD equations in a natural tetrad

Regarding that the spin condition $S^{\mu\nu}k_\nu = 0$ holds, we naturally tie the interpretation tetrad to k^μ. Proceeding as above, one assumes that $S^{\mu\nu}$ is spacelike ($S_{\mu\nu}S^{\mu\nu} = 2s^2 > 0$), which implies that it has a timelike eigenplane. Within such a plane, it is possible to find two independent null eigenvectors. Denote by l^μ "the other one," independent of k^μ, and normalize it by $k_\mu l^\mu = -1$. To complete the standard NP null tetrad, add two complex null vectors m^μ and $\bar{m}^\mu$, orthogonal to both k^μ and l^μ and normalized as $m_\mu \bar{m}^\mu = 1$.

The MPD equation of motion (64) for the massless case can now be written as

$$\begin{aligned}\dot{p}^\mu &= -\frac{1}{2}R^\mu{}_{\nu\kappa\lambda}k^\nu S^{\kappa\lambda} = \frac{s}{2}g^{\mu\rho}R_{\rho\nu\kappa\lambda}\epsilon^{\kappa\lambda\alpha\beta}k^\nu k_\alpha l_\beta \\ &= sg^{\mu\rho}\,{}^*R_{\rho\nu\alpha\beta}k^\nu k^\alpha l^\beta = -s\,{}^*C^\mu{}_{\nu\alpha\beta}k^\nu l^\alpha k^\beta,\end{aligned} \tag{79}$$

where $R^*_{\rho\nu\alpha\beta}$ and $^*R_{\rho\nu\alpha\beta}$ are the Riemann-tensor right and left duals [as in paper I, Eq. (39), we have used that they are equal in the vacuum case; this does not depend on the value of cosmological constant]. Since $^*R^\mu{}_{\nu\alpha\beta} = {}^*C^\mu{}_{\nu\alpha\beta} + \frac{\Lambda}{3}\epsilon^\mu{}_{\nu\alpha\beta}$, the cosmological constant drops out completely due to the presence of $k^\nu k^\beta$.

One can first decompose the MPD equation of motion directly in the NP tetrad, while employing the Weyl-scalar relations summarized in paper I, Eqs. (A1)–(A4):

$$k_\mu \dot{p}^\mu = -s\,{}^*C_{\mu\nu\alpha\beta}k^\mu k^\nu l^\alpha k^\beta = 0, \tag{80}$$

$$l_\mu \dot{p}^\mu = -s\,{}^*C_{\mu\nu\alpha\beta}l^\mu k^\nu l^\alpha k^\beta = 2s\,\mathrm{Im}\Psi_2, \tag{81}$$

$$m_\mu \dot{p}^\mu = -s\,{}^*C_{\mu\nu\alpha\beta}m^\mu k^\nu l^\alpha k^\beta = -\mathrm{i}s\Psi_1, \tag{82}$$

$$\bar{m}_\mu \dot{p}^\mu = -s\,{}^*C_{\mu\nu\alpha\beta}\bar{m}^\mu k^\nu l^\alpha k^\beta = \mathrm{i}s\bar{\Psi}_1. \tag{83}$$

It may, however, be more natural to escape the complex results by writing the last two components as projected onto the (real) orthonormal vectors (77) rather than onto their complex null counterparts. Since

$$e^\mu_{(2)} = \frac{1}{\sqrt{2}}(m^\mu + \bar{m}^\mu), \qquad e^\mu_{(3)} = \frac{1}{\sqrt{2}\mathrm{i}}(m^\mu - \bar{m}^\mu),$$

we find easily, in lieu of (82) and (83),

$$e^{(2)}_\mu \dot{p}^\mu = \sqrt{2}s\,\mathrm{Im}\Psi_1, \tag{84}$$

$$e^{(3)}_\mu \dot{p}^\mu = -\sqrt{2}s\,\mathrm{Re}\Psi_1. \tag{85}$$

In order to parallel the decomposition made in the massive case, one can also introduce orthonormal vectors

$$V^\mu = \frac{1}{\sqrt{2}}(k^\mu + l^\mu), \qquad e^\mu_{(1)} = \frac{1}{\sqrt{2}}(k^\mu - l^\mu)$$

and add the corresponding projections instead of (80) and (81),

$$-V_\mu \dot{p}^\mu = e^{(1)}_\mu \dot{p}^\mu = -\sqrt{2}s\,\mathrm{Im}\Psi_2. \tag{86}$$

The vector V^μ is a most natural timelike direction with which the massless problem can be connected; clearly, $e^\mu_{(1)}$ represents the corresponding spin vector (its unit form)—it is orthogonal to V^μ and belongs to the spin-bivector eigenplane ($S^{\mu\nu}e^{(1)}_\nu = 0$).

Equations (84), (85), and (86) show that the projections of the massless pole-dipole MPD equation onto the natural tetrad based on the worldline tangent k^μ are very simple and determined just by Ψ_1 and Ψ_2. In comparison with equations

$$e^{\hat{1}}_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2,$$
$$e^{\hat{2}}_\mu \dot{p}^\mu = -s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1),$$
$$e^{\hat{3}}_\mu \dot{p}^\mu = -s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1),$$

obtained for massive particles and the Mathisson-Pirani spin condition (paper I), the massless case does not contain the Ψ_3 scalar. If one takes the advantage of the remaining freedom of the spin bivector subjected to only the null spin condition $S^{\mu\nu}k_\nu = 0$ (see Sec. IV A) and *chooses* its second null eigendirection l^μ to be given by the highest-multiplicity PND of the Weyl tensor (provided that $k_\mu l^\mu \neq 0$, of course), then, depending on the Petrov type, some of the Weyl scalars can be eliminated. In particular, besides $\Psi_4 = 0$ (note again that we take l^μ as the *second* vector of the NP tetrad), Ψ_3/Ψ_3, and Ψ_2/Ψ_3, Ψ_2 and Ψ_1 can thus be made to vanish in type-II/type-III/type-N space-times. Hence, since the MPD-equation projections contain Ψ_1 and Ψ_2, they only simplify in the type-III or type-N case.

F. Properties of the natural *orthonormal* tetrad

Let us check some more properties of the above-introduced natural orthonormal tetrad

$$V^\mu, \quad e^\mu_{(1)} =: \frac{S^\mu}{s}, \qquad e^\mu_{(2)} = \frac{p^\mu_\perp}{\mathcal{M}}, \qquad e^\mu_{(3)} = \frac{s\dot{l}^\mu}{\mathcal{M}}. \tag{87}$$

First, provided that the particle's geodesic worldline is affinely parametrized, $\dot{k}^\mu = 0$, we see that

$$\dot{V}^\mu = \frac{\dot{l}^\mu}{\sqrt{2}} = \frac{\mathcal{M}e^\mu_{(3)}}{\sqrt{2}s}. \tag{88}$$

One also easily relates the (null) spin s^μ to the newly introduced "spin with respect to V^μ" (denoted by S^μ),

$$s^\mu \equiv sk^\mu = \frac{s}{\sqrt{2}}(V^\mu + e^\mu_{(1)}) = \frac{1}{\sqrt{2}}(sV^\mu + S^\mu). \tag{89}$$

As $\dot{s}^\mu = s\dot{k}^\mu = 0$, we have

$$\dot{e}^\mu_{(1)} = -\dot{V}^\mu = -\frac{\mathcal{M}e^\mu_{(3)}}{\sqrt{2}s}. \tag{90}$$

Finally, regarding that

$$\dot{p}^\mu_\perp = \dot{p}^\mu + k^\mu l_\nu \dot{p}^\nu = (\delta^\mu_\nu + k^\mu l_\nu + l^\mu k_\nu)\dot{p}^\nu, \tag{91}$$

one finds, from orthonormality of the basis,

$$\dot{e}^\mu_{(2)} = \frac{e^{(3)}_\nu \dot{p}^\nu}{\mathcal{M}} e^\mu_{(3)}, \tag{92}$$

$$\dot{e}^\mu_{(3)} = \frac{\mathcal{M}}{s}k^\mu - \frac{e^{(3)}_\nu \dot{p}^\nu}{\mathcal{M}} e^\mu_{(2)}. \tag{93}$$

Having introduced V^μ and S^μ, we can express the spin bivectors alternatively as[6]

$$S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}V^\gamma S^\delta, \qquad {}^*S^{\mu\nu} = S^\mu V^\nu - V^\mu S^\nu \tag{94}$$

and write, similarly as in paper I (Sec. II. C), equations for $\dot{V}^\mu$ and $\dot{S}^\mu$ in terms of k^μ and p^μ. Actually, multiplying $\dot{S}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}\dot{V}^\gamma S^\delta + \epsilon_{\alpha\beta\gamma\delta}V^\gamma \dot{S}^\delta$ by $\epsilon^{\mu\nu\alpha\beta}V_\nu$ and $\epsilon^{\mu\nu\alpha\beta}S_\nu$, we obtain, respectively,

$$\dot{S}^\mu = \epsilon^{\mu\nu\alpha\beta}V_\nu k_\alpha p_\beta = -{}^*\dot{S}^{\mu\nu}V_\nu, \tag{95}$$

$$s^2\dot{V}^\mu = \epsilon^{\mu\nu\alpha\beta}S_\nu k_\alpha p_\beta = -{}^*\dot{S}^{\mu\nu}S_\nu, \tag{96}$$

where we have already regarded that $s\dot{V}^\mu = -\dot{S}^\mu \sim e^\mu_{(3)}$ and $\dot{s} = 0$. Note that the above equations can also be obtained very straightforwardly by differentiating

$$S^\mu = -{}^*S^{\mu\nu}V_\nu, \qquad s^2V^\mu = -{}^*S^{\mu\nu}S_\nu \tag{97}$$

and that, thanks to $S_\mu S^\mu = s^2$, the magnitude of S^μ is automatically constant along k^μ.

G. Special cases of motion

The massless spinning-particle problem turns out to be quite constrained; it offers much less freedom for various special performances than the massive case. Let us still mention two cases which arise naturally.

[6]Note that the above-introduced spin S^μ thus fixes the spin bivector *uniquely*, in contrast to the null spin s^μ introduced by (67).

1. $p^\mu \sim k^\mu$ circumstance

Notice, finally, that the tetrad (87) would be meaningless if p^μ belonged to the eigenplane of $S^{\mu\nu}$ (i.e., if it were some spacelike combination of k^μ and l^μ), because then $s^2\dot{l}^\mu \equiv -S^{\mu\alpha}p_\alpha = 0$ and, consequently, $\dot{S}_{\alpha\beta}=0$ and $p^\mu_\perp \equiv S^{\mu\nu}\dot{l}_\nu = 0$. In such a case, all the vectors k^μ, l^μ, s^μ, V^μ, S^μ, and p^μ would lie in the spin-bivector eigenplane, and most of them would be parallel transported along the representative worldline: $\dot{k}^\mu = 0$, $\dot{l}^\mu = 0$, $\dot{s}^\mu = 0$, $\dot{V}^\mu = 0$, $\dot{S}^\mu = 0$. However, as already noted below Eq. (74) and as best seen from Eq. (70), such a circumstance would imply $p^\alpha = -p^\beta l_\beta k^\alpha$, so $\mathcal{M} = 0$ and $\dot{p}^\alpha = -\dot{p}^\beta l_\beta k^\alpha$; i.e., both p^α and $\dot{p}^\alpha$ would also have to be lightlike and proportional to k^μ. According to Eq. (79), this would require ${}^*C^\mu{}_{\nu\alpha\beta}k^\nu l^\alpha k^\beta$ to be lightlike, which definitely does not hold for *generic* motion in generic space-time. Using the metric decomposition

$$g^{\mu\alpha} = -k^\mu l^\alpha - l^\mu k^\alpha + m^\mu \bar{m}^\alpha + \bar{m}^\mu m^\alpha \tag{98}$$

and regarding that the first two terms yield zero in the scalar product below, one can rewrite the requirement as

$$\begin{aligned} 0 &= ({}^*C_{\mu\nu\kappa\lambda}k^\nu l^\kappa k^\lambda)g^{\mu\alpha}({}^*C_{\alpha\beta\gamma\delta}k^\beta l^\gamma k^\delta) \\ &= 2({}^*C_{\mu\nu\kappa\lambda}m^\mu k^\nu l^\kappa k^\lambda)({}^*C_{\alpha\beta\gamma\delta}\bar{m}^\alpha k^\beta l^\gamma k^\delta) \\ &= 2\Psi_1\bar{\Psi}_1, \end{aligned} \tag{99}$$

which is only satisfied for $\Psi_1 = 0$, i.e., if (i) either the particle moves in the direction (k^μ) of the double PND of a Petrov-type-II space-time, (ii) or the space-time is of type N (and one aligns with its quadruple PND the second vector l^μ of the NP tetrad).

2. Stationary situation

The only basic scalar involved which may not be constant is $\mathcal{M}$. Consider now the case when it *is* constant, $\dot{\mathcal{M}} = 0$, but when $\mathcal{M} \neq 0$, so p^μ is spacelike (if p^μ were lightlike, it would immediately lead to $p^\mu \sim k^\mu$, which has already been mentioned above). From (66) one infers—in both cases—that $\dot{S}^{\alpha\beta}\dot{p}_\beta = 0$, which implies that $\dot{p}^\mu$ belongs to the eigenplane of $\dot{S}^{\mu\nu}$. This eigenplane is spanned by k^μ and $\dot{l}^\mu$, so $\dot{p}^\mu$ has to be given by their combination, say $\dot{p}^\mu = \alpha k^\mu + \mathcal{M}\beta\dot{l}^\mu$. In particular, $\dot{p}^\mu_\perp$ must be proportional to $\dot{l}^\mu$, since it does not have any component proportional to k^μ by definition. Actually, the latter also follows, given $\dot{\mathcal{M}} = 0$, from (92).[7]

A related consequence of $\dot{\mathcal{M}} = 0$ is of course $p_\mu\dot{p}^\mu = 0$. Writing $\dot{p}^\mu$ as (79), inserting the metric (98), and using $k_\sigma p^\sigma = 0$ and

$$\dot{l}_\sigma p^\sigma = 0 \;\Rightarrow\; m_\sigma p^\sigma = \bar{m}_\sigma p^\sigma = \frac{e^{(2)}_\sigma p^\sigma}{\sqrt{2}} = \frac{\mathcal{M}}{\sqrt{2}},$$

one can express the $p_\mu\dot{p}^\mu = 0$ circumstance as a simple condition for the type of space-time, because in terms of the Weyl scalars computed in our NP tetrad, it reads

$$\begin{aligned} 0 = p_\mu\dot{p}^\mu &= -s\,{}^*C_{\mu\nu\alpha\beta}p^\mu k^\nu l^\alpha k^\beta \\ &= -s\,{}^*C_{\mu\nu\alpha\beta}(m^\mu\bar{m}_\sigma + \bar{m}^\mu m_\sigma)p^\sigma k^\nu l^\alpha k^\beta \\ &= -\frac{\mathcal{M}s}{\sqrt{2}}{}^*C_{\mu\nu\alpha\beta}(m^\mu + \bar{m}^\mu)k^\nu l^\alpha k^\beta \end{aligned} \tag{100}$$

$$= \frac{\mathcal{M}s}{\sqrt{2}}(-\mathrm{i}\Psi_1 + \mathrm{i}\bar{\Psi}_1) = \sqrt{2}\mathcal{M}s\,\mathrm{Im}\Psi_1, \tag{101}$$

where, in the last row, Eqs. (82) and (83) have been used.

The coefficients of the $\dot{p}^\mu = \alpha k^\mu + \mathcal{M}\beta\dot{l}^\mu$ relation can also be found in terms of the Weyl scalars: multiplying it by l_μ and $\dot{l}_\mu$, we have, respectively,

$$l_\mu\dot{p}^\mu = -\alpha\ldots = 2s\,\mathrm{Im}\Psi_2, \tag{102}$$

$$\dot{l}_\mu\dot{p}^\mu = \mathcal{M}\beta\frac{\mathcal{M}^2}{s^2}\ldots = -\sqrt{2}\mathcal{M}\,\mathrm{Re}\Psi_1. \tag{103}$$

V. CONCLUDING REMARKS

We have continued the study of a spinning-particle motion in the pole-dipole approximation. After treating, in Ref. [1], the MPD equation of motion in an orthonormal tetrad tied to the reference observer (denoted V^μ), i.e., in a tetrad involving as a time leg the vector which fixes the spin supplementary condition ($S^{\mu\nu}V_\nu = 0$), we considered the tetrad tied to the tangent of the worldline that represents the particle's history (denoted u^μ). Both possibilities lead to usable formulations of the problem, with the latter (proposed in the present paper) being slightly less efficient, because u^μ cannot be freely chosen (in contrast to V^μ). In both cases, we showed how the MPD equation decomposes if representing the curvature terms in the language of Weyl-tensor scalars obtained in the NP null tetrads naturally associated with the orthonormal ones. In the case of decomposing the MPD equation in the u^μ-based tetrad, we also showed how the projections look when computing the Weyl scalars in a different NP tetrad (different than that naturally associated with the orthonormal u^μ-based tetrad), namely, the one tied to a vector V^μ that can be freely chosen.

Expressing the MPD-equation components in terms of the Weyl scalars, one can infer whether and how the exercise simplifies in particular Petrov types, provided that the NP tetrad can be aligned with the highest-multiplicity PND. Such an alignment is of course more problematic for

[7] Therefore, if $\dot{\mathcal{M}} = 0$, then $\dot{p}^\mu_\perp$ can be used, after normalization, as the $e^\mu_{(3)}$ vector of the interpretation tetrad equally as well as $\dot{l}^\mu$.

the u^μ-based tetrad (if one does not want to necessarily resort to the $S^{\mu\nu}u_\nu = 0$ spin condition) which is much less flexible. Another item was to see how the problem depends on the spin supplementary condition. We saw, in particular, that for the most advantageous option $u^\mu \| p^\mu$, the interpretation tetrads we had suggested (as given intrinsically by the geometry of the problem itself) were not available (two of their vectors turned zero) and suggested simple alternatives (which on the contrary do not work for the $S^{\mu\nu}u_\nu = 0$ condition).

The second part of the present paper was devoted to spinning particles with zero rest mass. For them, the worldlines are null geodesics, the spin vector is also lightlike (and proportional to the worldline tangent), the momentum is *spacelike* (or null in a certain limit, which, however, only corresponds to a specific motion in type-II fields), and the Mathisson-Pirani spin condition follows necessarily. In spite of these important differences, a similar analysis can be performed as in the massive case; in particular, a certain intrinsic interpretation tetrad can again be proposed. The respective decomposition of the MPD equation of motion is considerably simpler than in the massive case; it contains only Ψ_1 and Ψ_2 scalars and not the cosmological constant. Even (some of) these are eliminated in type-III or type-N space-times if the second null eigendirection l^μ of the spin bivector is identified with the main PND of the background curvature (this is possible thanks to the less restricting nature of the null Mathisson-Pirani condition), and of course in the case when the particle moves, at least at a given point, along a PND.

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[1] O. Semerák and M. Šrámek, Spinning particles in vacuum spacetimes of different curvature types, Phys. Rev. D **92**, 064032 (2015).

[2] G. S. Hall, *Symmetries and Curvature Structure in General Relativity* (World Scientific, Singapore, 2004).

[3] M. Bailyn and S. Ragusa, Pole-dipole model of massless particles, Phys. Rev. D **15**, 3543 (1977).

[4] M. Bailyn and S. Ragusa, Pole-dipole model of massless particles. II, Phys. Rev. D **23**, 1258 (1981).

[5] C. Duval and H. H. Fliche, A conformal invariant model of localized spinning test particles, J. Math. Phys. (N.Y.) **19**, 749 (1978).

[6] B. Mashhoon, Massless spinning test particles in a gravitational field, Ann. Phys. (N.Y.) **89**, 254 (1975).

[7] D. Bini, C. Cherubini, A. Geralico, and R. T. Jantzen, Massless spinning test particles in algebraically special vacuum space-times, Int. J. Mod. Phys. D **15**, 737 (2006).