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Averaging in cosmology based on Cartan scalars

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Abstract. Averaging scheme based on the use of Cartan scalars is presented in the cosmological context. The resulting method is applied to LTB cosmological model and the correlation terms appearing due to averaging are physically explained.

1. Introduction

Homogeneous cosmology is a successful but highly simplified model. The real inhomogeneous universe should be suitably averaged to obtain the corresponding homogeneous model. This is a nontrivial task from two perspectives. First, Einstein equations are nonlinear leading to the appearance of a correlation terms which can mimic various sources (e.g. dark energy). Second, there is no established method for averaging tensors covariantly, this can be circumvented by concentrating on scalars only.

Many approaches to averaging were developed in the past. The most popular approach to averaging is Buchert's method of averaging scalar part of Einstein equations [1], [2]. All of Einstein equations are averaged in the case of Macroscopic Gravity [3], [4], at the same time the Cartan structure equations are averaged. Isometric embedding of a 2-sphere into Euclidian space is used for averaging by Korzyñski [5]. In [6] the Weitzenböck connection was used for the definition of correct averaging of tensor fields. Coley [7] investigated averaging of scalar invariants constructed from the Riemann tensor and finite number of its covariant derivatives.

Here we propose to use Cartan scalars to covariantly average both geometry and Einstein equations. Originally, the theory of Cartan scalars was developed to solve the equivalence problem for geometries [8], [9] and therefore it is suitable to unambiguously describe the given spacetime and the process of averaging which is suitably trivial for scalars. We also average the left hand side of Einstein equations (which can be expressed using Cartan scalars) and we propose the procedure for obtaining the correlation term.

Usually, correlation term is defined as a difference between Einstein tensor defined via macroscopic highly symmetric ("averaged") metric and average of Einstein tensor computed using unaveraged metric

$$R^{\mu}{}_{\nu}\left(\overline{g_{\alpha\beta}}\right) - \frac{1}{2}R\left(\overline{g_{\alpha\beta}}\right)\delta^{\mu}{}_{\nu} + C^{\mu}{}_{\nu} = 8\pi T^{\mu}{}_{\nu}\left(\overline{g_{\alpha\beta}}\right)$$

$$C^{\mu}{}_{\nu} = \left\langle R^{\mu}{}_{\nu}\right\rangle - \frac{1}{2}\left\langle R\right\rangle\delta^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}\left(\overline{g_{\alpha\beta}}\right) - \frac{1}{2}R\left(\overline{g_{\alpha\beta}}\right)\delta^{\mu}{}_{\nu}.$$

2. Cartan scalars

They are constructed as tetrad projections of Riemann tensor and the finite number of its covariant derivatives and they completely (locally) specify the geometry of a manifold. Cartan scalars are true scalars on the bundle of frames $F(\mathcal{M})$ and for a fixed tetrad they behave as scalars on the manifold as well. We can use them to obtain the dimension of an isometry group and the corresponding algebra of Killing vectors. And as mentioned in introduction, they form a classical approach to the equivalence problem.

First we review the Cartan-Karlhede approach for construction of Cartan scalars. Let $\mathcal{M}$ be a n-dimensional differentiable manifold with a metric

$$\mathbf{g} = \eta_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j,$$

Frame $\boldsymbol{\omega}^i$ is fixed up to the generalized rotations

$$\boldsymbol{\omega}^i = \omega^i_\nu(x^\mu, \xi^\Upsilon)\mathbf{d}x^\nu,$$

where ξ^Υ denote the coordinates of a Lorentz group G. Bundle of frames $F(\mathcal{M})$ is locally isomorphic to $\mathcal{U} \times G, \mathcal{U} \subset \mathcal{M}$ and we introduce enlarged exterior derivative $\mathbf{d} = \mathbf{d}_x + \mathbf{d}_\xi$. This gives Cartan structure equeations

$$\mathbf{d}\boldsymbol{\omega}^i = \boldsymbol{\omega}^j \wedge {\boldsymbol{\omega}^i}_j,$$

$$\mathbf{d}{\boldsymbol{\omega}^i}_j = -{\boldsymbol{\omega}^i}_k \wedge {\boldsymbol{\omega}^k}_j + \frac{1}{2}{R^i}_{jkl}\boldsymbol{\omega}^k \wedge \boldsymbol{\omega}^l,$$

with a condition $\eta_{ik}{\boldsymbol{\omega}^k}_j + \eta_{jk}{\boldsymbol{\omega}^k}_i = 0$.

Generation of covariant derivatives of the Riemann tensor is performed by successive application of $\mathbf{d}$

$$\begin{aligned}
\mathbf{d}R_{ijkl} &= R_{mjkl}\boldsymbol{\omega}^m_i + R_{imkl}\boldsymbol{\omega}^m_j + R_{ijml}\boldsymbol{\omega}^m_k + R_{ijkm}\boldsymbol{\omega}^m_l + R_{ijkl;m}\boldsymbol{\omega}^m, \\
\mathbf{d}R_{ijkl;n} &= R_{mjkl;n}\boldsymbol{\omega}^m_i + R_{imkl;n}\boldsymbol{\omega}^m_i + ... + R_{ijkl;nm}\boldsymbol{\omega}^m, \\
&. \\
&.
\end{aligned}$$

Now, R^p denotes the set $\left\{R_{ijkm}, R_{ijkm;n_1}, ..., R_{ijkm;n_1...n_p}\right\}$ where p is such that R^{p+1} contains no additional functionally independent element (over $F(\mathcal{M})$, with dependence defined using $\mathbf{d}f \sim \mathbf{d}g$). The set R^p characterizes the geometry up to isometry and the elements are Cartan scalars. R^p has to satisfy (nonlinear) constraints so we can form a smaller subset R'^p that can be used to generate the full set.

3. Averaging

Geometry of $\mathcal{M}$ is now characterized by the set R^p of scalars which can be naturally averaged over a given domain $\mathcal{D}$ without any problems related to covariant tensor averaging

$$\langle f \rangle (x) = \frac{1}{V_\mathcal{D}} \int\limits_\mathcal{D} f\,(x + x')d^N x'.$$

New geometry $\langle \mathcal{M} \rangle$ describes a manifold identical as a set but with averaged geometry given by averaged Cartan scalars. However, naively averaging R^p would result in a set potentially violating nonlinear constraints and therefore not generating viable geometry. To overcome this we will use the restricted set R'^p to obtain a new set $\langle R'^p \rangle$ and then we generate the whole set $\langle R^p \rangle$ from $\langle R'^p \rangle$ using constraints. Then there exists a metric tensor $\langle g_{\mu\nu} \rangle$ (or equivalently $\langle \boldsymbol{\omega}^i \rangle$)

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giving the same Cartan scalars. Such metric may be constructed using the averaged Cartan scalars.

Averaging generally leads to decrease in the number of independent functions in $\langle R^p \rangle$, enlargement of isotropy group of the new spacetime $\langle \mathcal{M} \rangle$ and a new algebra of Killing vectors.

Left hand side of Einstein equations can be projected on tetrad of the Cartan-Karlhede algorithm to obtain expression in Cartan scalars enabling straightforward averaging.

We have two roads to averaging with Cartan scalars. Direct method described above - which is very difficult. And indirect method which is performed by comparing averaged scalars with scalars of known highly symmetric spacetimes (FRLW etc.), possibly getting conditions for their equivalence and correlation tensor

$$C^{\mu}{}_{\nu} = \langle R^{\mu}{}_{\nu} \rangle - \frac{1}{2} \langle R \rangle \delta^{\mu}{}_{\nu} - R^{\mu}{}_{\nu} \left(\overline{g_{\alpha\beta}}\right) - \frac{1}{2} R \left(\overline{g_{\alpha\beta}}\right) \delta^{\mu}{}_{\nu}$$

and note that the conserved stress-energy tensor has the following form

$$^{(ef)}T^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} \left(\overline{g_{\alpha\beta}}\right) - C^{\mu}{}_{\nu}. \tag{1}$$

This technique passes the basic test of correct averaging: constant curvature spacetimes are preserved.

4. Applications - LTB metric

Lemaître-Tolman-Bondi (LTB) metric is an exact spherically symmetric solution with inhomogeneous dust $T_{\mu\nu} = \rho u_\mu u_\nu$. The line element reads

$$ds^2 = -dt^2 + \frac{(R'(t,r))^2}{1+2E(r)} dr^2 + R^2(t,r)(d\theta^2 + \sin^2(\theta) d\phi^2).$$

Einstein equations reduce to the following

$$R_{,t}^2 = 2E + \frac{2M(r)}{R} + \frac{\Lambda}{3} R^2, \quad 4\pi\rho = \frac{M'}{R' R^2},$$

where function $E(r)$ determines a curvature of spatial section $t = const.$ and $M(r)$ is the gravitational mass contained within the comoving radius r. The first of the above equations can be integrated to obtain

$$\int_0^R \frac{d\tilde{R}}{\sqrt{2E + \frac{2M}{\tilde{R}} + \frac{1}{3}\Lambda \tilde{R}^2}} = t - t_B(r),$$

where $t_B(r)$ is the bang time function meaning the Big Bang is not simultaneous in all points.

4.1. First model

We will consider $E = 0$ and assume the ansatz

$$R(t,r) = A(t,r) \exp \psi(t,r),$$

where $\psi(t,r)$ is a quickly varying function, and $\psi << \psi_{,x} \sim \psi_{,xy} \sim \psi_{,xyz}$, where x, y and z denote time or radial coordinate; additionally $\psi_{,x} >> A(t,r)$ and its derivatives. We consider this tetrad

$$\begin{aligned}
\boldsymbol{\omega}^0 &= \frac{1}{\sqrt{2}}(dt + R_{,r}dr), \\
\boldsymbol{\omega}^1 &= \frac{1}{\sqrt{2}}(dt - R_{,r}dr), \\
\boldsymbol{\omega}^2 &= \frac{1}{\sqrt{2}}(Rd\theta + iR\sin\theta d\phi), \\
\boldsymbol{\omega}^3 &= \frac{1}{\sqrt{2}}(Rd\theta - iR\sin\theta d\phi).
\end{aligned}$$

After following the above described prescription for averaging we obtain following nontrivial zero-order Cartan scalars

$$\begin{gathered}
\psi_2 = -\tfrac{1}{6}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{6}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} + \tfrac{1}{6}R^{-1}R_{,tt} - \tfrac{1}{6}R^{-2}(R_{,t})^2, \\
\phi_{00'} = \phi_{22'} = \tfrac{1}{2}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} - \tfrac{1}{2}R^{-1}R_{,tt}, \\
\phi_{11'} = -\tfrac{1}{4}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{4}R^{-2}(R_{,t})^2, \\
\Lambda = \tfrac{1}{12}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{6}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} + \tfrac{1}{6}R^{-1}R_{,tt} + \tfrac{1}{12}R^{-2}(R_{,t})^2.
\end{gathered}$$

In the leading order all averages are equal to zero except

$$\langle\Lambda\rangle = \frac{1}{2}\langle\psi_{,t}^2\rangle. \tag{2}$$

First order scalars are (in the leading order) all equal to zero. Comparison with known solutions evidently leads to de Sitter space, correlation term is in the form of a positive cosmological constant.

4.2. Onion model

We consider approximate LTB solution by Biswas, Mansouri and Notari - a spacetime with radial shells of overdense and underdense regions. It is given by $E(r) > 0$, $a(t,r) := \frac{R(t,r)}{r}$ with

$$a(t,r) := \left(\frac{6}{\pi}\right)^{1/3} t^{2/3}(1 + Lt^{2/3}\frac{1}{r}\sin^2\pi r)$$

and can be interpreted as a perturbation of the flat dust FRW if we consider L as a small parameter.

Cartan scalar Λ reads

$$\begin{aligned}
\Lambda(r,t,L) = \frac{1}{12}\frac{1}{a^2(a+a_{,r})r}[&3a^2a_{,tt} + a^2ra_{,ttr} + 3a(a_{,t})^2 \\
&+2a_{,tt}aa_{,r}r + 2a_{,t}raa_{,tr} + (a_{,t})^2ra_{,r} + a_{,r}rK + 3Ka + aK_{,r}].
\end{aligned}$$

Function $K(r,L)$ is related to curvature function $E(r)$ by

$$K(r,L) = -\frac{2E(r)}{r^2} = \frac{-L}{\pi r}\sin\pi r\sin\pi r \tag{3}$$

Here we perform spatial averaging only and expand in the powers of L (coefficients A, B, C, D are given by averaging)

$$\langle\Lambda\rangle \approx \frac{A}{t^2} + \frac{B}{t^{4/3}}L + \frac{C}{t^{2/3}}L^2 + \frac{D}{t^0}L^3$$

Individual terms as interpreted with respect to Einstein-de Sitter model have the following meaning (beyond first term they represent correlation terms):

(i) standard dust
(ii) behaves like curvature
(iii) causes acceleration equivalent to $p = -\frac{2}{3}\rho$
(iv) cosmological constant

Other zero order Cartan scalars are at most linear in L. However this means that the interpretation given above should be treated with caution beyond second term.

5. Conclusion

We have seen that using Cartan scalars for averaging provides conceptually straightforward method. However, it presents challenging technical (for direct method) and interpretation problems (for indirect one). The correlation term is just a cosmological constant in the simple model. However it is rather complex in the more realistic Onion model.

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