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# Geodesics in nonexpanding impulsive gravitational waves with $\Lambda$ , part I

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## Abstract

We investigate the geodesics in the entire class of nonexpanding impulsive gravitational waves propagating in an (anti-)de Sitter universe using the distributional form of the metric. Employing a five-dimensional embedding formalism and a general regularisation technique, we prove the existence and uniqueness of the geodesics crossing the wave impulse, leading to a completeness result. We also derive the explicit form of the geodesics, thereby confirming previous results derived in a heuristic approach.

Keywords: impulsive gravitational waves, geodesics, low regularity

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Impulsive  $pp$ -waves have been studied for several decades and have become textbook examples of exact radiative spacetimes modelling short but intense bursts of gravitational radiation propagating in a Minkowski background (see, e.g. [16, section 20.2] and the references therein). Such geometries were introduced by Roger Penrose using his ‘scissors and paste method’ (see, e.g. [25]), leading to the *distributional Brinkmann form* of the metric

$$ds^2 = dx^2 + dy^2 - 2dudv + f(x, y)\delta(u)du^2, \quad (1.1)$$

i.e., impulsive limits of sandwich  $pp$ -waves [5]. Alternatively, almost at the same time, Aichelburg and Sexl in [1] obtained a special impulsive  $pp$ -wave as the ultrarelativistic limit

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of the Schwarzschild geometry and several authors have applied the same approach to other solutions of the Kerr–Newman family (see, e.g. [4, chapter 4] and [27, section 3.5.1] for an overview). This procedure of boosting static sources to the speed of light was later generalised to the case of a non-vanishing cosmological constant  $\Lambda$  in the pioneering work [19] by Hotta and Tanaka (see also [28, 29]), which lead to an increased interest in nonexpanding impulsive waves in cosmological de Sitter and anti-de Sitter backgrounds. The Penrose ‘scissors and paste method’ for non-vanishing  $\Lambda$  was described in [30, 38], while impulsive limits in the Kundt class were considered in [26] and elsewhere, see [16, section 20.3] for an overview.

Generally, nonexpanding impulsive waves in all backgrounds of constant curvature can be described by a continuous and a distributional form of the metric tensor. To give a brief discussion of these, we start with the conformally flat form of Minkowski ( $\Lambda = 0$ ) and (anti-) de Sitter ( $\Lambda \neq 0$ ) background spacetimes

$$ds_0^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V}}{\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^2}. \tag{1.2}$$

As in [30], here  $\mathcal{U}, \mathcal{V}$  are the usual null and  $\eta, \bar{\eta}$  the usual complex spatial coordinates. Now, for  $\mathcal{U} > 0$ , we perform the transformation

$$\mathcal{U} = U, \quad \mathcal{V} = V + H + UH_{,Z}H_{,\bar{Z}}, \quad \eta = Z + UH_{,\bar{Z}}, \tag{1.3}$$

where  $H(Z, \bar{Z})$  is an arbitrary real-valued function. Combining this with the background line element (1.2) for  $U < 0$  in which we set  $\mathcal{U} = U, \mathcal{V} = V, \eta = Z$ , we obtain the *continuous* form of the metric

$$ds^2 = \frac{2 |dZ + U_+(H_{,Z\bar{Z}}d\bar{Z} + H_{,\bar{Z}Z}dZ)|^2 - 2 dU dV}{\left[1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV + U_+G)\right]^2}, \tag{1.4}$$

where<sup>4</sup>  $G(Z, \bar{Z}) = ZH_{,Z} + \bar{Z}H_{,\bar{Z}} - H$  and  $U_+ \equiv U_+(U) = 0$  for  $U \leq 0$  and  $U_+(U) = U$  for  $U \geq 0$ . This ‘kink-function’  $U_+$  is Lipschitz continuous, hence the spacetime (apart from possible poles of  $H$ , which indeed occur in physically realistic models, see, e.g. [1, 19], and section 2, below) is locally Lipschitz. Recall that a locally Lipschitz metric possesses a locally bounded connection and hence a distributional curvature, which, in general, is unbounded. In fact, the discontinuity in the derivatives of the metric introduces impulsive components in the Weyl and curvature tensors (see [30]), and the metric (1.4) explicitly describes impulsive waves in de Sitter, anti-de Sitter or Minkowski backgrounds.

For  $\Lambda = 0$ , (1.4) reduces to the classic Rosen form of impulsive  $pp$ -waves (cf. [16, section 17.5]). In this special case, the geodesic equation has been rigorously solved in [23] using Carathéodory’s solution concept (see, e.g. [12, chapter 1]), which allows us to deal with the locally bounded but discontinuous right hand side of the equation. The geodesics thereby obtained coincide with the limits of the geodesics for the distributional form (1.1) calculated in [21], which have been previously derived heuristically (e.g. in [2, 11, 35]).

However, to deal with the geodesic equation for the full class of nonexpanding impulsive waves for arbitrary  $\Lambda$ , that is the complete metric (1.4), the more sophisticated solution concept of Filippov [12, chapter 2] has been applied recently in [32]. Building on a general result for all locally Lipschitz spacetimes [40], existence, uniqueness and global  $C^1$ -regularity of the geodesics has been established. This, in particular, justifies the  $C^1$ -matching procedure that has been used before to explicitly derive the geodesics in this and similar situations [30, 31, 33–35, 41].

<sup>4</sup> This choice of sign of  $G$  is in accordance with [31], which is our main point of reference, but different from more recent papers, e.g. [32].

On the other hand, the *distributional form* of the impulsive metric arises by writing the transformation relating (1.2) and (1.4) in a combined way for all  $U$ , by using the Heaviside function  $\Theta$ , as

$$U = U, \quad \mathcal{V} = V + \Theta H + U_+ H_{,z} H_{,z}, \quad \eta = Z + U_+ H_{,z}. \quad (1.5)$$

Of course, (1.5) is discontinuous in the coordinate  $\mathcal{V}$  and merely Lipschitz continuous in  $\eta$  across  $\{U = 0\}$  but applying it *formally* to (1.4), we arrive at

$$ds^2 = \frac{2 d\eta d\bar{\eta} - 2 dU d\mathcal{V} + 2H(\eta, \bar{\eta})\delta(U)dU^2}{\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - U\mathcal{V})\right]^2}, \quad (1.6)$$

which has the striking advantage of coinciding with the background metric  $ds_0^2$  (1.2) off the impulse located at  $U = 0$ . This, however, comes at the price of introducing a distributional coefficient in the metric, which leads us out of the Geroch–Traschen class [15] of metrics (of regularity  $W_{\text{loc}}^{1,2} \cap L_{\text{loc}}^\infty$ ), which guarantees the existence and stability of the curvature in distributions (see also [24, 42]). However, due to its simple geometrical structure, the metric (1.6) nevertheless allows to calculate the curvature as a distribution, again leading to impulsive components in the Weyl and curvature tensors [30]. Also, in the Minkowski background, the metric (1.6) reduces to the Brinkmann form of impulsive  $pp$ -waves (1.1).

Clearly, a mathematically sound treatment of the transformation (1.5) is a delicate matter. In the special case of  $pp$ -waves, this has been achieved in [22] by using nonlinear distributional geometry ([17, chapter 3]) based on algebras of generalised functions [8]. More precisely, the ‘discontinuous coordinate change’ was shown to be the distributional limit of a ‘generalised diffeomorphism’, a concept further studied in [9, 10]. The key to these results was provided by a nonlinear distributional analysis of the geodesics in the metric (1.1), and, in particular, an existence, uniqueness and completeness result for the geodesic equation in nonlinear generalised functions [21, 39].

This suggests that a first step towards the long-term goal of understanding the transformation (1.5) for  $\Lambda \neq 0$  is to reach a mathematically sound understanding of the geodesics in the distributional metric (1.6). The geodesic equation for (1.6), however, displays a very singular behaviour including terms proportional to the square of the Dirac- $\delta$  (cf. [38]). For this reason, the authors of [31] have employed a five-dimensional formalism [29, 30] of embedding (anti-)de Sitter space into a five-dimensional  $pp$ -wave spacetime (see section 2 below). In this approach, the geodesic equation takes a form that is distributionally accessible at all, however, not mathematically rigorously. In the absence of a valid solution concept for this nonlinear distributional equation, a natural ansatz was used to derive the geodesics and to study them in detail in [31 section 4–5]. Nevertheless, a desirable nonlinear distributional analysis of the geodesic equation in the  $\Lambda \neq 0$  cases, which will eventually lead to a mathematical understanding of the transformation (1.5), has been missing to date.

In this paper, we provide such an analysis. Thereby, we follow [31] in employing the five-dimensional formalism. However, we will not use any theory of nonlinear distributions, leaving the detailed study of nonexpanding impulsive waves in an (anti-)de Sitter universe as (nonlinear) distributional geometries for a subsequent paper. Instead, we will employ a regularisation approach and view (1.6) as a spacetime with a short but finitely extended impulse (i.e., a generic sandwich wave with support in a regularisation strip, which we will also call the ‘wave zone’) and employ an analysis in the spirit of [36] where impulsive limits in a class of  $pp$ -wave type spacetimes with a more general wave surface but vanishing  $\Lambda$  ([6, 7, 13, 14]) have been considered.

We will detail this regularisation approach in the next section after introducing the five-dimensional formalism. In particular, we will replace the Dirac- $\delta$  in the metric (2.1) below by a very general regularisation  $\delta_\varepsilon$ , thereby ensuring that our results are *regularisation independent* (within the class of so-called *model delta nets*). Then, in section 3, we will employ a fixed point argument to show that the regularised equations have unique smooth solutions that cross the regularisation strip. This will lead to our main result on completeness of non-expanding impulsive gravitational waves in a cosmological background. The technical proofs allowing for the application of the fixed point theorem are deferred to appendix A. In section 4, we study the boundedness properties of the regularised geodesics, which are essential when dealing with their limits in section 5. There, we show that the solutions of the regularised geodesic equation converge, as the regularisation parameter goes to zero, to geodesics of the background (anti-)de Sitter spacetime, which have to be matched appropriately across the impulse and have been derived previously in [31]. Overly technical calculations are collected in appendix B.

## 2. The geodesic equation for $\Lambda \neq 0$

In this section, we detail our regularisation approach and derive the respective geodesic equations. To begin with, however, we recall the five-dimensional formalism of [29, 30] for the full class of nonexpanding impulsive waves with non-vanishing  $\Lambda$ . To this end, we start with the five-dimensional impulsive *pp*-wave spacetime  $\bar{M}$  with metric (extending (1.1))

$$ds^2 = dZ_2^2 + dZ_3^2 + \sigma dZ_4^2 - 2dUdV + H(Z_2, Z_3, Z_4)\delta(U)dU^2 \quad (2.1)$$

and consider the four-dimensional (anti-)de Sitter hyperboloid  $(M, g)$  given by the constraint

$$Z_2^2 + Z_3^2 + \sigma Z_4^2 - 2UV = \sigma a^2, \quad (2.2)$$

where  $a := \sqrt{3/(\sigma\Lambda)}$ ,  $\sigma := \text{sign}(\Lambda) = \pm 1$  and  $U = \frac{1}{\sqrt{2}}(Z_0 + Z_1)$ ,  $V = \frac{1}{\sqrt{2}}(Z_0 - Z_1)$  are null-coordinates<sup>5</sup>. Here,  $(Z_0, \dots, Z_4)$  are global Cartesian coordinates of  $\mathbb{R}^5$ . The impulse is located on the null hypersurface  $\{U = 0\}$  given by

$$Z_2^2 + Z_3^2 + \sigma Z_4^2 = \sigma a^2, \quad (2.3)$$

which is a *nonexpanding* 2-sphere in the de Sitter universe ( $\Lambda > 0$ ) and a hyperboloidal 2-surface in the anti-de Sitter universe ( $\Lambda < 0$ ). Various four-dimensional coordinate parametrizations of these spacetimes have been considered, e.g. in [28]. Physically, the spacetime (2.1), (2.2) describes impulsive gravitational waves as well as impulses of null matter. Purely gravitational waves occur in the case where the vacuum field equations are satisfied. It was demonstrated in [29, 30] that such solutions can be explicitly written as

$$H(z, \phi) = b_0 Q_1(z) + \sum_{m=1}^{\infty} b_m Q_1^m(z) \cos[m(\phi - \phi_m)], \quad (2.4)$$

where  $z = Z_4/a$ ,  $\tan \phi = Z_3/Z_2$  and  $Q_1^m(z)$  are associated Legendre functions of the second kind generated by the relation  $Q_1^m(z) = (-\sigma)^m |1 - z^2|^{m/2} \frac{d^m}{dz^m} Q_1(z)$ . The first term for  $m = 0$ , i.e.,  $Q_1(z) = \frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1$ , represents the simplest axisymmetric Hotta–Tanaka solution [19]. The components with  $m \geq 1$  describe nonexpanding impulsive gravitational

<sup>5</sup> These coordinates are different from those used in the metric (1.4). Since in this paper we will not use the continuous form (1.4), we simplify the notation by not distinguishing them by a bar (as we did in [32]).

waves in an (anti-)de Sitter universe generated by null point sources with an  $m$ -pole structure, localized on the wave front at the singularities  $z = \pm 1$ . See [27, 29, 30] for more details.

Now, the geodesics  $\gamma$  of  $M$  with tangent  $T$  are characterized by the condition that their  $\overline{M}$ -acceleration  $A = \overline{\nabla}_T T$  is everywhere normal to  $M$ . Denoting by  $N$  the normal vector to  $M$  in  $\overline{M}$  with  $g(N, N) = \sigma$ , we hence obtain

$$\overline{\nabla}_T T = -\sigma g(T, \overline{\nabla}_T N)N. \quad (2.5)$$

Using this identity and the constraint (2.2), the explicit form of the geodesic equation was given in [31, equation (28)] as

$$\begin{aligned} \ddot{U} &= -\frac{1}{3} \Lambda U e, \\ \ddot{V} - \frac{1}{2} H \delta' \dot{U}^2 - \delta^{pq} H_{,p} \delta \dot{Z}_q \dot{U} &= -\frac{1}{3} \Lambda V \left( e + \frac{1}{2} G \delta \dot{U}^2 \right), \\ \ddot{Z}_i - \frac{1}{2} H_{,i} \delta \dot{U}^2 &= -\frac{1}{3} \Lambda Z_i \left( e + \frac{1}{2} G \delta \dot{U}^2 \right), \\ \ddot{Z}_4 - \frac{\sigma}{2} H_{,4} \delta \dot{U}^2 &= -\frac{1}{3} \Lambda Z_4 \left( e + \frac{1}{2} G \delta \dot{U}^2 \right), \end{aligned} \quad (2.6)$$

where

$$G := \delta^{pq} Z_p H_{,q} - H, \quad \text{and} \quad e := g(T, T) = \pm 1, 0 \quad (2.7)$$

denotes the normalisation constant for spacelike ( $e = 1$ ), timelike ( $e = -1$ ) and null ( $e = 0$ ) geodesics. Note that  $\dot{\phantom{x}}$  denotes the derivative with respect to an affine parameter  $t$ , which we have suppressed in the equations. Here and in the following, we also adopt the convention that Greek indices  $\alpha, \beta$  take all values 0, 1, 2, 3, 4 while the indices  $p, q, r$  are restricted to the values 2, 3, 4 and  $i, j$  run from 2 to 3 only.

Obviously, these equations reduce to the geodesic equations of the (anti-)de Sitter background off the impulse located at  $\{U = 0\}$ . Also observe that the first equation decouples from the rest of the system and can be easily integrated. Consequently,  $U$  can be used as a parameter of the remaining equations, a fact that is essential for the analysis of the system (2.6) presented in [31, section 4]. In fact, the geodesics of the (anti-)de Sitter background in front and behind the wave impulsive are matched by using a natural ansatz for solutions in the entire spacetime. However, this procedure has to be viewed as being only heuristic since the solution's  $Z_p$ -components are (assumed to be) continuous but not  $C^1$ , while the  $V$ -component is (assumed to be) discontinuous. Consequently, the solutions cannot be plugged back into the equations due to the occurrence of undefined products of distributions, and so the question arises in which sense they actually solve the equations, see the discussion at the end of section 4 of [31]. This situation is similar to the one encountered for impulsive  $pp$ -waves with  $\Lambda = 0$  and we refer to the discussion in [39, section II] as well as to the general discussion in [18].

To resolve this open problem, we now employ a regularisation approach and detail the setting we are working with. To begin with, we replace the Dirac- $\delta$  in the line element (2.1) by a fairly general class of smooth approximations called *model delta nets*. Thus, we chose an arbitrary smooth function  $\rho$  on  $\mathbb{R}$  with unit integral and its support contained in  $[-1, 1]$ . Then, for  $0 < \varepsilon \leq 1$  set

$$\delta_\varepsilon(x) := \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right). \quad (2.8)$$

We now consider, for fixed  $\varepsilon \in (0, 1]$ , the five-dimensional sandwich wave

$$ds_\varepsilon^2 = dZ_2^2 + dZ_3^2 + \sigma dZ_4^2 - 2dUdV + H(Z_2, Z_3, Z_4)\delta_\varepsilon(U)dU^2, \quad (2.9)$$

and define the spacetime of our interest as  $(M, g_\varepsilon)$  given by the constraint (2.2), i.e.,

$$F(U, V, Z_2, Z_3, Z_4) := -2UV + Z_2^2 + Z_3^2 + \sigma Z_4^2 - \sigma a^2 = 0. \quad (2.10)$$

Observe that while the differential  $dF = 2(-V, -U, Z_2, Z_3, \sigma Z_4)$  is independent of  $\varepsilon$ , the normal vector  $N_\varepsilon^\alpha := g_\varepsilon^{\alpha\beta} dF_\beta$  depends on  $\varepsilon$ . Indeed, we choose to work with the *non-normalised* normal vector  $N_\varepsilon$  to  $M$  given by

$$N_\varepsilon = (U, V + HU\delta_\varepsilon(U), Z_p) \quad \text{with} \quad g_\varepsilon(N_\varepsilon, N_\varepsilon) = \sigma a^2 - U^2 H\delta_\varepsilon(U). \quad (2.11)$$

The non-zero Christoffel symbols of (2.9) are given by

$$\Gamma_\varepsilon^V{}_{UU} = -\frac{1}{2}H\delta'_\varepsilon(U), \quad \Gamma_\varepsilon^V{}_{Up} = -\frac{1}{2}H_{,p}\delta_\varepsilon(U), \quad (2.12)$$

$$\Gamma_\varepsilon^i{}_{UU} = -\frac{1}{2}H_{,i}\delta_\varepsilon(U), \quad \Gamma_\varepsilon^4{}_{UU} = -\frac{1}{2}\sigma H_{,4}\delta_\varepsilon(U), \quad (2.13)$$

and the geodesics of  $(M, g_\varepsilon)$  are now characterised by

$$\bar{\nabla}_{T_\varepsilon}^\varepsilon T_\varepsilon = -g_\varepsilon(T_\varepsilon, \bar{\nabla}_{T_\varepsilon}^\varepsilon N_\varepsilon) \frac{N_\varepsilon}{g_\varepsilon(N_\varepsilon, N_\varepsilon)}, \quad (2.14)$$

where (suppressing the parameter) we write  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  for the geodesics with tangent  $T_\varepsilon = (\dot{U}_\varepsilon, \dot{V}_\varepsilon, \dot{Z}_{p\varepsilon})$  and  $\bar{\nabla}^\varepsilon$  denotes the Levi-Civita connection of (2.9). By a straightforward calculation, we now obtain

$$g_\varepsilon(T_\varepsilon, \bar{\nabla}_{T_\varepsilon}^\varepsilon N_\varepsilon) = e + \frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon(U_\varepsilon, Z_{p\varepsilon}) - \dot{U}_\varepsilon (H(Z_{p\varepsilon})\delta_\varepsilon(U_\varepsilon)U_\varepsilon), \quad (2.15)$$

where we have used the abbreviations

$$\tilde{G}_\varepsilon(U, Z_r) := \delta^{pq} H_{,p}(Z_r)\delta_\varepsilon(U)Z_q + H(Z_r)\delta'_\varepsilon(U)U, \quad \text{and} \quad e := g_\varepsilon(T_\varepsilon, T_\varepsilon) = \pm 1, 0. \quad (2.16)$$

Observe that since the  $g_\varepsilon$ -norm of the tangent vector  $T_\varepsilon$  is constant along the geodesic  $\gamma_\varepsilon$ , we have chosen it to be also constant in  $\varepsilon$ , which means that we have fixed the normalisation independently of  $\varepsilon$ . Finally, we obtain the following explicit form of the geodesic equations

$$\begin{aligned}
\ddot{U}_\varepsilon &= - \left( e + \frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon) \right) \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon}, \\
\ddot{V}_\varepsilon - \frac{1}{2} H \delta'_\varepsilon \dot{U}_\varepsilon^2 - \delta^{pq} H_{,p} \delta_\varepsilon \dot{Z}_q^\varepsilon \dot{U}_\varepsilon &= - \left( e + \frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon) \right) \frac{V_\varepsilon + H \delta_\varepsilon U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon}, \\
\ddot{Z}_{i\varepsilon} - \frac{1}{2} H_{,i} \delta_\varepsilon \dot{U}_\varepsilon^2 &= - \left( e + \frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon) \right) \frac{Z_{i\varepsilon}}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon}, \\
\ddot{Z}_{4\varepsilon} - \frac{\sigma}{2} H_{,4} \delta_\varepsilon \dot{U}_\varepsilon^2 &= - \left( e + \frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon) \right) \frac{Z_{4\varepsilon}}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon},
\end{aligned} \tag{2.17}$$

where we have again suppressed the parameter  $t$  as well as the dependencies on the variables. However, note that

$$\begin{aligned}
\delta_\varepsilon &= \delta_\varepsilon(U_\varepsilon(t)), \quad \delta'_\varepsilon = \delta'_\varepsilon(U_\varepsilon(t)), \\
\tilde{G}_\varepsilon &= \tilde{G}_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t)), \quad H = H(Z_{p\varepsilon}(t)), \quad \text{and} \quad H_{,p} = H_{,p}(Z_{q\varepsilon}(t)).
\end{aligned} \tag{2.18}$$

The right hand sides of these equations are considerably more complicated than their ‘distributional counterparts’ in (2.6), the reason being that in the regularised equations the distributional identities  $\delta(U)U = 0$  and  $\delta'(U)U = -\delta(U)$  do not apply. Indeed, the lack of the first one leads to the more complicated form of the normal vector (see (2.11)) and is reflected in the second summand in the denominator of (2.17). On the other hand, the lack of the second one is responsible for the different form of the terms involving  $\tilde{G}_\varepsilon$  compared to the ones involving  $G$  (cf. (2.7)) in (2.6), to which they reduce in the limit  $\varepsilon \rightarrow 0$ . Finally, the terms proportional to  $(H\delta_\varepsilon U_\varepsilon)'$ , which occur in all four equations, vanish in the limit  $\varepsilon \rightarrow 0$  again due to the first identity. The same holds true for the term proportional to  $\tilde{G}_\varepsilon U_\varepsilon$  contained in the first equation. In this sense, the equations (2.17) converge weakly to (2.6) as  $\varepsilon \rightarrow 0$ .

The more complicated form of the system (2.17), in particular, results in the fact that the  $U_\varepsilon$  equation does *not decouple* from the rest of the system and consequently  $U$  cannot be used as a parameter along the geodesics. This issue greatly complicates our analysis. However, the  $V_\varepsilon$  equation is still linear and decoupled, hence can be simply integrated once the rest of the system is solved.

### 3. Existence, uniqueness and completeness of geodesics

In this section, we prove an existence and uniqueness result for solutions of the initial value problem for (2.17) that additionally guarantees that the geodesics that enter the sandwich region  $U \in [-\varepsilon, \varepsilon]$  at one side exist long enough to leave it on the other side. This allows us to obtain global solutions of the geodesic equations, since outside the strip  $\{-\varepsilon \leq U \leq \varepsilon\}$  the spacetime coincides with the background (anti-)de Sitter universe.

Observe that for fixed  $\varepsilon$ , the equations are smooth and hence a local solution is guaranteed to exist. However, the time of existence might depend on  $\varepsilon$  and in principle could even shrink to zero as  $\varepsilon \rightarrow 0$ . So the main objective here is to provide a result that guarantees that the time of existence is independent of  $\varepsilon$ , and large enough such that the solutions pass through the regularisation sandwich (at least for all small  $\varepsilon$ ). To this end, we employ a fixed point argument in the spirit of [36, appendix A] based on Weissinger’s fixed point theorem [43]. However, the significant increase in the complexity of the equations forces the use of new ideas to derive the required estimates. In particular, since it is not possible to use the  $U$ -coordinate as a parameter along the geodesics, the ‘singular terms’ such as  $\delta_\varepsilon$  are composed with the  $U$ -coordinate of the solution  $U_\varepsilon$ , see (2.18). We have separated the technical proofs



preparing the grounds for the application of the fixed point theorem from the main line of arguments and have deferred them to appendix A.

Let us start by giving the general setup. Consider any geodesic

$$\gamma = (U, V, Z_p) \quad (3.1)$$

on the background (anti-)de Sitter universe *without* impulsive wave but reaching  $U = 0$ . All other geodesics are not of interest to the present analysis and will be dealt with separately. We choose an affine parameter  $t$  in such a way that  $U(t = 0) = 0$  and assume  $\dot{\gamma}$  to be normalised by  $e = \pm 1, 0$ . Such geodesics can explicitly be written as

$$U = t, \quad U = a\dot{U}^0 \sinh(t/a), \quad U = a\dot{U}^0 \sin(t/a), \quad (3.2)$$

in the cases  $\sigma e = 0$ ,  $\sigma e < 0$  and  $\sigma e > 0$ , respectively, see [31, equation (33)]. Recall that the case  $\sigma e = 0$  corresponds to null geodesics in both de Sitter and anti-de Sitter space, while the case  $\sigma e < 0$  corresponds to timelike geodesics in de Sitter as well as to spacelike geodesics in anti-de Sitter spacetime, and finally  $\sigma e > 0$  describes spacelike geodesics in de Sitter and timelike geodesics in anti-de Sitter space. Without loss of generality, we assume the constant  $\dot{U}^0$  to be positive<sup>6</sup> so that in all three cases  $U$  is increasing (at least for  $t \in [-a\pi/2, a\pi/2]$ ). It is thus most convenient to prescribe initial data at  $t = 0$ , that is, we set

$$\gamma(t = 0) =: (0, V^0, Z_p^0), \quad \dot{\gamma}(t = 0) =: (\dot{U}^0 > 0, \dot{V}^0, \dot{Z}_p^0), \quad (3.3)$$

where the constants satisfy the constraints

$$\begin{aligned} (Z_2^0)^2 + (Z_3^0)^2 + \sigma(Z_4^0)^2 - 2U^0 V^0 &= \sigma a^2, \\ Z_2^0 \dot{Z}_2^0 + Z_3^0 \dot{Z}_3^0 + \sigma Z_4^0 \dot{Z}_4^0 - V^0 \dot{U}^0 - U^0 \dot{V}^0 &= 0, \end{aligned} \quad (3.4)$$

which are simply consequences of the fact that we are dealing with  $\gamma$  on the (anti-)de Sitter manifold, see (2.2). Note, however, that  $U^0 = 0$ , so the last term on the left hand side of either condition actually vanishes. In addition, the normalisation condition

$$-2\dot{U}^0 \dot{V}^0 + (\dot{Z}_2^0)^2 + (\dot{Z}_3^0)^2 + \sigma(\dot{Z}_4^0)^2 = e \quad (3.5)$$

holds.

Now, we start to think of  $\gamma$  as a geodesic in the impulsive wave spacetime (2.1), (2.2) ‘*in front*’ of the impulse, that is, for  $U < 0$ . Also,  $\gamma$  is a geodesic in the regularised spacetime (2.9), (2.10) ‘*in front*’ of the sandwich wave, that is, for  $U \leq -\varepsilon$ . We will call it the ‘*seed geodesic*’ and denote the affine parameter time when  $\gamma$  enters this regularisation wave region by  $\alpha_\varepsilon$ ,

$$U(t = \alpha_\varepsilon) = -\varepsilon. \quad (3.6)$$

Observe that, by continuity of  $\gamma$ ,  $\alpha_\varepsilon \rightarrow 0$  from below as  $\varepsilon \rightarrow 0$ . More precisely, we have  $\alpha_\varepsilon = -\varepsilon$ ,  $\alpha_\varepsilon = -a \operatorname{Arcsinh}(\varepsilon/a\dot{U}^0)$  and  $\alpha_\varepsilon = -\arcsin(\varepsilon/a\dot{U}^0)$ , respectively, for the three cases in (3.2), leading to  $\alpha_\varepsilon = -\varepsilon/\dot{U}^0 + O(\varepsilon^3)$  in the latter cases and hence overall

$$-C\varepsilon \leq \alpha_\varepsilon < 0, \quad (3.7)$$

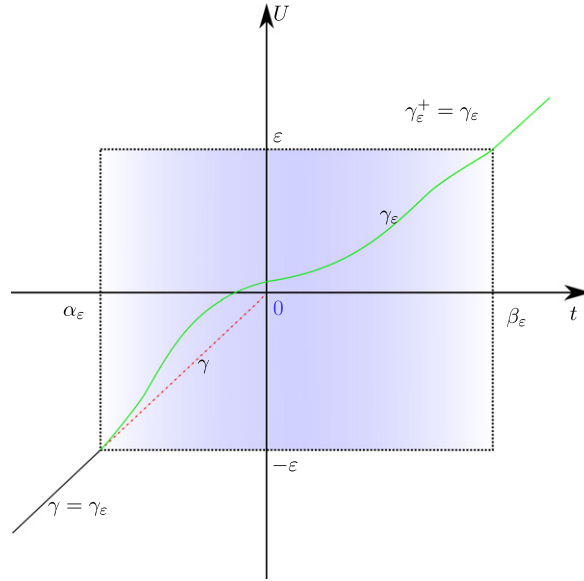
for some positive constant  $C$ .

To investigate the geodesics in the regularised spacetime (2.9), (2.10), which is the main topic of this paper, we follow  $\gamma$  up to the beginning of the wave zone, i.e., up to  $t = \alpha_\varepsilon$ , and then extend it (smoothly) to a geodesic

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon}) \quad (3.8)$$

by solving the regularised geodesic equations (2.17). This means  $\gamma_\varepsilon$  at  $\alpha_\varepsilon$  assumes the data (see figure 1)

<sup>6</sup> The time-reversed case with  $\dot{U}^0 < 0$  can be treated in complete analogy.



**Figure 1.** The  $U$ -component of the ‘seed geodesic’  $\gamma$  is depicted in black until it reaches the regularisation sandwich at parameter time  $t = \alpha_\varepsilon$ , i.e.,  $U(\alpha_\varepsilon) = -\varepsilon$ . In the background spacetime, it would continue as the dotted red line to  $U = 0$  at  $t = 0$ ; however, in the regularised spacetime, it continues as  $\gamma_\varepsilon$  of (3.8) (depicted in green), solving the equations (2.17) with data (3.9). Theorem 3.2 guarantees that  $\gamma_\varepsilon$  (for  $\varepsilon$  small) leaves the regularisation sandwich at  $t = \beta_\varepsilon$  and continues as background geodesic  $\gamma_\varepsilon^+$  of (3.14) with data (3.13).

$$\gamma_\varepsilon(\alpha_\varepsilon) := \gamma(\alpha_\varepsilon) =: (-\varepsilon, V_\varepsilon^0, Z_{p\varepsilon}^0), \quad \dot{\gamma}_\varepsilon(\alpha_\varepsilon) := \dot{\gamma}(\alpha_\varepsilon) =: (\dot{U}_\varepsilon^0, \dot{V}_\varepsilon^0, \dot{Z}_{p\varepsilon}^0). \tag{3.9}$$

Observe that from the smoothness of  $\gamma$ , the data  $\gamma_\varepsilon(\alpha_\varepsilon)$  and  $\dot{\gamma}_\varepsilon(\alpha_\varepsilon)$  converge to  $\gamma(0)$  and  $\dot{\gamma}(0)$ , respectively, as  $\varepsilon \rightarrow 0$ . In fact, by a mean value argument and (3.7) we even have

$$\begin{aligned} |(-\varepsilon, V_\varepsilon^0, Z_{p\varepsilon}^0) - (0, V^0, Z_p^0)| &\leq \sup_{\alpha_\varepsilon \leq t \leq 0} \|\dot{\gamma}(t)\|_h |\alpha_\varepsilon| \leq C\varepsilon, \\ |(\dot{U}_\varepsilon^0, \dot{V}_\varepsilon^0, \dot{Z}_{p\varepsilon}^0) - (\dot{U}^0, \dot{V}^0, \dot{Z}_p^0)| &\leq \sup_{\alpha_\varepsilon \leq t \leq 0} \|\dot{\gamma}(t)\|_h |\alpha_\varepsilon| \leq C\varepsilon, \end{aligned} \tag{3.10}$$

where  $h$  is any Riemannian background metric and  $C$  is again a generic constant.

Based on theorem A.6 proved in appendix A, we may now state and prove our main results on the existence, uniqueness and completeness of the geodesics in the *regularised* spacetime (2.9), (2.10):

**Theorem 3.1 (Existence and uniqueness).** *Consider the geodesic equations (2.17) with initial data (3.9). Then, for all  $\varepsilon$  small enough (more precisely, for all  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  is constrained by (A.14)), there exists a unique smooth solution  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ , where  $\eta$  is independent of  $\varepsilon$  (and explicitly given by (A.12)).*

**Proof.** As noted in the appendix, it suffices to first solve the (simplified) model system (A.1) for  $(u_\varepsilon, z_\varepsilon)$ . Identifying  $(U_\varepsilon, Z_{p\varepsilon})$  with  $(u_\varepsilon, z_\varepsilon)$ , the initial data (3.3) and (3.9) transfer to the

data (A.3), (A.2). Then, (3.10) implies (A.4) and theorem A.6 applies to provide a unique smooth solution  $(U_\varepsilon, Z_{p\varepsilon})$  of (2.17), (3.9) on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ , with  $\eta$  given by (A.12).

Finally, we solve the linear equation for  $V_\varepsilon$  to obtain the claimed smooth solution  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ .  $\square$

Next, we make sure that the solutions just obtained, which by construction enter the wave zone at  $U_\varepsilon = -\varepsilon$  at parameter time  $t = \alpha_\varepsilon$  with positive speed  $\dot{U}_\varepsilon^0$ , in fact *leave the sandwich region*, that is, they reach  $U_\varepsilon = \varepsilon$  within their time of existence  $\eta$ . Consequently, they naturally extend to the background (anti-)de Sitter universe ‘behind’ the sandwich region. Observe that here it is vital that  $\eta$  in (A.12) is independent of  $\varepsilon$ .

**Theorem 3.2 (Extension of geodesics).** *The unique smooth geodesics  $\gamma_\varepsilon$  of theorem 3.1 extend to geodesics of the background (anti-)de Sitter spacetime ‘behind’ the sandwich wave zone.*

**Proof.** Let  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  be the unique solution of (2.17), (3.9) given by theorem 3.1. By the definition of the ‘solution space’  $\mathfrak{X}_\varepsilon$  (see (A.5)), we obtain

$$U_\varepsilon(\alpha_\varepsilon + \eta) = -\varepsilon + \int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \eta} \dot{U}_\varepsilon(s) ds \geq -\varepsilon + \frac{\eta}{2} \dot{U}^0 \geq -\varepsilon + 3\varepsilon \geq \varepsilon, \tag{3.11}$$

since  $\varepsilon \leq \eta \dot{U}^0/6$  from (A.13).

So for such  $\varepsilon$ , the geodesic  $\gamma_\varepsilon$  leaves the wave zone and extends to a geodesic of the background spacetime since the geodesic equations (2.17) coincide with the geodesic equations of the background (anti-)de Sitter spacetime for  $U_\varepsilon \geq \varepsilon$ .  $\square$

Recall that by construction the global geodesics  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  with data (3.9) of theorem 3.2 for  $t \leq \alpha_\varepsilon$ , i.e. ‘in front’ of the sandwich, coincide for all (small)  $\varepsilon$  with the single ‘seed geodesic’  $\gamma$  with data (3.3). However, ‘behind’ the sandwich, the geodesics  $\gamma_\varepsilon$  for each  $\varepsilon$  will coincide with a *different* geodesic of the *background* spacetime. To make this observation more precise, we define the affine parameter time when the geodesic  $\gamma_\varepsilon$  leaves the sandwich wave zone by  $\beta_\varepsilon$ ,

$$U_\varepsilon(t = \beta_\varepsilon) = \varepsilon \tag{3.12}$$

and denote the corresponding values of  $\gamma_\varepsilon$  at  $\beta_\varepsilon$  by

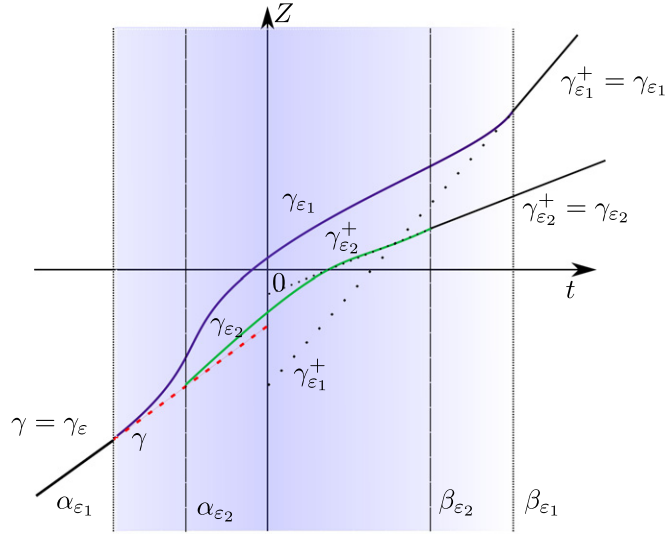
$$\gamma_\varepsilon(\beta_\varepsilon) =: (\varepsilon, V_\varepsilon^{0+}, Z_{p\varepsilon}^{0+}), \quad \dot{\gamma}_\varepsilon(\beta_\varepsilon) =: (\dot{U}_\varepsilon^{0+}, \dot{V}_\varepsilon^{0+}, \dot{Z}_{p\varepsilon}^{0+}). \tag{3.13}$$

Then, for  $t \geq \beta_\varepsilon$ , the geodesic  $\gamma_\varepsilon$  will coincide with the geodesic

$$\gamma_\varepsilon^+ = (U_\varepsilon^+, V_\varepsilon^+, Z_{p\varepsilon}^+) \tag{3.14}$$

of the *background* (anti-)de Sitter space with the data (3.13), see figure 1 and also figure 2. Observe that the data (3.13) is normalised and constrained, more precisely, we have:

**Remark 3.3 (Preservation of constraints and normalisation).** The fact that the data (3.3) of the ‘seed geodesic’  $\gamma$  in (3.1) is constrained and normalised, i.e., it satisfies (3.4) and (3.5), implies that the data (3.9) of  $\gamma_\varepsilon$  is also constrained and normalised. Clearly, these conditions are propagated by  $\gamma_\varepsilon$  being a solution to (2.17). Moreover, at  $t = \beta_\varepsilon$  the regularised metric  $g_\varepsilon$



**Figure 2.** The  $Z$ -components of two solutions  $\gamma_{\varepsilon_1}$  (purple) and  $\gamma_{\varepsilon_2}$  (green) of the regularised equations (2.17) with the same ‘seed geodesic’  $\gamma$  of (3.1) are depicted for  $\varepsilon_1 > \varepsilon_2$ . The regularisation sandwich is given by  $[\alpha_{\varepsilon_1}, \beta_{\varepsilon_1}]$  and  $[\alpha_{\varepsilon_2}, \beta_{\varepsilon_2}]$ , respectively. The dotted red line represents the  $Z$ -components of  $\gamma$ , while the black dotted lines are said components of  $\gamma_{\varepsilon_1}^+$ .

and the background metric  $g$  coincide and so the data (3.13) of  $\gamma_{\varepsilon}^+$  is constrained and normalised with respect to the background spacetime.

While the preservation of the constraints confirms the consistency of our construction, the preservation of the normalisation, in particular, implies that the causal character of  $\gamma_{\varepsilon}$  (and  $\gamma_{\varepsilon}^+$ ) is the same as that of the ‘seed’  $\gamma$ .

Note that the geodesic  $\gamma_{\varepsilon}^+$  ‘behind’ the regularisation sandwich depends on  $\varepsilon$  (only) via this initial data. Interestingly, as we will detail in section 5, for  $t > 0$  the family of geodesics  $\gamma_{\varepsilon}$  of the regularised spacetime converges for  $\varepsilon \rightarrow 0$  to a unique geodesic  $\gamma^+$  in the background with data given by the limits of (3.13). This will explicitly describe the effect of the impulsive gravitational wave on the geodesics in the (anti-)de Sitter universe.

In the remainder of this section, we will formulate completeness results for the regularised spacetimes. First, we remark that actually our results allow us to make the smallness assumption on  $\varepsilon$  precise:  $\varepsilon$  has to be smaller than  $\varepsilon_0$ , constrained by (A.14). This, however, means that the specific  $\varepsilon$  for which, on a certain geodesic,  $\gamma_{\varepsilon}$  becomes complete depends on its data (3.3), i.e., on the ‘seed geodesic’, and there is in general *no global*  $\varepsilon$  from which, on *all* geodesics, the spacetime is complete. A ‘global’ completeness result in the spirit of [37], however, can be obtained by using the geometric theory of generalised functions [17 chapter 3] and we reserve its detailed presentation for future work.

To formulate completeness results in our current setting, the dependence of  $\varepsilon$  on the data discussed above also makes it necessary to be careful about global effects. Indeed, geodesics in the background spacetimes with  $\sigma_e > 0$ , that is, spacelike geodesics in de Sitter space and timelike geodesics in anti-de Sitter space, are periodic. Consequently, geodesics  $\gamma_{\varepsilon}$  in the regularised spacetimes constructed from such ‘seed geodesics’  $\gamma$  will share their causal character (see remark 3.3) and hence cross the wave zone infinitely often and we (have to) (re)apply theorems 3.1, 3.2 again and again. However, note that the geodesics may enter the regularisation sandwich region

each time with different data. So the  $\varepsilon$  for which theorem 3.2 applies may in principal become smaller and smaller on successive crossings with no positive infimum. In such a case, given an initial geodesic  $\gamma$  as in (3.1) and given any finite number  $N$  of crossings, we can specify an  $\varepsilon$  for which the geodesic  $\gamma_\varepsilon$  extends to cross the wave zone  $N$  times. However, we cannot give a (global)  $\varepsilon$  for which the geodesic  $\gamma_\varepsilon$  extends to all (positive) values of its affine parameter. Consequently, we prefer to avoid multiple crossings of the impulse in the formulation of our results by restricting to causal geodesics in the de Sitter case (neglecting unphysical tachyons only) and working with the universal covering spacetime in the case of anti-de Sitter space.

Note also that in our discussion so far (see the specification of the ‘seed geodesics’  $\gamma$  at the beginning of this section), we have exclusively dealt with geodesics with non-constant  $U$ -components. Hence, it remains to deal with geodesics travelling parallel to the surface  $\{U = 0\}$ . In the case where  $U = \text{const} \neq 0$ , the geodesic will never enter the sandwich region of the regularised spacetime, provided  $\varepsilon$  is small enough. Staying entirely on the constant curvature background, such a geodesic clearly is complete. To discuss the geodesics with  $U = 0$ , we observe that the surface  $\{U = 0\}$  is totally geodesic (not only in the background but also) in the regularised spacetime, which can be seen from the  $U$ -component of the geodesic equations (2.17) (see also [32], the discussion prior to theorem 3.6). Hence, such geodesics have trivial  $U$ -components and consequently the system (2.17) reduces to the background geodesic equations, which again leads to completeness. So, we finally arrive at:

**Theorem 3.4 (Causal completeness for positive  $\Lambda$ ).** *Every causal geodesic in the entire class of regularised nonexpanding impulsive gravitational waves propagating in a de Sitter universe (i.e., (2.9), (2.10) with  $\Lambda > 0$  and a smooth profile function  $H$ ) is complete, provided the regularisation parameter  $\varepsilon$  is small enough.*

**Theorem 3.5 (Completeness for negative  $\Lambda$ ).** *Every geodesic in the entire class of regularised nonexpanding impulsive gravitational waves propagating in the universal cover of an anti-de Sitter universe (i.e., (2.9), (2.10), with  $\Lambda < 0$  and a smooth profile function  $H$ ) is complete, provided the regularisation parameter  $\varepsilon$  is small enough.*

**Remark 3.6 (Non-smooth profiles  $H$ ).** In the case where the profile function  $H$  in the metric (2.9) is non-smooth—which, in fact, occurs in physically interesting models where  $H$  possesses poles on the wave front at  $z = \pm 1$ , see (2.4)—our method still applies, but some care is needed. Indeed, if a ‘seed geodesic’  $\gamma$  hits the surface at  $U = 0$  away from the poles we may work on an open subset of the spacetime with the poles of  $H$  removed. We only have to choose the constant  $C_1$  in (A.5) so that it is small enough that the curves in the ‘solution space’  $\mathcal{X}_\varepsilon$  stay away from the poles, and then theorems 3.1 and 3.2 still apply. The only ‘seed geodesics’  $\gamma$  that do *not* allow for such a treatment are those that *directly head* at the poles (i.e.,  $\gamma$  of (3.1) hits the pole at  $\gamma(0)$ ), which is in complete agreement with physical expectations.

#### 4. Boundedness properties of geodesics

In this section, we establish the boundedness properties of the global geodesics in the regularised spacetimes obtained in the previous section. In particular, we will prove local boundedness of  $\gamma_\varepsilon$  and of some components of its velocity *uniformly in  $\varepsilon$* . These properties will be essential in the next section where we derive the limits of  $\gamma_\varepsilon$ .

To begin with, observe that the fixed point argument of the appendix already gives uniform boundedness of the  $U$ - and  $Z_p$ -components together with their first order derivatives. On the other hand, the  $V$ -component was not involved in the fixed point argument and we have to establish its

boundedness properties by using the  $V$ -component of the geodesic equation (2.17). Since this equation involves a  $\delta'$  term, which is *not* multiplied by  $U_\varepsilon$ , the  $V$ -component of  $\dot{\gamma}_\varepsilon$  is not uniformly bounded in the regularisation sandwich. However, we will show that  $\dot{V}_\varepsilon(\beta_\varepsilon)$ , i.e., the  $V$ -speed when the geodesic *leaves* the regularisation sandwich is uniformly bounded.

**Proposition 4.1 (Uniform boundedness of geodesics).** *The global geodesics  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  of Theorem 3.2 satisfy*

- (i)  $U_\varepsilon$  and  $\dot{U}_\varepsilon$  are locally uniformly bounded in  $\varepsilon$ ,
- (ii)  $Z_{p\varepsilon}$  and  $\dot{Z}_{p\varepsilon}$  are locally uniformly bounded in  $\varepsilon$ ,
- (iii)  $V_\varepsilon$  is locally uniformly bounded in  $\varepsilon$  and
- (iv)  $\dot{V}_\varepsilon(\beta_\varepsilon)$  is uniformly bounded in  $\varepsilon$ .

Observe that from lemma A.2 the time  $\beta_\varepsilon$  when the geodesic leaves the regularisation strip satisfies  $\beta_\varepsilon \leq \alpha_\varepsilon + 4\varepsilon/\dot{U}^0 \rightarrow 0$ .

**Proof.** Items (i) and (ii) are immediate from theorem A.6.

To deal with the  $V$ -component, we write

$$|V_\varepsilon(t)| \leq |V^0| + |\dot{V}^0| |t - \alpha_\varepsilon| + \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |\ddot{V}_\varepsilon(r)| \, dr ds \tag{4.1}$$

and estimate each term in the differential equation (2.17) for  $V_\varepsilon$ . To begin with, we claim that

$$V_\varepsilon \text{ is bounded on } [\alpha_\varepsilon, \beta_\varepsilon]. \tag{4.2}$$

Indeed, proceeding as in the proof of proposition A.3 and, in particular, using (A.5) and lemmata A.1, A.2, we obtain

$$\begin{aligned} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |\ddot{V}_\varepsilon(r)| \, dr ds &\leq \left(\frac{4\varepsilon}{\dot{U}^0}\right)^2 \left(\frac{1}{2} \|H\|_\infty \frac{1}{\varepsilon^2} \|\rho'\|_\infty \frac{9}{4} (\dot{U}^0)^2 \right. \\ &\quad \left. + 3 \|DH\|_\infty \frac{1}{\varepsilon} \|\rho\|_\infty (|\dot{Z}^0| + C_2) \frac{3}{2} \dot{U}^0\right) \\ &\quad + \frac{2}{a^2} \left(\frac{4\varepsilon}{\dot{U}^0}\right)^2 \left(1 + \frac{9}{8} (\dot{U}^0)^2 \|\tilde{G}_\varepsilon\|_\infty + \frac{3}{2} \dot{U}^0 \|(H\delta_\varepsilon U_\varepsilon)\|_\infty\right) \|H\|_\infty \|\rho\|_\infty \\ &\quad + \frac{2}{a^2} \left(1 + \frac{9}{8} (\dot{U}^0)^2 \|\tilde{G}_\varepsilon\|_\infty + \frac{3}{2} \dot{U}^0 \|(H\delta_\varepsilon U_\varepsilon)\|_\infty\right) \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |V_\varepsilon(r)| \, dr ds \\ &\leq C + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |V_\varepsilon(r)| \left(1 + \frac{\chi_\varepsilon(r)}{\varepsilon}\right) \, dr ds, \end{aligned} \tag{4.3}$$

where  $C$  is some generic constant and  $\chi_\varepsilon$  is the characteristic function of  $[\alpha_\varepsilon, \beta_\varepsilon]$ . Moreover, we have used  $\|\tilde{G}_\varepsilon\|_\infty = O(1/\varepsilon) = \|(H\delta_\varepsilon U_\varepsilon)\|_\infty$ . So, overall we obtain from a generalization of Gronwall's inequality due to Bykov [3, theorem 11.1]<sup>7</sup>

$$|V_\varepsilon| \leq C(1 + \varepsilon) \exp\left(\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \int_{\alpha_\varepsilon}^s C\left(1 + \frac{1}{\varepsilon}\right) \, dr ds\right) \leq C e^C, \tag{4.4}$$

establishing the claim (4.2).

Now, we may prove (iv): Writing  $\dot{V}_\varepsilon(\beta_\varepsilon)$  also as an integral and using the boundedness of  $V_\varepsilon$  on  $[\alpha_\varepsilon, \beta_\varepsilon]$ , we may proceed as in (4.3). Again, we obtain uniform boundedness of all the

<sup>7</sup> Note that it suffices that  $\chi_\varepsilon$  is integrable rather than continuous.

terms but the first one, which involves the  $\delta'_\varepsilon$  term. To estimate this one, we use integration by parts to obtain

$$\begin{aligned} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} H(Z_\varepsilon(s)) \delta'_\varepsilon(U_\varepsilon(s)) \dot{U}_\varepsilon^2(s) ds &= \int_{-\varepsilon}^\varepsilon H(Z_\varepsilon(U_\varepsilon^{-1}(r))) \delta'_\varepsilon(r) \dot{U}_\varepsilon(U_\varepsilon^{-1}(r)) dr \\ &= \delta_\varepsilon(r) H(Z_\varepsilon(U_\varepsilon^{-1}(r))) \dot{U}_\varepsilon(U_\varepsilon^{-1}(r)) \Big|_{-\varepsilon}^\varepsilon - \int_{-\varepsilon}^\varepsilon \delta_\varepsilon(r) (H(Z_\varepsilon(U_\varepsilon^{-1}(r))) \dot{U}_\varepsilon(U_\varepsilon^{-1}(r)))' dr. \end{aligned} \tag{4.5}$$

Now, the first term vanishes and the second one is bounded independently of  $\varepsilon$  by (i), (ii), the fact that  $\dot{U}_\varepsilon^{-1}$  is uniformly bounded away from zero by (A.5), and since  $\ddot{U}_\varepsilon$  is uniformly bounded by (2.17). This establishes (iv).

To prove (iii), it remains to show that  $V_\varepsilon$  is bounded on any compact interval disjoint from  $(\alpha_\varepsilon, \beta_\varepsilon)$ . This follows from the fact that  $\gamma_\varepsilon$  outside of  $(\alpha_\varepsilon, \beta_\varepsilon)$  solves the geodesic equation of the background spacetime of constant curvature, and that  $\gamma_\varepsilon(\beta_\varepsilon)$  and  $\dot{\gamma}_\varepsilon(\beta_\varepsilon)$  are uniformly bounded by (i), (ii), (iv) and (4.2).  $\square$

### 5. Limiting geodesics

In this final section, we consider the limit  $\varepsilon \rightarrow 0$  of the unique global smooth geodesics  $\gamma_\varepsilon$  of the regularised spacetime (2.9), (2.10) obtained in section 3 (theorems 3.1, 3.2). This physically amounts to *explicitly determining the geodesics of the distributional form* (2.1), (2.2) of all nonexpanding impulse gravitational waves propagating in an (anti-)de Sitter universe. In particular, we will prove that the geodesics  $\gamma_\varepsilon$  converge to the geodesics of the background (anti-)de Sitter spacetime with appropriate *but different* data on either side of the impulse ( $U < 0$  and  $U > 0$ , respectively), which from a global point of view amounts to a convergence of  $\gamma_\varepsilon$  to the geodesics of the background, which have to be *matched appropriately* across the impulse. The technical calculation of the limits is given in appendix B.

To make our claims on the convergence of  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  precise, we introduce the following notation for the geodesics of the background (anti-)de Sitter universe: Let  $\gamma = (U, V, Z_p)$  be a ‘seed geodesic’ as in (3.1), that is,  $U(t=0) = 0$  and  $\gamma$  assumes the data (3.3), i.e.,

$$\gamma(0) = (0, V^0, Z_p^0), \quad \text{and} \quad \dot{\gamma}(0) = (\dot{U}^0 > 0, \dot{V}^0, \dot{Z}_p^0), \tag{5.1}$$

where the constants satisfy the constraints (3.4) and are normalised as in (3.5). Furthermore, let  $\gamma^+ = (U^+, V^+, Z_p^+)$  be a geodesic of the background again crossing  $U = 0$  at  $t = 0$ , i.e., with  $U^+(t=0) = 0$  and with data

$$\gamma^+(0) = (0, B, Z_p^0), \quad \text{and} \quad \dot{\gamma}^+(0) = (\dot{U}^0, C, A_p), \tag{5.2}$$

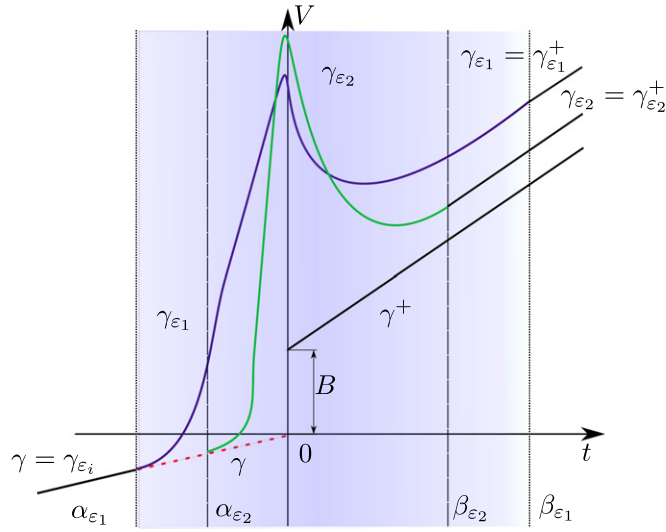
where we define

$$A_p := \lim_{\varepsilon \rightarrow 0} \dot{Z}_{p\varepsilon}(\beta_\varepsilon), \quad B = \lim_{\varepsilon \rightarrow 0} V_\varepsilon(\beta_\varepsilon), \quad C = \lim_{\varepsilon \rightarrow 0} \dot{V}_\varepsilon(\beta_\varepsilon). \tag{5.3}$$

Recall that  $\beta_\varepsilon \leq \alpha_\varepsilon + 4\varepsilon/\dot{U}^0 \rightarrow 0$  is defined to be the time when the regularised geodesic  $\gamma_\varepsilon$  leaves the regularisation strip, i.e.,  $U_\varepsilon(\beta_\varepsilon) = \varepsilon$ . Finally, we define  $\tilde{\gamma} = (\tilde{U}, \tilde{V}, \tilde{Z}_p)$  by

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t), & t \leq 0 \\ \gamma^+(t), & t > 0. \end{cases} \tag{5.4}$$

We will show that  $\gamma_\varepsilon$  converges to the ‘matched geodesics’  $\tilde{\gamma}$  of the impulsive spacetime, which from now on we will also call the ‘limiting geodesics’ with ‘past branch’  $\gamma$  and ‘future



**Figure 3.** The  $V$ -components of  $\gamma_{\varepsilon_1}$  (purple) and  $\gamma_{\varepsilon_2}$  (green) for  $\varepsilon_1 > \varepsilon_2$  are depicted. They converge to the ‘limiting geodesic’  $\tilde{\gamma}$  the ‘future branch’  $\gamma^+$  of which is separated from its ‘past branch’  $\gamma$  (black outside and dotted red inside the regularisation sandwich) by the ‘jump’  $B$  calculated in proposition 5.3. The ‘jump’  $C$  in  $\dot{V}$  is indicated by the different  $V$ -slopes of the ‘past branch’  $\gamma$  and the ‘future branch’  $\gamma^+$ .

branch’  $\gamma^+$ , see also figure 3. Note that the respective notion of convergence of the individual components of  $\gamma_\varepsilon$  will differ, subject to the regularity of the respective components of the ‘limiting geodesics’. Indeed,  $\tilde{\gamma} = (\tilde{U}, \tilde{V}, \tilde{Z}_p)$  has a smooth first component  $U$ , while  $Z_p$  is continuous with a finite jump in  $\dot{Z}_p$  (determined from  $A_p$ ) across the impulse at  $t = 0$ , and  $V$  is even discontinuous across  $t = 0$  with a finite jump in  $V$  and  $\dot{V}$  (determined from the coefficients  $B$  and  $C$ , respectively).

Observe that, at the moment, we only know the limits in (5.3) to exist for subsequences (by the uniform boundedness of  $\dot{Z}_{p\varepsilon}$ ,  $V_\varepsilon$  and  $\dot{V}_\varepsilon(\beta_\varepsilon)$ , cf. proposition 4.1) and hence  $A_p$ ,  $B$  and  $C$  need not be uniquely defined. We will, however, prove convergence and we will derive explicit expressions for  $A_p$ ,  $B$  and  $C$  in proposition 5.3, below. But first, we state and prove the main assertion on the limits of the geodesics in the regularised spacetime:

**Theorem 5.1.** *The geodesics  $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$  of the regularised spacetime derived in theorem 3.2 converge to the ‘limiting geodesics’  $\tilde{\gamma}$  of (5.4) in the following sense:*

- (i)  $U_\varepsilon \rightarrow \tilde{U}$  in  $\mathcal{C}^1$ ,
- (ii)  $Z_{p\varepsilon} \rightarrow \tilde{Z}_p$  locally uniformly,  $\dot{Z}_{p\varepsilon} \rightarrow \dot{\tilde{Z}}_p$  in distributions and uniformly on compact intervals not containing  $t = 0$ ,
- (iii)  $V_\varepsilon \rightarrow \tilde{V}$  in distributions and in  $\mathcal{C}^1$  on compact intervals not containing  $t = 0$ .

Observe that  $\dot{\tilde{Z}}_p$  is discontinuous across  $t = 0$ , hence convergence of  $\dot{Z}_{p\varepsilon}$  cannot be uniform on any interval containing  $t = 0$ <sup>8</sup>, and the same holds true for  $V_\varepsilon$  and  $\dot{V}_\varepsilon$ .

<sup>8</sup> Unless  $t = 0$  is the right endpoint of the interval. This, however, is an artefact due to our choice in setting  $\tilde{\gamma}(0) = \gamma(0)$ , cf. (5.4).



**Proof.** First, we consider the  $U$ -component of  $\gamma_\varepsilon$  in the interval  $[\alpha_\varepsilon, \beta_\varepsilon]$ , where we have from the geodesic equations (2.17) resp. (2.6)

$$|U_\varepsilon(t) - U(t)| \leq \varepsilon + |e| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left| \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon} - \frac{U}{\sigma a^2} \right| dr ds + \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left| \frac{\frac{1}{2} U_\varepsilon \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - U_\varepsilon \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon} \right| dr ds =: \varepsilon + |e|I + II. \tag{5.5}$$

To estimate  $I$ , observe that (cf. the proof of lemma A.1)

$$\left| \frac{1}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon} - \frac{1}{\sigma a^2} \right| \leq \frac{2}{a^4} \varepsilon \|H\|_\infty \|\rho\|_\infty \leq C\varepsilon, \tag{5.6}$$

with  $C$  as a generic constant, and consequently from (A.5)

$$I \leq \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left( \left| \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H \delta_\varepsilon} - \frac{U_\varepsilon}{\sigma a^2} \right| + \left| \frac{U_\varepsilon}{\sigma a^2} - \frac{U}{\sigma a^2} \right| \right) dr ds \leq \eta^2(\varepsilon + C_1)C\varepsilon + \frac{1}{a^2} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |U_\varepsilon - U| dr ds. \tag{5.7}$$

For the term  $II$ , we obtain, as in the proof of proposition A.3 (cf. (A.16), (A.18)), that  $II \leq C\varepsilon$ . Hence, overall

$$|U_\varepsilon(t) - U(t)| \leq C\varepsilon + \frac{|e|}{a^2} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |U_\varepsilon - U| dr ds, \tag{5.8}$$

and so again by Bykov’s inequality  $|U_\varepsilon(t) - U(t)| = O(\varepsilon)$ . In the same way, we see that also  $|\dot{U}_\varepsilon(t) - \dot{U}(t)| = O(\varepsilon)$  and so

$$\sup_{\alpha_\varepsilon \leq t \leq \beta_\varepsilon} |U_\varepsilon(t) - U(t)| + |\dot{U}_\varepsilon(t) - \dot{U}(t)| \rightarrow 0. \tag{5.9}$$

We now turn to the  $Z_p$ -components on the interval  $[\alpha_\varepsilon, \beta_\varepsilon]$ . We have

$$\sup_{\alpha_\varepsilon \leq t \leq \beta_\varepsilon} |Z_{p\varepsilon}(t) - \tilde{Z}_p(t)| \leq \sup_{\alpha_\varepsilon \leq t \leq \beta_\varepsilon} |Z_{p\varepsilon}(t) - Z_{p\varepsilon}^0| + \sup_{\alpha_\varepsilon \leq t \leq \beta_\varepsilon} |Z_{p\varepsilon}^0 - \tilde{Z}_p(t)| \rightarrow 0 \quad (\varepsilon \rightarrow 0), \tag{5.10}$$

where  $Z_{p\varepsilon}^0 = Z_p(\alpha_\varepsilon)$  was defined in (3.9). Indeed, by continuity of  $\tilde{Z}_p$ , the second term converges to zero since  $\alpha_\varepsilon, \beta_\varepsilon \rightarrow 0$  and the first term can be estimated by using the differential equation

$$|Z_{p\varepsilon}(t) - Z_{p\varepsilon}^0| \leq \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \int_{\alpha_\varepsilon}^s \left| \frac{DH \delta_\varepsilon U_\varepsilon^2}{2} - \frac{eZ_p^\varepsilon + \frac{1}{2}Z_p^\varepsilon \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - Z_p^\varepsilon \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)}{\sigma a^2 - U^2 H \delta_\varepsilon} \right| dr ds + |\dot{Z}_{p\varepsilon}^0|(\alpha_\varepsilon + \beta_\varepsilon) \leq C\varepsilon. \tag{5.11}$$

Here, we have used that the inner integral is bounded by (A.27)–(A.29).

Now, we finish the proof of (i) and establish the claim of uniform convergence in (ii) and (iii). First, note that by construction there is nothing to show for  $t \leq 0$ . For  $t \geq \beta_\varepsilon$ , we use the continuous dependence of solutions to ordinary differential equations on the data. Indeed, for such  $t$ , both  $\gamma_\varepsilon$  and  $\tilde{\gamma} = \gamma^+$  are solutions to the same differential equation, albeit, with different

data, which is given for  $\gamma_\varepsilon$  at  $t = \beta_\varepsilon$  by (5.2) and for  $\gamma^+$  at  $t = 0$  by (3.13). More precisely, for all  $T > 0$  (which implies  $T > \beta_\varepsilon$  when  $\varepsilon$  is small), we have

$$\begin{aligned} & \sup_{\beta_\varepsilon \leq t \leq T} (|\gamma_\varepsilon(t) - \gamma^+(t)|, |\dot{\gamma}_\varepsilon(t) - \dot{\gamma}^+(t)|) \\ & \leq \max(|\gamma_\varepsilon(\beta_\varepsilon) - \gamma^+(\beta_\varepsilon)|, |\dot{\gamma}_\varepsilon(\beta_\varepsilon) - \dot{\gamma}^+(\beta_\varepsilon)|) e^{TL}, \end{aligned} \quad (5.12)$$

where  $L$  is a Lipschitz constant for the right hand side of the geodesic equation of the background on a suitable compact set. Note that such a set exists due to the boundedness properties of  $\gamma_\varepsilon$  established in proposition 4.1, i.e.,  $\dot{\gamma}_\varepsilon(\beta_\varepsilon)$  is uniformly bounded. Finally, for the terms in the maximum in (5.12), we have

$$\begin{aligned} |U_\varepsilon(\beta_\varepsilon) - U^+(\beta_\varepsilon)| &\rightarrow 0, & |\dot{U}_\varepsilon(\beta_\varepsilon) - \dot{U}^+(\beta_\varepsilon)| &\rightarrow 0 & \text{from (5.9), and} \\ |Z_{p\varepsilon}(\beta_\varepsilon) - Z_p^+(\beta_\varepsilon)| &\rightarrow 0 & & & \text{from (5.10),} \end{aligned} \quad (5.13)$$

whereas for the remaining terms, we write

$$\begin{aligned} |V_\varepsilon(\beta_\varepsilon) - V^+(\beta_\varepsilon)| &\leq |V_\varepsilon(\beta_\varepsilon) - V^+(0)| + |V^+(0) - V^+(\beta_\varepsilon)|, \\ |\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(\beta_\varepsilon)| &\leq |\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(0)| + |\dot{V}^+(0) - \dot{V}^+(\beta_\varepsilon)|, \\ |\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(\beta_\varepsilon)| &\leq |\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(0)| + |\dot{Z}_p^+(0) - \dot{Z}_p^+(\beta_\varepsilon)|. \end{aligned} \quad (5.14)$$

Now, in each line the last term on the right hand side goes to zero due to the smoothness of  $\gamma^+$ , while for the respective first terms, we have by our choice of data (5.2), (5.3)

$$\begin{aligned} |V_\varepsilon(\beta_\varepsilon) - V^+(0)| &= |V_\varepsilon(\beta_\varepsilon) - \mathbf{B}| \rightarrow 0, \\ |\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(0)| &= |\dot{V}_\varepsilon(\beta_\varepsilon) - \mathbf{C}| \rightarrow 0, \\ |\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(0)| &= |\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \mathbf{A}_p| \rightarrow 0. \end{aligned} \quad (5.15)$$

Finally, to prove the distributional convergence in (iii) thanks to the uniform convergence of  $V_\varepsilon$  established above, we only have to consider the integral

$$\int_{\alpha_\varepsilon}^{\beta_\varepsilon} (V_\varepsilon(s) - V(s)) \varphi(s) ds \quad (5.16)$$

for a test function  $\varphi$  on  $\mathbb{R}$ . This, however, converges to zero due to the local uniform boundedness of  $V_\varepsilon$  established in proposition 4.1(iii). In the case of the distributional convergence in (ii), we argue in precisely the same manner, now using proposition 4.1(ii).  $\square$

**Remark 5.2 (Normalisation and constraints in the limit).** The convergence result provided by theorem 5.1 also guarantees the preservation of the normalisation, i.e., the normalisation of the ‘seed geodesic’  $\gamma$ , which by remark 3.3 carries over to the regularised geodesics  $\gamma_\varepsilon$  (and hence to  $\gamma_\varepsilon^+$ ), and also carries over to the ‘future branch’ of the ‘limiting geodesic’  $\gamma^+$ . To demonstrate this, we write

$$\begin{aligned} e &= g_\varepsilon(\gamma_\varepsilon(\beta_\varepsilon))(\dot{\gamma}_\varepsilon(\beta_\varepsilon), \dot{\gamma}_\varepsilon(\beta_\varepsilon)) = g(\dot{\gamma}_\varepsilon(\beta_\varepsilon), \dot{\gamma}_\varepsilon(\beta_\varepsilon)) \\ &= -2\dot{U}_\varepsilon(\beta_\varepsilon)\dot{V}_\varepsilon(\beta_\varepsilon) + (\dot{Z}_{2\varepsilon}(\beta_\varepsilon))^2 + (\dot{Z}_{3\varepsilon}(\beta_\varepsilon))^2 + \sigma(\dot{Z}_{4\varepsilon}(\beta_\varepsilon))^2 \\ &\rightarrow -2\dot{U}^0\mathbf{C} + \mathbf{A}_2^2 + \mathbf{A}_3^2 + \sigma\mathbf{A}_4^2 = g(\dot{\gamma}^+(0), \dot{\gamma}^+(0)). \end{aligned} \quad (5.17)$$

Here, the first equality follows from remark 3.3 and the second one follows from the fact that, at  $\gamma_\varepsilon(\beta_\varepsilon)$ , the regularised metric agrees with the (constant) background metric. Finally, convergence is due to theorem 5.1(i) and our choice of the data (5.2), (5.3).

Also, by a similar (actually simpler) argument, the constraints carry over from  $\gamma_\varepsilon$  to  $\gamma^+$ , which again confirms the consistency of our construction.

To end this section and the entire paper, we now explicitly evaluate the limits in (5.3), thereby showing that the ‘limiting geodesics’ (5.4) of the smooth global geodesics  $\gamma_\varepsilon$  of theorem 3.2 in the regularised spacetime (2.9), (2.10) coincide with the geodesics (2.6) of the distributional spacetime (2.1), (2.2) derived previously in [31].

**Proposition 5.3.** *The constants  $A_p$ ,  $B$  and  $C$  determining the data for the ‘future branch’  $\gamma^+$  of the ‘limiting geodesics’  $\tilde{\gamma}$  are explicitly given by*

$$\begin{aligned} A_i &:= \lim_{\varepsilon \rightarrow 0} \dot{Z}_{i\varepsilon}(\beta_\varepsilon) = \frac{1}{2} \dot{U}^0 \left( H_{,i}(Z_r^0) + \frac{Z_i^0}{\sigma a^2} (H(Z_r^0) - \delta^{pq} Z_p^0 H_{,q}(Z_r^0)) \right) + \dot{Z}_i^0, \\ A_4 &:= \lim_{\varepsilon \rightarrow 0} \dot{Z}_{4\varepsilon}(\beta_\varepsilon) = \frac{1}{2} \dot{U}^0 \left( \sigma H_{,4}(Z_r^0) + \frac{Z_4^0}{\sigma a^2} (H(Z_r^0) - \delta^{pq} Z_p^0 H_{,q}(Z_r^0)) \right) + \dot{Z}_4^0, \\ B &:= \lim_{\varepsilon \rightarrow 0} V_\varepsilon(\beta_\varepsilon) = \frac{1}{2} H(Z_p^0) + V^0, \\ C &:= \lim_{\varepsilon \rightarrow 0} \dot{V}_\varepsilon(\beta_\varepsilon) = \dot{V}^0 + \frac{\dot{U}^0}{8} \left( H_{,2}(Z_r^0)^2 + H_{,3}(Z_r^0)^2 + \sigma H_{,4}(Z_r^0)^2 \right. \\ &\quad \left. + \frac{1}{\sigma a^2} H(Z_r^0)^2 - \frac{1}{\sigma a^2} (\delta^{pq} Z_p^0 H_{,q}(Z_r^0))^2 \right) \\ &\quad - \frac{\dot{U}^0}{2\sigma a^2} \left( \delta^{pq} Z_p^0 H_{,q}(Z_r^0) - H(Z_r^0) \right) V^0 + \frac{1}{2} \delta^{pq} H_{,p}(Z_r^0) \dot{Z}_q^0. \end{aligned} \quad (5.18)$$

The calculation is rather technical and we sketch the main points in appendix B. Finally, we remark that our results are fully compatible with the ones in [31]:

**Remark 5.4.** To simplify the comparison of the results on the ‘limiting geodesics’ of proposition 5.3 with the heuristically derived geodesics of the impulsive wave spacetime of [31, equations (38), (39)], we remark that in [31] the geodesics were restricted to  $V^0 = \dot{Z}^0 = 0$  and  $\dot{U}^0 = 1$ . Moreover, recalling that  $1/(\sigma a^2) = \Lambda/3$  and using the notations  $G(0) = G(Z_p^0) = \delta^{pq} Z_p^0 H_{,q}(Z_r^0) - H(Z_r^0)$  and  $H(0) = H(Z_p^0)$  of [31], equations (5.18) take the form

$$\begin{aligned} A_i &= \frac{1}{2} \left( H_{,i}(0) - \frac{\Lambda}{3} Z_i^0 G(0) \right), \quad A_4 = \frac{1}{2} \left( \sigma H_{,4}(0) - \frac{\Lambda}{3} Z_4^0 G(0) \right), \quad B = \frac{1}{2} H(0), \\ C &= \frac{1}{8} \left( H_{,2}(0)^2 + H_{,3}(0)^2 + \sigma H_{,4}(0)^2 + \frac{\Lambda}{3} H(0)^2 - \frac{\Lambda}{3} (\delta^{pq} Z_p^0 H_{,q}(0))^2 \right) + \dot{V}^0. \end{aligned} \quad (5.19)$$

Finally, taking into account that  $\varepsilon = \text{sign}(\Lambda)$  in [31] as well as the slightly different definition of  $C$  in [31, equation (37)], we see that (5.19) indeed agrees with equation (38) of [31].

## Summary

In this paper, we have rigorously investigated all geodesics in the entire class of non-expanding impulsive gravitational waves propagating in the (anti-)de Sitter universe, thus extending many previous studies of geodesic motion in the class of impulsive  $pp$ -wave spacetimes with vanishing cosmological constants. Following [31], we employed the

*distributional form of the metric in the context of a five-dimensional embedding formalism.* We have applied a regularisation technique, replacing the Dirac- $\delta$  by a general class of smooth functions—the model delta nets (2.8). Since we have not used any special property of the regularising net, our results are completely *regularisation independent* within this class. In physical terms, this means that the formal distributional form of the impulsive metric (2.1), (2.2) is understood as a limit of a family of spacetimes with ever shorter but ever stronger sandwich gravitational waves (2.9), (2.10) of an *arbitrary smooth profile*  $\delta_\varepsilon$ .

Although the resulting regularized geodesic equations (2.17) form a highly complicated coupled system, we were able to prove in section 3 the existence and uniqueness of geodesics crossing the wave impulse, leading to completeness results, see theorems 3.1, 3.2 and 3.4, 3.5. Observe that, in particular, we prove that the geodesics of the regularised spacetime that hit the wave zone and hence interact *nonlinearly* with the impulse actually cross it. This is physical information not provided by the approach of [31] in which this feature is built into the heuristic ansatz.

Our proof is based on the application of a fixed point theorem, the technical details of which can be found in appendix A. There, we have extended the range of applicability of this kind of fixed point technique, originating in [22] and generalised in [36], to a far more involved situation where the higher order ‘contraction estimate’ contains a term proportional to  $1/\varepsilon$ , cf. (A.32). In this way, we have pushed on the crucial point of the method to the estimate (A.29), cf. the remark below the proof of proposition A.3. This raises hopes that these techniques can be extended to the even wilder ‘four-dimensional’ distributional form of the metric (1.6) in the future.

In section 4, we studied the boundedness properties of the global geodesics in the regularised spacetimes. These technical results, summarized in proposition 4.1, were essential for finding their limits in section 5. Due to the complexity of the system of geodesic equations (2.17), it seemed advisable to simplify the ‘usual arguments’ in this limiting procedure. We have done so by abstracting from the concrete form of the (limiting) geodesics and repeatedly using the continuous dependence of solutions of ODEs on their data. In this way, we were able to show that, as the regularisation parameter goes to zero, the solutions of the regularised geodesic equation converge to unique geodesics of the background (anti-)de Sitter spacetime (theorem 5.1), which have to be matched appropriately across the impulse. In fact, we rigorously derived the explicit form of these matching conditions (some technical related calculations are contained in appendix B). The resulting coefficients (5.18) of proposition 5.3 fully agree with the previous results derived by a heuristic approach in [31]. Remarkably, the impulsive limit is completely independent of the specific regularisation, i.e., in the limit  $\varepsilon \rightarrow 0$  it is *the same for any smooth profile* of the sandwich gravitational waves.

Finally, as mentioned in the introduction in [32], we have recently investigated the complete family of nonexpanding impulsive gravitational waves propagating in spaces of constant curvature (Minkowski, de Sitter and anti-de Sitter universes) employing the (Lipschitz) *continuous form of the metric* (1.4). Using Filippov’s solution concept for differential equations with discontinuous right hand side, we proved the existence and uniqueness of continuously differentiable geodesics. In section 4 of [32], we explicitly derived such geodesics by using a  $C^1$ -matching procedure resulting in specific matching conditions, namely equations (4.4)–(4.10) of [32].

A natural question thus arises about the mutual consistency of the two results, both obtained in a rigorous way but starting from two different forms of the metric, namely the continuous form of the metric (employed in [32]) and the distributional form of the metric in the context of the five-dimensional embedding formalism (employed here and in [31]). In fact, it was shown in the recent work [20] that the *matching conditions* of [31, 32] are fully

*equivalent* when appropriate coordinate transformations are applied. This result confirms that both our approaches are consistent. It follows that the understanding of geodesics in the complete family of spacetimes with nonexpanding impulsive gravitational waves and any cosmological constant now rests on firm mathematical grounds.

These results set the stage for a sound mathematical analysis of the ‘discontinuous coordinate transformations’ between the continuous and the distributional forms of the metric. Together with the results of [20], it now seems feasible to rigorously relate the continuous form of the metric (1.4) to the ‘five-dimensional’ distributional form (2.1), (2.2). On the other hand, the technical advances of the fixed point technique made here might eventually bring into reach a direct approach to the mathematical intricacies of the transformation (1.5).

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## Appendix A. The fixed point argument

In this appendix, we detail the fixed point argument used to prove a suitable existence and uniqueness result for solutions of the regularised geodesic equations (2.17) with data (3.9), which additionally guarantees the solutions to live long enough to leave the regularisation sandwich.

To do so, we only have to prove existence of the  $U_\varepsilon$  and  $Z_{p\varepsilon}$ -components, since the equation for  $V_\varepsilon$  decouples and is linear, and hence can be solved on the domain of existence of  $(U_\varepsilon, Z_{p\varepsilon})$ . Moreover, the sign difference between the  $Z_{i\varepsilon}$  equations and the  $Z_{4\varepsilon}$  equation can safely be ignored in the estimates, leading to the fixed point argument. Therefore, in this appendix, we only (have to) deal with the following *simplified model system*

$$\begin{aligned} \ddot{u}_\varepsilon &= - \left( e + \frac{1}{2} \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{u}_\varepsilon (H \delta_\varepsilon u_\varepsilon) \right) \frac{u_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon}, \\ \ddot{z}_\varepsilon - \frac{1}{2} DH \delta_\varepsilon \dot{u}_\varepsilon^2 &= - \left( e + \frac{1}{2} \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{u}_\varepsilon (H \delta_\varepsilon u_\varepsilon) \right) \frac{z_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon}, \end{aligned} \quad (\text{A.1})$$

where  $H = H(z_\varepsilon)$  is a smooth function on  $\mathbb{R}^3$ ,  $DH$  denotes its gradient, and  $\tilde{G}_\varepsilon(u_\varepsilon, z_\varepsilon) := DH(z_\varepsilon) \delta_\varepsilon(u_\varepsilon) z_\varepsilon + H(z_\varepsilon) \delta'_\varepsilon(u_\varepsilon) u_\varepsilon$ . We will also frequently use the notation  $x_\varepsilon = (u_\varepsilon, z_\varepsilon)$ .

We begin by setting up the *initial data*. Let  $\eta > 0$  and let  $J_\varepsilon = [\alpha_\varepsilon, \alpha_\varepsilon + \eta]$  be the parameter interval in which we look for solutions. In accordance with the strategy employed in section 3, we will pose initial data at  $t = \alpha_\varepsilon$  and compare it to fixed data (corresponding to the initial data of the ‘seed geodesic’ at  $t = 0$ ). So, let

$$\begin{aligned}
x_\varepsilon^0 &= (u_\varepsilon^0, z_\varepsilon^0) \in \mathbb{R} \times \mathbb{R}^3 \quad \text{and} \quad \dot{x}_\varepsilon^0 := (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \in \mathbb{R} \times \mathbb{R}^3 \text{ be given and set} \\
x_\varepsilon(\alpha_\varepsilon) &= (u_\varepsilon(\alpha_\varepsilon), z_\varepsilon(\alpha_\varepsilon)) = (u_\varepsilon^0, z_\varepsilon^0) \quad \text{and} \quad \dot{x}_\varepsilon(\alpha_\varepsilon) = (\dot{u}_\varepsilon(\alpha_\varepsilon), \dot{z}_\varepsilon(\alpha_\varepsilon)) = (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0)
\end{aligned} \tag{A.2}$$

and additionally let

$$u^0, \dot{u}^0 \in \mathbb{R}, z^0, \dot{z}^0 \in \mathbb{R}^3 \text{ be given and write } x^0 := (u^0, z^0), \dot{x}^0 := (\dot{u}^0, \dot{z}^0). \tag{A.3}$$

As detailed in section 3, we exclusively deal with data satisfying

$$u_\varepsilon^0 = -\varepsilon, \dot{u}_\varepsilon^0 > 0 \text{ and } u^0 = 0, \dot{u}^0 > 0, \text{ with the additional assumption (cf. (3.10))}$$

$$x_\varepsilon^0 \rightarrow x^0 \text{ and } \dot{x}_\varepsilon^0 \rightarrow \dot{x}^0 \text{ as } \varepsilon \rightarrow 0. \tag{A.4}$$

We will apply our fixed point argument on a *complete metric space*, which we will call the ‘solution space’ and which is given as the closed subset of  $\mathcal{C}^1(J_\varepsilon, \mathbb{R}^4)$  defined by

$$\begin{aligned}
\mathfrak{X}_\varepsilon &:= \{x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{C}^1(J_\varepsilon, \mathbb{R}^4) : x_\varepsilon(\alpha_\varepsilon) = x_\varepsilon^0, \dot{x}_\varepsilon(\alpha_\varepsilon) = \dot{x}_\varepsilon^0 \text{ and} \\
&\|x_\varepsilon - x^0\|_\infty \leq C_1, \|\dot{z}_\varepsilon - \dot{z}^0\|_\infty \leq C_2, \dot{u}_\varepsilon \in \left[\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0\right]\}.
\end{aligned} \tag{A.5}$$

Observe that we have ‘centred’ the functions in  $\mathfrak{X}_\varepsilon$  around the ‘fixed’ initial data (A.3), while the prospective solutions are required to assume the  $\varepsilon$ -dependent data (A.2) at  $t = \alpha_\varepsilon$ . Also note that the final condition forces  $\dot{u}_\varepsilon$  to stay positive, which is the essential ingredient that forces the solutions to leave the regularisation sandwich. Now, we arrange the constants as follows: First, let  $C_1 > 0$  and set

$$\begin{aligned}
C_2 &:= 1 + \max \left\{ 9\dot{u}^0 \|DH\|_\infty \|\rho\|_1, \frac{36}{a^2} \dot{u}^0 (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty), \right. \\
&\left. \frac{48}{a^2} (|z^0| + C_1) \left( 1 + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right) \right\},
\end{aligned} \tag{A.6}$$

where  $\|H\|_\infty$  and  $\|DH\|_\infty$  are taken over the closed Euclidean ball  $B_{C_1}(z^0)$ . Also,  $\rho$  is as in (2.8). Observe that the space  $\mathfrak{X}_\varepsilon$  only depends on  $\varepsilon$  via the domain  $J_\varepsilon$  and the initial data (A.2).

Next, we define the *solution operator*  $A_\varepsilon$  acting on  $\mathfrak{X}_\varepsilon$  for all  $t \in J_\varepsilon$  via  $A_\varepsilon(x_\varepsilon)(t) := (A_\varepsilon^1(x_\varepsilon)(t), A_\varepsilon^2(x_\varepsilon)(t))$  with

$$\begin{aligned}
A_\varepsilon^1(x_\varepsilon)(t) &= \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{eu_\varepsilon + \frac{1}{2}u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - u_\varepsilon \dot{u}_\varepsilon (H\delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} \, dr \, ds + \dot{u}_\varepsilon^0 (t - \alpha_\varepsilon) - \varepsilon, \\
A_\varepsilon^2(x_\varepsilon)(t) &:= \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left( \frac{1}{2} DH\delta_\varepsilon \dot{u}_\varepsilon^2 - \frac{ez_\varepsilon + \frac{1}{2}z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - z_\varepsilon \dot{u}_\varepsilon (H\delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} \right) \, dr \, ds \\
&\quad + \dot{z}_\varepsilon^0 (t - \alpha_\varepsilon) + z_\varepsilon^0,
\end{aligned} \tag{A.7}$$

where again we have suppressed the dependence of  $\delta_\varepsilon$ ,  $\tilde{G}_\varepsilon$  and  $H$ , as well as their derivatives, on the variables.

Our next step will be to show that the operator  $A_\varepsilon$  takes  $\mathfrak{X}_\varepsilon$  to  $\mathfrak{X}_\varepsilon$ , see proposition A.3, below. We begin with two preliminary results. First, we bound the term in the denominator of  $A_\varepsilon$  from below.

**Lemma A.1.** Suppose  $z \in B_{C_1}(z^0)$ , then for all  $u \in \mathbb{R}$

$$\left| \frac{1}{\sigma a^2 - u^2 H(z) \delta_\varepsilon(u)} \right| \leq \frac{2}{a^2}, \quad (\text{A.8})$$

provided  $\varepsilon \leq a^2/(2 \|\rho\|_\infty \|H\|_\infty)$ .

**Proof.** First, in the case  $|u| > \varepsilon$ , we have  $u \notin \text{supp}(\delta_\varepsilon)$  and consequently  $\frac{1}{|\sigma a^2 - u^2 H \delta_\varepsilon|} = \frac{1}{a^2}$ . Second, in the case  $|u| \leq \varepsilon$ , we have  $|u^2 H(z) \delta_\varepsilon(u)| \leq \varepsilon \|H\|_\infty \|\rho\|_\infty \leq a^2/2$  and therefore in both cases

$$\frac{1}{|\sigma a^2 - u^2 H \delta_\varepsilon|} \leq \frac{2}{a^2}.$$

□

The second preliminary result shows that the conditions imposed on  $\dot{u}_\varepsilon$  in (A.5), i.e., that  $\dot{u}_\varepsilon \geq \dot{u}^0/2$ , prevents the  $u$ -component from slowing down too much in the sense that  $u_\varepsilon(t)$  leaves the sandwich region early enough. To state the result in a precise way, we define for  $x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathfrak{X}_\varepsilon$  the set

$$\Gamma_\varepsilon(x_\varepsilon) \equiv \Gamma_\varepsilon(u_\varepsilon) := \{t \in J_\varepsilon : |u_\varepsilon(t)| \leq \varepsilon\} \subseteq J_\varepsilon, \quad (\text{A.9})$$

which is the maximal set where the terms in (A.1), (A.7) involving  $\delta_\varepsilon$  or  $\delta'_\varepsilon$  are non-vanishing. We now have:

**Lemma A.2.** The diameter of  $\Gamma_\varepsilon(u_\varepsilon)$  is bounded for all  $x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathfrak{X}_\varepsilon$  by

$$\text{diam}(\Gamma_\varepsilon(u_\varepsilon)) \leq \frac{4\varepsilon}{\dot{u}^0}. \quad (\text{A.10})$$

**Proof.** For  $x_\varepsilon \in \mathfrak{X}_\varepsilon$ , let  $t \in \Gamma_\varepsilon(x_\varepsilon)$ , which implies  $|u_\varepsilon(t)| \leq \varepsilon$  and so

$$\varepsilon \geq u_\varepsilon(t) = u_\varepsilon^0 + \int_{\alpha_\varepsilon}^t \dot{u}_\varepsilon(\tau) d\tau \geq u_\varepsilon^0 + \frac{1}{2} \dot{u}^0 (t - \alpha_\varepsilon). \quad (\text{A.11})$$

However, this implies  $t \leq \alpha_\varepsilon + 2(\varepsilon - u_\varepsilon^0)/\dot{u}^0 = \alpha_\varepsilon + 4\varepsilon/\dot{u}^0$ . □

Now, we may state and prove that  $A_\varepsilon(\mathfrak{X}_\varepsilon) \subseteq \mathfrak{X}_\varepsilon$  provided  $\eta$  is chosen appropriately and  $\varepsilon$  is small enough.

**Proposition A.3.** Set

$$\eta := \min \left\{ 1, \frac{a^2}{24\dot{u}^0}, \frac{C_1}{\frac{3}{2} + \dot{u}^0}, \frac{2C_1}{54 \|\rho\|_1 \|DH\|_\infty \dot{u}^0}, \frac{a^2 C_1}{12(|z^0| + C_1)}, \frac{a^2 C_2}{8(|z^0| + C_1)}, \right. \\ \left. \frac{C_1 a^2}{54} (\dot{u}^0 (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty))^{-1}, \frac{C_1}{6(1 + |z^0|)}, \right. \\ \left. \frac{C_1 a^2}{72} \left( (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2) + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty)) \right)^{-1} \right\} \quad (\text{A.12})$$

and

$$\begin{aligned} \varepsilon'_0 := \min & \left\{ \frac{a^2}{2 \|\rho\|_\infty \|H\|_\infty}, \frac{a^2}{72\dot{u}^0} (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty)^{-1}, \right. \\ & \frac{a^2}{96} \left( 3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2) + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right)^{-1}, \\ & \left. (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2))^{-1}, \frac{\eta \dot{u}^0}{6}, \eta \right\}. \end{aligned} \tag{A.13}$$

Now, choose  $\varepsilon_0$  such that

$$\begin{aligned} 0 < \varepsilon_0 \leq \varepsilon'_0, \quad \text{and} \\ |\dot{u}_\varepsilon^0 - \dot{u}^0| \leq \frac{1}{8}, \quad |z_\varepsilon^0 - z^0| \leq \frac{C_1}{6}, \quad \text{and} \quad |z_\varepsilon^0 - z^0| \leq 1 \quad \text{for all} \quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{A.14}$$

Then, for all  $\varepsilon \leq \varepsilon_0$ , the operator  $A_\varepsilon$  maps  $\mathfrak{X}_\varepsilon$  to  $\mathfrak{X}_\varepsilon$ .

Observe that from (A.4) there exists  $\varepsilon_0$ , which guarantees the estimates in (A.14) hold.

**Proof.** We begin by estimating

$$\frac{d}{dt} A_\varepsilon^1(x_\varepsilon)(t) = - \int_{\alpha_\varepsilon}^t \frac{e u_\varepsilon + \frac{1}{2} u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - u_\varepsilon \dot{u}_\varepsilon (H \delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} ds + \dot{u}_\varepsilon^0 \tag{A.15}$$

and proceed term by term beginning with the latter two under the integral, which will be seen to vanish as  $\varepsilon \rightarrow 0$ . Indeed, we have for all  $\varepsilon \leq \varepsilon_0$  from the definition of  $\mathfrak{X}_\varepsilon$  (A.5) and from lemma A.1

$$\begin{aligned} & \left| \int_{\alpha_\varepsilon}^t \frac{u_\varepsilon \dot{u}_\varepsilon^2 (DH \delta_\varepsilon z_\varepsilon + H \delta'_\varepsilon u_\varepsilon)'}{2(\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon)} ds \right| \\ & \leq \frac{1}{a^2} \text{diam}(\Gamma_\varepsilon(u_\varepsilon)) \varepsilon \left( \frac{3}{2} \dot{u}^0 \right)^2 \left( 3 \|DH\|_\infty \|\rho\|_\infty \frac{1}{\varepsilon} (|z^0| + C_1) + \|H\|_\infty \frac{1}{\varepsilon^2} \|\rho'\|_\infty \varepsilon \right) \\ & \leq \frac{9\dot{u}^0}{a^2} \varepsilon (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty) \leq \frac{1}{8}, \end{aligned} \tag{A.16}$$

where for the second inequality we have used lemma A.2 and for the third we have used

$$\varepsilon_0 \leq \frac{a^2}{72\dot{u}^0} (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty)^{-1}. \tag{A.17}$$

Similarly, we have

$$\begin{aligned} & \left| \int_{\alpha_\varepsilon}^t \frac{u_\varepsilon \dot{u}_\varepsilon (H \delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} ds \right| \\ & \leq \frac{12}{a^2} \varepsilon \left( 3\varepsilon \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2) + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right) \leq \frac{1}{8}, \end{aligned} \tag{A.18}$$

where the final estimate again follows from our assumptions on  $\varepsilon_0$ .



Finally, to estimate the first term under the integral in (A.15), we write for  $u_\varepsilon \in \mathfrak{X}_\varepsilon$

$$u_\varepsilon(t) = u_\varepsilon^0 + \int_{\alpha_\varepsilon}^t \dot{u}_\varepsilon(s) ds \leq -\varepsilon + \eta \frac{3}{2} \dot{u}^0 \leq \frac{3}{2} \dot{u}^0 \eta. \quad (\text{A.19})$$

Since  $-\varepsilon \leq u_\varepsilon(t)$  and from the last condition on  $\varepsilon_0$  in (A.14), we obtain  $|u_\varepsilon(t)| \leq \frac{3}{2} \dot{u}^0 \eta$  and hence

$$\left| \int_{\alpha_\varepsilon}^t \frac{e u_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} ds \right| \leq \frac{2}{a^2} \int_{\alpha_\varepsilon}^t |u_\varepsilon(s)| ds \leq \frac{3}{a^2} \dot{u}^0 \eta^2 \leq \frac{3}{a^2} \dot{u}^0 \eta \leq \frac{1}{8}, \quad (\text{A.20})$$

where we have used that  $\eta \leq 1$  and  $\eta \leq a^2/(24\dot{u}^0)$ , cf. (A.12). Thus, from  $|\dot{u}_\varepsilon^0 - \dot{u}^0| \leq \frac{1}{8}$ , we obtain overall  $\|\frac{d}{dt} A_\varepsilon^1(x_\varepsilon) - \dot{u}^0\|_\infty \leq \frac{1}{2}$ , i.e.,  $\frac{d}{dt} A_\varepsilon^1(x_\varepsilon)(t) \in [\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0]$  for all  $t \in J_\varepsilon$ .

Moreover, by using the above estimates, integrating once more and using  $\varepsilon \leq \eta$ , we find that

$$\|A_\varepsilon^1(x_\varepsilon) - u^0\|_\infty \leq \varepsilon + \eta \frac{3}{8} + \eta \dot{u}_\varepsilon^0 \leq \eta \left( \frac{3}{2} + \dot{u}^0 \right) \leq C_1, \quad (\text{A.21})$$

due to the assumption that  $\eta \leq C_1 / (\frac{3}{2} + \dot{u}^0)$ .

Now, we turn to the ‘spatial component’  $A_\varepsilon^2$  of the solution operator. We have to show that

$$\begin{aligned} & \|A_\varepsilon^2(x_\varepsilon) - z^0\|_\infty \\ &= \left\| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left( \frac{1}{2} DH \delta_\varepsilon \dot{u}_\varepsilon^2 - \frac{e z_\varepsilon + \frac{1}{2} z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - z_\varepsilon \dot{u}_\varepsilon (H \delta_\varepsilon u_\varepsilon)}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} \right) dr ds + z_\varepsilon^0(t - \alpha_\varepsilon) \right\|_\infty \leq C_1 \end{aligned} \quad (\text{A.22})$$

and again proceed term by term. To begin with, we note the following auxiliary estimate

$$\begin{aligned} & \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| ds = \frac{2}{\dot{u}^0} \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| \frac{\dot{u}^0}{2} ds \\ & \leq \frac{2}{\dot{u}^0} \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| \dot{u}_\varepsilon(s) ds = \frac{2}{\dot{u}^0} \int_{-\varepsilon}^{u_\varepsilon(t)} |\delta_\varepsilon(r)| dr \leq \frac{2}{\dot{u}^0} \|\rho\|_1. \end{aligned} \quad (\text{A.23})$$

Now, we once more use the definition of  $\mathfrak{X}_\varepsilon$

$$\begin{aligned} & \frac{1}{2} \left| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s DH \delta_\varepsilon \dot{u}_\varepsilon^2 dr ds \right| \leq \frac{2}{\dot{u}^0} \|\rho\|_1 \|DH\|_\infty \left( \frac{3}{2} \dot{u}^0 \right)^2 \eta \\ & = \frac{9}{2} \|\rho\|_1 \|DH\|_\infty \dot{u}^0 \eta \leq \frac{C_1}{6}, \end{aligned} \quad (\text{A.24})$$

where we have made use of  $\eta \leq C_1 / ((2/54) \|\rho\|_1 \|DH\|_\infty \dot{u}^0)$ . Similarly, since  $\eta \leq (a^2 C_1) / (12(|z^0| + C_1))$ , we obtain

$$\left| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{e z_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} dr ds \right| \leq \frac{2}{a^2} (|z^0| + C_1) \eta^2 \leq \frac{2}{a^2} (|z^0| + C_1) \eta \leq \frac{C_1}{6}. \quad (\text{A.25})$$

Furthermore, we estimate

$$\begin{aligned} & \left| \frac{1}{2} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon} dr ds \right| \\ & \leq \frac{9}{a^2} \dot{u}^0 (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty) \eta \leq \frac{C_1}{6}, \end{aligned}$$

where we have used  $\eta \leq \frac{C_1 a^2}{54} (\dot{u}^0 (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty))^{-1}$ , and finally

$$\begin{aligned}
 & \left| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{z_\varepsilon \dot{u}_\varepsilon (H\delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} dr ds \right| \\
 & \leq \frac{12}{a^2} (|z^0| + C_1) \left( 3\varepsilon \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2) + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right) \eta \\
 & \leq \frac{C_1}{6},
 \end{aligned} \tag{A.26}$$

where we have used the final condition on  $\eta$  in (A.12). This establishes (A.22) by using the penultimate condition on  $\eta$  in (A.12) together with  $|z_\varepsilon^0 - z^0| \leq C_1/6$ .

It remains to show that  $\|\frac{d}{dt}A_\varepsilon^2(x_\varepsilon) - \dot{z}^0\|_\infty \leq C_2$ . As in (A.24), (A.25), we estimate

$$\begin{aligned}
 & \frac{1}{2} \left| \int_{\alpha_\varepsilon}^t DH\delta_\varepsilon(u_\varepsilon) \dot{u}_\varepsilon^2 ds \right| \leq \frac{9}{4} \|\rho\|_1 \|DH\|_\infty \dot{u}^0 \leq \frac{C_2}{4}, \\
 & \left| \int_{\alpha_\varepsilon}^t \frac{e z_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} ds \right| \leq \frac{2}{a^2} (|z^0| + C_1) \eta \leq \frac{C_2}{4},
 \end{aligned} \tag{A.27}$$

where we have used the first condition on  $C_2$  in (A.6) and the sixth one on  $\eta$  in (A.12).

For the remaining two terms, we have

$$\begin{aligned}
 & \left| \frac{1}{2} \int_{\alpha_\varepsilon}^t \frac{z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} ds \right| \\
 & \leq \frac{36}{4a^2} \dot{u}^0 (|z^0| + C_1) (3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty) \leq \frac{C_2}{4},
 \end{aligned} \tag{A.28}$$

where we have used the second condition on  $C_2$  in (A.6), and

$$\begin{aligned}
 & \left| \int_{\alpha_\varepsilon}^t \frac{z_\varepsilon \dot{u}_\varepsilon (H\delta_\varepsilon u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} ds \right| \\
 & \leq \frac{12}{a^2} (|z^0| + C_1) \left( 3\varepsilon \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_2) + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right) \\
 & \leq \frac{12}{a^2} (|z^0| + C_1) \left( 1 + \frac{3}{2} \dot{u}^0 \|H\|_\infty (\|\rho'\|_\infty + \|\rho\|_\infty) \right) \leq \frac{C_2}{4}.
 \end{aligned} \tag{A.29}$$

Here, we have used the fourth condition on  $\varepsilon_0$  in (A.14) as well as the final condition on  $C_2$  in (A.6).  $\square$

Observe that in the estimate (A.29), it is absolutely *vital* that the term in the second line involving  $C_2$  is proportional to  $\varepsilon$ —otherwise we would end up in a circle and our method would fail.

Our next step is to prove that the solution operator  $A_\varepsilon$  has a fixed point on  $\mathfrak{X}_\varepsilon$ . To this end, we need the following technical preparation.

**Lemma A.4.** *There exist constants  $\tilde{C}$  and  $\tilde{C}'$  (independent of  $\varepsilon$ ) such that for all  $x_\varepsilon, x_\varepsilon^* \in \mathfrak{X}_\varepsilon$ , we have*

$$(i) \left| \int_{\alpha_\varepsilon}^t (\delta_\varepsilon(u_\varepsilon)u_\varepsilon - \delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*) ds \right| \leq \tilde{C} \|u_\varepsilon - u_\varepsilon^*\|_\infty, \text{ and}$$

$$(ii) \left| \int_{\alpha_\varepsilon}^t (\delta'_\varepsilon(u_\varepsilon)u_\varepsilon^2 - \delta'_\varepsilon(u_\varepsilon^*)(u_\varepsilon^*)^2) ds \right| \leq \tilde{C}' \|u_\varepsilon - u_\varepsilon^*\|_\infty.$$

**Proof.** To prove (i), we first consider the case  $\|u_\varepsilon - u_\varepsilon^*\|_\infty \leq \varepsilon$ . We have from (A.23)

$$\begin{aligned} & \left| \int_{\alpha_\varepsilon}^t (\delta_\varepsilon(u_\varepsilon)u_\varepsilon - \delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*) ds \right| \\ & \leq \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon)u_\varepsilon - \delta_\varepsilon(u_\varepsilon)u_\varepsilon^*| ds \\ & \quad + \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon)u_\varepsilon^* - \delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*| ds \\ & \leq \frac{2}{\dot{u}^0} \|\rho\|_1 \|u_\varepsilon - u_\varepsilon^*\|_\infty + \int_{\Gamma_\varepsilon(x_\varepsilon) \cup \Gamma_\varepsilon(x_\varepsilon^*)} |\delta_\varepsilon(u_\varepsilon) - \delta_\varepsilon(u_\varepsilon^*)| |u_\varepsilon^*| ds. \end{aligned} \tag{A.30}$$

Now, the last integral is non-vanishing only if  $|u_\varepsilon| \leq \varepsilon$  or  $|u_\varepsilon^*| \leq \varepsilon$ , hence we have in any case  $|u_\varepsilon^*| \leq 2\varepsilon$ . Since both  $x_\varepsilon, x_\varepsilon^* \in \mathfrak{X}_\varepsilon$ , lemma A.2 applies so that using a mean value argument we may bound the integral by  $\frac{8}{\dot{u}^0} \|\rho'\|_\infty \|u_\varepsilon - u_\varepsilon^*\|_\infty$ .

In the case  $\|u_\varepsilon - u_\varepsilon^*\|_\infty > \varepsilon$ , we obtain again from lemma A.2

$$\begin{aligned} & \left| \int_{\alpha_\varepsilon}^t (\delta_\varepsilon(u_\varepsilon)u_\varepsilon - \delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*) ds \right| \leq \int_{\Gamma_\varepsilon(x_\varepsilon)} |\delta_\varepsilon(u_\varepsilon)u_\varepsilon| ds + \int_{\Gamma_\varepsilon(x_\varepsilon^*)} |\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*| ds \\ & \leq \frac{8}{\dot{u}^0} \|\rho\|_\infty \varepsilon < \frac{8}{\dot{u}^0} \|\rho\|_\infty \|u_\varepsilon - u_\varepsilon^*\|_\infty. \end{aligned} \tag{A.31}$$

So, we may choose  $\tilde{C} = \frac{2}{\dot{u}^0} \max(\|\rho\|_1 + 4 \|\rho'\|_\infty, 4 \|\rho\|_\infty)$  and (i) is proved. In addition, (ii) is proved analogously with the choice  $\tilde{C}' = \frac{4}{\dot{u}^0} \max(4 \|\rho'\|_\infty + \|\rho''\|_\infty, 2 \|\rho'\|_\infty)$ .  $\square$

We finally prove the key estimates, which will allow the application of Weissinger’s fixed point theorem.

**Proposition A.5.** *There exists a sequence of positive real numbers  $(\alpha_n)_n$  (depending on  $\rho, \rho', \rho'', H, DH, D^2 H$  and  $\dot{u}^0$ , but independent of  $\varepsilon$ ) with  $\sum_{n \in \mathbb{N}} \alpha_n < \infty$  such that for all  $x_\varepsilon, x_\varepsilon^* \in \mathfrak{X}_\varepsilon$  with  $\varepsilon \leq \varepsilon_0$  of (A.14) and  $\eta$  as in (A.12) and all  $n \in \mathbb{N}$ , we have*

$$\|(A_\varepsilon)^n(x_\varepsilon) - (A_\varepsilon)^n(x_\varepsilon^*)\|_{C^1} \leq \frac{1}{\varepsilon} \alpha_n \|x_\varepsilon - x_\varepsilon^*\|_{C^1}. \tag{A.32}$$

**Proof.** It suffices to show  $\|A_\varepsilon(x_\varepsilon) - A_\varepsilon(x_\varepsilon^*)\|_{C^1} \leq (C/\varepsilon) \|x_\varepsilon - x_\varepsilon^*\|_{C^1}$  for some appropriate constant  $C$ , since for higher powers we then may use

$$\int_{\alpha_\varepsilon}^{t_{2n}} \dots \int_{\alpha_\varepsilon}^{t_1} 1 dt dt_1 \dots dt_{2n-1} \leq \frac{\eta^{2n}}{(2n)!} \tag{A.33}$$

to obtain a converging series.

We again proceed term by term, skipping some of the details of the (by now) routine estimates and only stress the technical key points.

We start with the first term in  $\|A_\varepsilon^1(x_\varepsilon) - A_\varepsilon^1(x_\varepsilon^*)\|_{C^1}$ . By writing the two summands on a common denominator, we obtain

$$\begin{aligned}
& \left| \int_{\alpha_\varepsilon}^t \frac{eu_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)} ds - \frac{eu_\varepsilon^*}{\sigma a^2 - (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)} ds \right| \\
& \leq \frac{4}{a^4} \int_{\alpha_\varepsilon}^t a^2 |u_\varepsilon - u_\varepsilon^*| ds + \frac{4}{a^4} \int_{\alpha_\varepsilon}^t \left| u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*) - u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon^*) \right| ds \\
& \quad + \frac{4}{a^4} \int_{\alpha_\varepsilon}^t \left| u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon^*) - u_\varepsilon^2 u_\varepsilon^* H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon) \right| ds \\
& \leq \frac{4}{a^2} \eta \|u_\varepsilon - u_\varepsilon^*\|_\infty + \frac{4}{a^4} (|u^0| + C_1)^2 \\
& \quad \times \left( \text{Lip}(H) \|z_\varepsilon - z_\varepsilon^*\|_\infty \frac{4}{\dot{u}^0} \|\rho\|_\infty + \|H\|_\infty \tilde{C} \|u_\varepsilon - u_\varepsilon^*\|_\infty \right) \\
& \leq \frac{4}{a^2} \left( \eta + \frac{1}{a^2} (1 + C_1)^2 (\text{Lip}(H)) \frac{4}{\dot{u}^0} \|\rho\|_\infty + \|H\|_\infty \tilde{C} \right) \|x_\varepsilon - x_\varepsilon^*\|_{C^1}, \tag{A.34}
\end{aligned}$$

where  $\text{Lip}(H)$  denotes the Lipschitz constant of  $H$  on  $B_{C_1}(z^0)$  and  $\tilde{C}$  is the constant given by lemma A.4.

For the second term, we need the following auxiliary estimate, which is proven by a combination of (i) and (ii) in lemma A.4:

$$\int_{\alpha_\varepsilon}^t |\tilde{G}u_\varepsilon - \tilde{G}^*u_\varepsilon^*| ds \leq C' \|x_\varepsilon - x_\varepsilon^*\|_{C^1}, \tag{A.35}$$

where

$$C' = \|D^2 H\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|DH\|_\infty ((|z^0| + C_1)\tilde{C} + \|\rho\|_\infty + \|\rho'\|_\infty) + \|H\|_\infty \tilde{C}'.$$

Abbreviating  $C_{\tilde{G}} := 3 \|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty$ , we are able to estimate

$$\begin{aligned}
& \left| \int_{\alpha_\varepsilon}^t \left( \frac{\frac{1}{2} u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}}{\sigma a^2 - u_\varepsilon^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)} - \frac{\frac{1}{2} u_\varepsilon^* (\dot{u}_\varepsilon^*)^2 \tilde{G}^*}{\sigma a^2 - (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)} \right) ds \right| \\
& \leq \frac{2}{a^4} \int_{\alpha_\varepsilon}^t a^2 \left| u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G} - u_\varepsilon^* (\dot{u}_\varepsilon^*)^2 \tilde{G}^* \right| \\
& \quad + \left| u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G} (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*) - u_\varepsilon^* (\dot{u}_\varepsilon^*)^2 \tilde{G}^* u_\varepsilon^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon) \right| ds \\
& \leq \frac{18(\dot{u}^0)}{a^4} \left( \frac{a^2 C'}{4} + \frac{a^2 C_{\tilde{G}}}{3\dot{u}^0} + \frac{C'}{4} \|H\|_\infty \|\rho\|_\infty + \frac{\tilde{C} C_{\tilde{G}}}{4} \|H\|_\infty \right. \\
& \quad \left. + \frac{C_{\tilde{G}}}{4} \text{Lip}(H) \|\rho\|_\infty + \frac{C_{\tilde{G}}}{3\dot{u}^0} \|H\|_\infty \|\rho\|_\infty \right) \|x_\varepsilon - x_\varepsilon^*\|_{C^1},
\end{aligned}$$

by using (A.35) and lemma A.4.

The final term in  $A_\varepsilon^1(x_\varepsilon)$ , i.e.,

$$\left| \int_{\alpha_\varepsilon}^t \left( \frac{u_\varepsilon \dot{u}_\varepsilon (H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)u_\varepsilon)'}{\sigma a^2 - u_\varepsilon^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)} - \frac{u_\varepsilon^* \dot{u}_\varepsilon^* (H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*)'}{\sigma a^2 - (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)} \right) ds \right| \tag{A.36}$$

can be estimated in perfect analogy to the previous terms by inserting and subtracting appropriate terms wherever necessary to arrive at an estimate proportional to  $\|x_\varepsilon - x_\varepsilon^*\|_{C^1}$ .

The ‘spatial component’  $A_\varepsilon^2$  of the solution operator can be treated in a similar way. The only new aspect when estimating  $\|A_\varepsilon^2(x_\varepsilon) - A_\varepsilon^2(x_\varepsilon^*)\|_{C^1}$  is the following. When bounding terms such as  $|\tilde{G} - \tilde{G}^*|$  by multiples of  $\|x_\varepsilon - x_\varepsilon^*\|_{C^1}$ , we find that they are no longer multiplied by  $u_\varepsilon$  and  $u_\varepsilon^*$ , respectively. Thus, we cannot use the auxiliary result (A.35) and consequently terms proportional to  $1/\varepsilon$  remain. (Note, however, that the occurrence of  $1/\varepsilon$  terms at this stage causes no problem at all in the application of the fixed point theorem, see below.) Summing up, we arrive at

$$\left\| \frac{d}{dt} A_\varepsilon(x_\varepsilon) - \frac{d}{dt} A_\varepsilon(x_\varepsilon^*) \right\|_{\infty} \leq \frac{1}{\varepsilon} C \|x_\varepsilon - x_\varepsilon^*\|_{C^1},$$

where  $C$  is some constant (as above, depending on  $H$ ,  $\rho$ , etc.). Furthermore, since  $\eta \leq 1$ , we obtain the same estimate for the zeroth order, i.e.,  $\|A_\varepsilon(x_\varepsilon) - A_\varepsilon(x_\varepsilon^*)\|_{\infty} \leq \frac{1}{\varepsilon} C \|x_\varepsilon - x_\varepsilon^*\|_{C^1}$ , and hence

$$\|A_\varepsilon(x_\varepsilon) - A_\varepsilon(x_\varepsilon^*)\|_{C^1} \leq \frac{1}{\varepsilon} C \|x_\varepsilon - x_\varepsilon^*\|_{C^1}.$$

Finally, for higher powers of  $A_\varepsilon$ , we obtain (using (A.33))

$$\|(A_\varepsilon)^n(x_\varepsilon) - (A_\varepsilon)^n(x_\varepsilon^*)\|_{C^1} \leq \frac{1}{\varepsilon} \alpha_n \|x_\varepsilon - x_\varepsilon^*\|_{C^1},$$

where  $\alpha_n := C \frac{\eta^{2n}}{(2n)!}$  ( $n \in \mathbb{N}$ ). □

At this point, we finally obtain the existence of a unique solution to (A.1) in  $\mathfrak{X}_\varepsilon$  for all fixed small  $\varepsilon$  by applying Weissinger’s fixed point theorem [43]. Note that the factor  $1/\varepsilon$  in the estimate (A.32) provided by proposition A.5 does not cause any trouble. Its only effect is that the approximating sequence  $(A_\varepsilon)^n(x_\varepsilon)$  converges to the fixed point more slowly as  $\varepsilon$  gets smaller. Nevertheless, we obtain a fixed point for every fixed (small)  $\varepsilon$ :

**Theorem A.6 (Existence and uniqueness).** *Consider the system (A.1) with initial data (A.2), satisfying (A.3), (A.4). Then, for all  $\varepsilon \leq \varepsilon_0$  where  $\varepsilon_0$  is constrained by (A.14) and for  $\eta$  given by (A.12), we have a unique smooth solution  $(u_\varepsilon, z_\varepsilon)$  on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ . Moreover,  $u_\varepsilon$  and  $z_\varepsilon$  as well as their first order derivatives are uniformly bounded in  $\varepsilon$ .*

**Proof.** Propositions A.3 and A.5 allow the application of Weissinger’s fixed point theorem ([43]) for fixed  $\varepsilon \leq \varepsilon_0$  and suitable  $\eta$ , thus providing a unique fixed point for the operator  $A_\varepsilon$  on the space  $\mathfrak{X}_\varepsilon$ , which in turn gives a unique  $C^1$ -solution  $x_\varepsilon = (u_\varepsilon, z_\varepsilon)$  on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$  to the system (A.1) with data (A.2). Moreover, since the right hand sides of (A.1) are smooth, the solution is smooth as well.

The solution obtained via the fixed point argument is unique in the space  $\mathfrak{X}_\varepsilon$  and thereby unique among all smooth solutions assuming this data by the usual argument from ODE theory.

Finally,  $u_\varepsilon$ ,  $\dot{u}_\varepsilon$ ,  $z_\varepsilon$  and  $\dot{z}_\varepsilon$  are bounded uniformly in  $\varepsilon$  on  $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$  by the very definition of  $\mathfrak{X}_\varepsilon$ . □

## Appendix B. Limits

In this appendix, we deal with the explicit form of the limits  $A_p = \lim_{\varepsilon \rightarrow 0} \dot{Z}_{p\varepsilon}(\beta_\varepsilon)$ ,  $B = \lim_{\varepsilon \rightarrow 0} V_\varepsilon(\beta_\varepsilon)$  and  $C = \lim_{\varepsilon \rightarrow 0} \dot{V}_\varepsilon(\beta_\varepsilon)$  as stated in proposition 5.3. Since the actual calculations are very technical, we only sketch the main points.

Again, the sign difference between the  $Z_r$ -components and  $Z_4$  is minor and to simplify the notation we will use a similar convention as in appendix A and write  $Z_\varepsilon$  and  $Z$  instead of  $Z_{p\varepsilon}$  and  $Z_p$  and analogously for their derivatives. Also, we will write  $DH$  instead of  $H_{,p}$ .

Starting with  $A_p$ , we use the differential equation (2.17) for  $Z_{p\varepsilon}$  and the uniform convergence of  $Z_{p\varepsilon}$  and  $\dot{U}_\varepsilon$  established in theorem 5.1 to show that

$$A = \lim_{\varepsilon \rightarrow 0} \dot{Z}_\varepsilon(\beta_\varepsilon) = \frac{1}{2} \dot{U}^0 \left( DH(Z^0) + \frac{Z^0}{\sigma a^2} (H(Z^0) - DH(Z^0)Z^0) \right) + \dot{Z}^0. \quad (\text{B.1})$$

To begin with, we express  $\dot{Z}_\varepsilon(\beta_\varepsilon)$  according to (2.17)

$$\begin{aligned} \dot{Z}_\varepsilon(\beta_\varepsilon) &= \dot{Z}_\varepsilon^0 + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \ddot{Z}_\varepsilon(r) \, dr \\ &= \dot{Z}_\varepsilon^0 + \frac{1}{2} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH \delta_\varepsilon \dot{U}^2 \, dr - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{eZ_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon) \delta_\varepsilon} \, dr \\ &\quad - \frac{1}{2} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot{U}_\varepsilon^2 DH \delta_\varepsilon Z_\varepsilon^2}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon) \delta_\varepsilon} \, dr + \frac{1}{2} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot{U}_\varepsilon^2 H \delta'_\varepsilon U_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon) \delta_\varepsilon} \, dr \\ &\quad + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot{U}_\varepsilon DH \dot{Z}_\varepsilon \delta_\varepsilon U_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon) \delta_\varepsilon} \, dr + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot{U}_\varepsilon^2 H \delta_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon) \delta_\varepsilon} \, dr \\ &=: \dot{Z}_\varepsilon^0 + \text{I}_\varepsilon + \text{II}_\varepsilon + \text{III}_\varepsilon + \text{IV}_\varepsilon + \text{V}_\varepsilon + \text{VI}_\varepsilon, \end{aligned} \quad (\text{B.2})$$

where we have used that

$$\frac{1}{2} \dot{U}_\varepsilon^2 \tilde{G}_\varepsilon - \dot{U}_\varepsilon (H(Z_\varepsilon) \delta_\varepsilon U_\varepsilon)' = \frac{1}{2} \dot{U}_\varepsilon^2 DH \delta_\varepsilon Z_\varepsilon - \frac{1}{2} \dot{U}_\varepsilon^2 H \delta'_\varepsilon U_\varepsilon - \dot{U}_\varepsilon DH \dot{Z}_\varepsilon \delta_\varepsilon U_\varepsilon - \dot{U}_\varepsilon^2 H \delta_\varepsilon. \quad (\text{B.3})$$

Proceeding term by term, we have

$$\begin{aligned} \left| \text{I}_\varepsilon - \frac{1}{2} DH(Z^0) \dot{U}^0 \right| &= \frac{1}{2} \left| \int_{-\varepsilon}^\varepsilon (DH(Z_\varepsilon(U_\varepsilon^{-1}(s))) \delta_\varepsilon(s) \dot{U}_\varepsilon(U_\varepsilon^{-1}(s)) - DH(Z^0) \delta_\varepsilon(s) \dot{U}^0) \, ds \right| \\ &\leq \frac{1}{2} \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} \left| DH(Z_\varepsilon(w)) \dot{U}_\varepsilon(w) - DH(Z^0) \dot{U}^0 \right| \|\rho\|_{L^1} \rightarrow 0, \end{aligned}$$

where we have used that  $U_\varepsilon^{-1}([-\varepsilon, \varepsilon]) = [\alpha_\varepsilon, \beta_\varepsilon]$  together with lemma A.2. The next term,  $\text{II}_\varepsilon$ , vanishes in the limit due to the uniform boundedness of the integrand, and the same holds true for  $\text{V}_\varepsilon$ . Now,  $\text{III}_\varepsilon$  can be treated as  $\text{I}_\varepsilon$ , additionally using (5.6) to conclude

$$\text{III}_\varepsilon \rightarrow -\frac{1}{2} \frac{\dot{U}^0 DH(Z^0)(Z^0)^2}{\sigma a^2}. \quad (\text{B.4})$$

We treat  $\text{IV}_\varepsilon$  by using  $\int \delta'_\varepsilon(s) s \, ds = -1$  to obtain

$$\begin{aligned} \left| \text{IV}_\varepsilon + \frac{\dot{U}^0 H(Z^0) Z^0}{2\sigma a^2} \right| &\leq \frac{1}{2} \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} \left| \frac{\dot{U}_\varepsilon(w) H(Z_\varepsilon(w)) Z_\varepsilon(w)}{\sigma a^2 - U_\varepsilon^2(w) H(Z_\varepsilon(w)) \delta_\varepsilon(U_\varepsilon(w))} - \frac{\dot{U}^0 H(Z^0) Z^0}{\sigma a^2} \right| \|\delta'_\varepsilon(s) s\|_{L^1} \rightarrow 0. \end{aligned} \quad (\text{B.5})$$

Finally, the limit of  $\mathbb{V}_\varepsilon$  is proportional to the limit of  $\mathbb{IV}_\varepsilon$ ,

$$\begin{aligned} & \left| \mathbb{V}_\varepsilon - \frac{\dot{U}^0 H(Z^0)Z^0}{\sigma a^2} \right| \\ & \leq \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} \left| \frac{\dot{U}_\varepsilon(w)H(Z_\varepsilon(w))Z_\varepsilon(w)}{\sigma a^2 - U_\varepsilon^2(w)H(Z_\varepsilon(w))\delta_\varepsilon(U_\varepsilon(w))} - \frac{\dot{U}^0 H(Z^0)Z^0}{\sigma a^2} \right| \|\rho\|_{L^1} \rightarrow 0. \end{aligned} \tag{B.6}$$

By adding up the terms and using (3.10), we establish (B.1).

The calculations for **B** are relatively simple. Using equation (2.17) for  $\dot{V}_\varepsilon$ , we write (cf. (4.3))

$$V_\varepsilon(\beta_\varepsilon) = V_\varepsilon^0 + \frac{1}{2} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \int_{\alpha_\varepsilon}^s H(Z_\varepsilon(r))\delta'_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon(r)^2 \, dr \, ds + O(\varepsilon). \tag{B.7}$$

We substitute twice, use  $\int_{-\varepsilon}^\varepsilon \int_{-\varepsilon}^s \delta'_\varepsilon(r) \, dr \, ds = 1$  and insert the appropriate terms to obtain

$$\begin{aligned} & \frac{1}{2} \left| \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \int_{\alpha_\varepsilon}^s H(Z_\varepsilon(r))\delta'_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon(r)^2 \, dr \, ds - H(Z^0) \right| \\ & = \frac{1}{2} \left| \int_{-\varepsilon}^\varepsilon \frac{1}{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))} \int_{-\varepsilon}^l H(Z_\varepsilon(U_\varepsilon^{-1}(\tau)))\delta'_\varepsilon(\tau)\dot{U}_\varepsilon(U_\varepsilon^{-1}(\tau)) \, d\tau \, dl \right. \\ & \quad \left. - \int_{-\varepsilon}^\varepsilon \frac{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))}{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))} \int_{-\varepsilon}^l H(Z^0)\delta'_\varepsilon(\tau) \, d\tau \, dl \right| \\ & \leq \frac{4 \|\rho'\|_\infty}{\dot{U}^0} \left( \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} |H(Z_\varepsilon(w))\dot{U}_\varepsilon(w) - H(Z^0)\dot{U}^0| + |H(Z^0)| \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} |\dot{U}_\varepsilon(w) - \dot{U}^0| \right), \end{aligned} \tag{B.8}$$

which goes to zero due to the uniform convergence of  $Z_\varepsilon$  and  $\dot{U}_\varepsilon$ , establishing the claimed form of **B**.

Finally, we turn to the calculation of **C**, which is the most demanding one. As above, we express  $\dot{V}_\varepsilon(\beta_\varepsilon)$  using the geodesic equation (2.17) to obtain

$$\begin{aligned} \dot{V}_\varepsilon(\beta_\varepsilon) & = \dot{V}_\varepsilon^0 + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{1}{2} H(Z_\varepsilon(r))\delta'_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon^2(r) \, dr \\ & \quad + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon(r)\dot{Z}_\varepsilon(r) \, dr \\ & \quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{e(V_\varepsilon(r) + H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon(r))}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)} \, dr \\ & \quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\left(\frac{1}{2}\dot{U}_\varepsilon^2(r)\tilde{G}_\varepsilon(r) - \dot{U}_\varepsilon(r)(H(Z_\varepsilon(r))U_\varepsilon(r)\delta_\varepsilon(U_\varepsilon(r)))\right)V_\varepsilon(r)}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)} \, dr \\ & \quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\left(\frac{1}{2}\dot{U}_\varepsilon^2(r)\tilde{G}_\varepsilon(r) - \dot{U}_\varepsilon(r)(H(Z_\varepsilon(r))U_\varepsilon(r)\delta_\varepsilon(U_\varepsilon(r)))\right)H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon(r)}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)} \, dr \\ & =: \dot{V}_\varepsilon^0 + \mathbb{I}_\varepsilon + \mathbb{II}_\varepsilon + \mathbb{III}_\varepsilon + \mathbb{IV}_\varepsilon + \mathbb{V}_\varepsilon. \end{aligned} \tag{B.9}$$

Note that  $\mathbb{III}_\varepsilon \rightarrow 0$  because the integrand is uniformly bounded. Now, we rewrite  $\mathbb{I}_\varepsilon$  substituting  $s = U_\varepsilon(r)$ , abbreviating  $w := U_\varepsilon^{-1}(s)$  and using equation (2.17) for  $\dot{U}_\varepsilon$

$$\begin{aligned}
\mathbb{I}_\varepsilon &= \frac{1}{2} \int_{-\varepsilon}^\varepsilon H(Z_\varepsilon(w)) \delta'_\varepsilon(s) \dot{U}_\varepsilon(w) ds = 0 - \frac{1}{2} \int_{-\varepsilon}^\varepsilon \delta_\varepsilon(s) (H(Z_\varepsilon(w)) \dot{U}_\varepsilon(w))' ds \\
&= -\frac{1}{2} \int_{-\varepsilon}^\varepsilon \delta_\varepsilon(s) DH(Z_\varepsilon(w)) \dot{Z}_\varepsilon(w) ds \\
&\quad + \frac{1}{2} \int_{-\varepsilon}^\varepsilon \delta_\varepsilon(s) H(Z_\varepsilon(w)) \frac{\left(\frac{1}{2} \dot{U}_\varepsilon(w) \tilde{G}_\varepsilon(w) - (H(Z_\varepsilon) \delta_\varepsilon U_\varepsilon)'(w)\right) s}{\sigma a^2 - s^2 H(Z_\varepsilon(w)) \delta_\varepsilon(s)} ds + O(\varepsilon). \quad (\text{B.10})
\end{aligned}$$

Now, the integrals on the right hand side of (B.10) combine with  $\mathbb{II}_\varepsilon$  and  $\mathbb{V}_\varepsilon$  to give

$$\dot{V}_\varepsilon(\beta_\varepsilon) = \dot{V}_\varepsilon^0 + \frac{1}{2} \mathbb{II}_\varepsilon + \mathbb{III}_\varepsilon + \mathbb{IV}_\varepsilon + \frac{1}{2} \mathbb{V}_\varepsilon. \quad (\text{B.11})$$

Next, we insert equation (2.17) for  $\dot{Z}_\varepsilon$  into  $\mathbb{II}_\varepsilon$  and follow the same pattern as before. The only remarkable new point is the occurrence of the regularisation-dependent term  $\int_{-\varepsilon}^\varepsilon \delta_\varepsilon(s)^2 s ds$ , the prefactors of which cancel after a long and tedious calculation, where we repeatedly use identities such as  $\int_{-\varepsilon}^\varepsilon \int_{-\varepsilon}^s \delta_\varepsilon(r) dr ds = \frac{1}{2}$ .

For example, we obtain for the term in (B.11) related to  $\dot{Z}_\varepsilon^0$

$$\begin{aligned}
&\left| \frac{1}{2} \int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH(Z_\varepsilon(r)) \delta_\varepsilon(U_\varepsilon(r)) \dot{U}_\varepsilon(r) \dot{Z}_\varepsilon(r) dr - \frac{1}{2} DH(Z^0) \dot{Z}^0 \right| \\
&= \frac{1}{2} \left| \int_{-\varepsilon}^\varepsilon \left( DH(Z_\varepsilon(U_\varepsilon^{-1}(s))) \delta_\varepsilon(s) \dot{Z}_\varepsilon(U_\varepsilon^{-1}(s)) - DH(Z^0) \delta_\varepsilon(s) \dot{Z}^0 \right) ds \right| \\
&\leq \sup_{w \in U_\varepsilon^{-1}([-\varepsilon, \varepsilon])} \frac{1}{2} \|\rho\|_{L^1} \left( |DH(Z_\varepsilon(w))| |\dot{Z}_\varepsilon^0 - \dot{Z}^0| + |\dot{Z}^0| |DH(Z_\varepsilon(w)) - DH(Z^0)| \right) \rightarrow 0.
\end{aligned}$$

## References

- [1] Aichelburg P C and Sexl R U 1971 On the gravitational field of a massless particle *Gen. Relativ. Gravit.* **2** 303–12
- [2] Balasin H 1997 Geodesics for impulsive gravitational waves and the multiplication of distributions *Class. Quantum Grav.* **14** 455–62
- [3] Bainov D and Simeonov P S 1992 *Integral Inequalities and Applications* vol 57 (Dordrecht: Springer)
- [4] Barrabès C and Hogan P A 2003 *Singular Null Hypersurfaces in General Relativity* (Singapore: World Scientific)
- [5] Bondi H, Pirani F A E and Robinson I 1959 Gravitational waves in general relativity: III. Exact plane waves *Proc. Roy. Soc. London Ser. A* **251** 519–33
- [6] Candela A M, Flores J L and Sánchez M 2003 On general plane fronted waves. Geodesics *Gen. Relativ. Gravit.* **35** 631–49
- [7] Candela A M, Flores J L and Sánchez M 2004 Geodesic connectedness in plane wave type spacetimes. A variational approach *Dyn. Syst. Appl.* **4** 458–64.
- [8] Colombeau J F 1985 *Elementary Introduction to New Generalized Functions* (Amsterdam: North Holland)
- [9] Erlacher E and Grosse M 2011 Inversion of a ‘discontinuous coordinate transformation’ in general relativity *Appl. Anal.* **90** 1707–28
- [10] Erlacher E and Grosse M 2013 Inversion of Colombeau generalized functions *Proc. Edinb. Math. Soc.* **56** 469–500
- [11] Ferrari V, Pendenza P and Veneziano G 1988 Beam-like gravitational waves and their geodesics *Gen. Relativ. Gravit.* **20** 1185–91



- [12] Filippov A F 1988 *Differential Equations with Discontinuous Righthand Sides* (Dordrecht: Kluwer Academic Publishers)
- [13] Flores J L and Sánchez M 2003 Causality and conjugate points in general plane waves *Class. Quantum Grav.* **20** 2275–91
- [14] Flores J L and Sánchez M 2006 On the geometry of *pp*-wave type spacetimes *Analytical and Numerical Approaches to Mathematical Relativity* vol 692 (Berlin: Springer) pp 79–98
- [15] Geroch R and Traschen J 1987 Strings and other distributional sources in general relativity *Phys. Rev. D* **36** 1017–31
- [16] Griffiths J B and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge: Cambridge University Press)
- [17] Grosser M, Kunzinger M, Oberguggenberger M and Steinbauer R 2001 *Geometric Theory of Generalized Functions* vol 537 (Dordrecht: Kluwer Academic Publishers)
- [18] Hájek O 1985 Book review: differential systems involving impulses *Bull. Am. Math. Soc.* **12** 272–9
- [19] Hotta M and Tanaka M 1993 Shock-wave geometry with non-vanishing cosmological constant *Class. Quantum Grav.* **10** 307–14
- [20] Karamazov M 2015 Impulsive gravitational waves *Bachelor Thesis, Faculty of Mathematics and Physics* (Prague: Charles University in Prague) pp 23–7
- [21] Kunzinger M and Steinbauer R 1999 A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves *J. Math. Phys.* **40** 1479–89
- [22] Kunzinger M and Steinbauer R 1999 A note on the Penrose junction conditions *Class. Quantum Grav.* **16** 1255–64
- [23] Lecke A, Steinbauer R and Švarc R 2014 The regularity of geodesics in impulsive *pp*-waves *Gen. Relativ. Gravit.* **46** 1648
- [24] LeFloch P and Madare C 2007 Definition and stability of Lorentzian manifolds with distributional curvature *Port. Math. (N.S.)* **64** 535–73
- [25] Penrose R 1972 The geometry of impulsive gravitational waves *General Relativity* ed L O'Raifeartaigh (Oxford: Clarendon Press) pp 101–15
- [26] Podolský J 1998 Non-expanding impulsive gravitational waves *Class. Quantum Grav.* **15** 3229–39
- [27] Podolský J 2002 Exact impulsive gravitational waves in spacetimes of constant curvature *Gravitation: Following the Prague Inspiration* ed O Semerák, J Podolský and M Žofka (Singapore: World Scientific) pp 205–46
- [28] Podolský J and Griffiths J B 1997 Impulsive gravitational waves generated by null particles in de Sitter and anti-de Sitter backgrounds *Phys. Rev. D* **56** 4756–67
- [29] Podolský J and Griffiths J B 1998 Impulsive waves in de Sitter and anti-de Sitter space-times generated by null particles with an arbitrary multipole structure *Class. Quantum Grav.* **15** 453–63
- [30] Podolský J and Griffiths J B 1999 Nonexpanding impulsive gravitational waves with an arbitrary cosmological constant *Phys. Lett. A* **261** 1–4
- [31] Podolský J and Ortaggio M 2001 Symmetries and geodesics in (anti-)de Sitter spacetimes with non-expanding impulsive gravitational waves *Class. Quantum Grav.* **18** 2689–706
- [32] Podolský J, Sämman C, Steinbauer R and Švarc R 2015 The global existence, uniqueness and  $C^1$ -regularity of geodesics in nonexpanding impulsive gravitational waves *Class. Quantum Grav.* **32** 025003
- [33] Podolský J and Steinbauer R 2003 Geodesics in spacetimes with expanding impulsive gravitational waves *Phys. Rev. D* **67** 064013
- [34] Podolský J and Švarc R 2010 Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant *Phys. Rev. D* **81** 124035
- [35] Podolský J and Veselý K 1999 Smearing of chaos in sandwich *pp*-waves *Class. Quantum Grav.* **16** 3599–618
- [36] Sämman C and Steinbauer R 2012 On the completeness of impulsive gravitational waves *Class. Quantum Grav.* **29** 245011
- [37] Sämman C and Steinbauer R 2015 Geodesic completeness of generalized space-times *Pseudo-Differential Operators and Generalized Functions (Operator Theory: Advances and Applications)* ed S Molahajloo, S Pilipović, J Toft and M W Wong 245 (Basel: Birkhäuser Verlag) pp 243–53
- [38] Sfetsos K 1995 On gravitational shock waves in curved spacetimes *Nucl. Phys. B* **436** 721–46

- [39] Steinbauer R 1998 Geodesics and geodesic deviation for impulsive gravitational waves *J. Math. Phys.* **39** 2201–12
- [40] Steinbauer R 2014 Every Lipschitz metric has  $C^1$ -geodesics *Class. Quantum Grav.* **31** 057001
- [41] Steinbauer R On the geometry of impulsive gravitational waves arXiv:[gr-qc/9809054v2](https://arxiv.org/abs/gr-qc/9809054v2)
- [42] Steinbauer R and Vickers J A 2009 On the Geroch–Traschen class of metrics *Class. Quantum Grav.* **26** 065001
- [43] Weissinger J 1952 Zur theorie und anwendung des iterationsverfahrens *Mathematische Nachrichten* **8** 193–212