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Comment on "Chaotic orbits for spinning particles in Schwarzschild spacetime"

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The astrophysical relevance of chaos for a test particle with spin moving in Schwarzschild spacetime was the objective of C. Verhaaren and E. W. Hirschmann in [Phys. Rev. D **81**, 124034 (2010)]. Even if the results of the study might appear to be qualitatively in agreement with similar works, the study presented in their work suffers both from theoretical and technical issues. These issues are discussed in this comment.

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I. ISSUE OF SCALING

The Mathisson-Papapetrou (MP) equations [1,2] describe the motion of an extended body in the pole-dipole approximation on a curved spacetime. In the MP description, the body has some internal degrees of freedom, which are constrained to fix its centroid, i.e., the wordline along which the body moves. This constraint is imposed when a spin supplementary condition (SSC) is chosen. In Ref. [3], the Tulczyjew (T) SSC has been chosen [4]. This makes Ref. [3] comparable with previous similar works since T SSC has been used in Ref. [5] (Schwarzschild background as in Ref. [3]) and in Refs. [6–8] (Kerr background).

When studying the MP equations with T SSC, one chooses the mass of the test particle to be described by the contraction of the four-momentum, i.e., $P^a P_a = -\mathcal{M}^2$, since this mass is a conserved quantity for T SSC (see, e.g., Ref. [9]). In Ref. [3], the mass is chosen with respect to the four-velocity, i.e., $P^a V_a = -\mu$, which is not a conserved quantity for T SSC (see, e.g., [9]). This choice brings along some complications when the spin is scaled with respect to μm as stated to be done in Ref. [3], where m is the mass of the central black hole (the notation of Ref. [3] is adopted throughout the comment). For example, the measure of the spin is a constant of motion for T SSC (see, e.g., Ref. [9]), but when one normalizes the spin with something that varies, the constancy of the spin ceases to be the case.

In order to understand these complications, let us discuss the scaling issue in more detail. The MP equations can be written in scale free units if the spin is scaled with respect to $m\mathcal{M}$, i.e.,

$$\frac{DP^\mu/\mathcal{M}}{d\tau/m} = -\frac{1}{2}(R^\mu{}_{\nu\kappa\lambda}m^2)V^\nu \frac{S^{\kappa\lambda}}{m\mathcal{M}},$$
$$\frac{DS^{\mu\nu}/(m\mathcal{M})}{d\tau/m} = (P^\mu V^\nu - V^\mu P^\nu)/\mathcal{M}, \quad (1)$$

where each quantity has been written with respect to its scale factor (see, e.g., Ref. [7]). It is easy to see that the scale factors cancel out. Now, if we follow the scalings suggested in Ref. [3], then

$$\frac{DP^\mu/\mathcal{M}}{d\tau/m} = -\frac{1}{2}(R^\mu{}_{\nu\kappa\lambda}m^2)V^\nu \frac{S^{\kappa\lambda}}{m\mu},$$
$$\frac{DS^{\mu\nu}/(m\mu)}{d\tau/m} = (P^\mu V^\nu - V^\mu P^\nu)/\mathcal{M}, \quad (2)$$

and we get

$$\frac{DP^\mu}{d\tau} = -\frac{\mathcal{M}}{2\mu}R^\mu{}_{\nu\kappa\lambda}V^\nu S^{\kappa\lambda},$$
$$\frac{DS^{\mu\nu}}{d\tau} = \frac{\mu}{\mathcal{M}}(P^\mu V^\nu - V^\mu P^\nu), \quad (3)$$

where the scales do not vanish.

One could argue that the scales would vanish if the momentum P^a was scaled with respect to μ and not with respect to $\mathcal{M}$. This is true, but in Ref. [3], it is said that $P^a P_a = -1$, which suggests either that the momentum in Ref. [3] is scaled with respect to $\mathcal{M}$ or that $P^a P_a = -\mathcal{M}^2/\mu^2 = -1$. The latter cannot be the case since during the evolution μ varies, while $\mathcal{M}$ is a constant, and in general $\mu \neq \mathcal{M}$ for T SSC (see, e.g., Ref. [9]). The rescaling of the momentum with respect to $\mathcal{M}$ is reflected on Eqs. (10), (13), and (14) in Ref. [3]. What is missing from Eqs. (13) and (14) is the rescaling of the spin four-vector S^a. S^a is the vector counterpart of S^{ab}; see, e.g., Eq. (10) in Ref. [3]. Equations (13) and (14) hold for the $m\mathcal{M}$ rescaling of the spin (e.g., the usual rescaling for T SSC used in Refs. [5,6]). But if the $m\mu$ rescaling of the spin was used in Ref. [3], as stated in Sec. II of Ref. [3], then the corresponding Eqs. (13) and (14) in Ref. [3] should include the ratios $\mu/\mathcal{M}$ as shown in the corresponding Eqs. (3) shown above. Thus, the rescaling implied by Eqs. (13) and (14) in Ref. [3] is inconsistent with the $m\mu$ rescaling of the spin stated in Ref. [3].

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This inconsistency is reflected also on Eqs. (15)–(17) of Ref. [3]. Furthermore, in Eq. (15) of Ref. [3], the first term P^a in the numerator should not share the denominator with the second term. Namely, Eq. (15) should read

$$V^\mu = \frac{\mu}{\mathcal{M}^2}\left(P^\mu - \frac{{}^*R^{*\nu\rho\kappa\lambda} S_\rho P_\kappa S_\lambda}{\mathcal{M}^2 + {}^*R^{*\alpha\beta\gamma\delta} S_\alpha P_\beta S_\gamma P_\delta}\right). \quad (4)$$

But this is probably just a typo. The main issue here is that the stated rescaling is in contradiction with the formulas presented.

Note also that if μ was considered constant, we would not be able to normalize the four-velocity so that $V_a V^a = -1$ in order to evolve the MP equations with T SSC. The variability of μ is what allows the four-velocity normalization (see, e.g., Ref. [9]).

Concluding on the spin scaling issue, if the $m\mu$ scaling was used, then the equations of motion used in Ref. [3] are wrong, and there is no point of further discussion. However, if the $\mu/\mathcal{M}$ scaling was used in Ref. [3], then the statements about the rescaling in Sec. II of Ref. [3] are wrong, and the equations of motion are correct. Assuming the latter, Ref. [3] is a reexamination of Ref. [5] for a larger parameter domain, and it is meaningful to discuss the chaos detection issue in the next section.

II. POINCARÉ SECTION AND THE LYAPUNOV NUMBER ISSUE

Poincaré sections are a useful tool to discern chaos from order in a 2 degrees of freedom Hamiltonian system. Regular orbits are represented by closed zero width smooth curves, while chaotic orbits are represented by scattered points covering a nonzero width space on the section. So, the Poincaré sections shown in Figs. 1 and 2 of Ref. [3] should indeed represent regular orbits.

However, the MP equations with T SSC have not been yet been described by a canonical Hamiltonian formalism (contrary to what is stated on page 3 in Ref. [3]), and the spin increases the phase space dimensionality [10]. So, one should be careful when interpreting 2D "Poincaré" sections for a test particle with spin moving in a Schwarzschild background. In the latter system, what one gets usually for regular orbits on a 2D section are projections of tori of which the dimensionality is higher than 3. These projections on a 2D section are represented by nonzero width curves in the case of regular orbits, and there is no safe way to discern chaos from order just from inspecting these projections on 2D sections. On such 2D sections, one has to detect a combination of nonlinear effects, like chains of islands of stability embedded in chaos (see, e.g., Figs. 4, 8, and 9 in Ref. [5]), in order to claim chaos detection. From Figs. 3 and 4 of Ref. [3], one cannot tell whether the orbits are regular or not, contrary to what is stated in Ref. [3] (see, e.g., the caption of Fig. 4 in Ref. [3]). Actually, in Ref. [8], there is a case (Fig. 3 in Ref. [8]) where an orbit looking like those in Figs. 3 and 4 of Ref. [3] was characterized as "chaotic mimic," because when such a 2D section was checked with other indicators of chaoticity, the other indicators implied that the orbit was regular.

One such indicator of chaoticity is the characteristic Lyapunov number λ. Lyapunov numbers were indeed employed to cross-check the results of Figs. 3 and 4 in Ref. [3]. However, defining the Lyapunov number for curved spacetimes is not a straightforward task as discussed in Sec. III of Ref. [3]. The main problem is how to define the norm of the deviation vector $\bar{\xi}$. Different norms $\xi = |\bar{\xi}|$ result in different values for the Lyapunov number. In Ref. [3], it is stated that for simplicity the Euclidean norm was preferred as the norm of ξ in that work, but no further information about the explicit form of the norm is provided. Namely, there are the questions of how the Euclidean norm was applied in the Schwarzschild coordinates of the orbit and how the spin was incorporated in the Euclidean norm of the deviation vector. Without the above explanations, the results of this work are not reproducible and ambiguous.

A standard way to find whether an orbit is chaotic or not by using Lyapunov numbers is the $\ln\lambda$ vs $\ln\tau$ plot (see, e.g., Ref. [8]). For a regular orbit, the deviation vector grows linearly, i.e., $\xi \propto \tau$, which means that $\lambda \propto \frac{\ln\tau}{\tau}$. On the logarithmic plot, this implies that for a regular orbit $\lim_{\tau\to 0} \ln\lambda \to -\infty$, i.e., $\lambda \to 0^+$, with a slope equal to -1. Note that the Lyapunov number is practically evaluated for finite time, and during this even for regular orbits $\lambda > 0$. If an orbit is evolved for time τ and τ is of the order of magnitude of $1/\lambda$, then we cannot tell whether this orbit is chaotic or not. For a chaotic orbit, the deviation vector grows exponentially $\xi \propto e^{\lambda\tau}$, which means that in the logarithmic plot we get a constant value $\ln\lambda$ after the Lyapunov time $\tau_\lambda = \lambda^{-1}$ is reached. In order to be sure that one gets a chaotic orbit, one has to evolve the orbit at least for 2 orders of magnitude more after τ_λ is reached, so λ is not any more comparable with $1/\tau$.

In Ref. [3], the above standard procedure is absent. The procedure to find the Lyapunov number in Ref. [3] is based on a phenomenological model (Eq. (28) in Ref. [3]) which is irrelevant with the basic principles describing the evolution of the deviation vector discussed in the above paragraph. The example in Fig. 6 of Ref. [3] evolves an orbit for $\tau = 10^5$ and predicts an orbit with a Lyapunov number $\lambda \approx 3.78710^{-4}$. For $\tau = 10^5$, $\lambda \propto \frac{\ln\tau}{\tau} \approx 6.4710^{-4}$. Namely, for the amount of time the orbit has been evolved in Fig. 6, the Lyapunov number has a value that is comparable with a value of λ corresponding to a regular orbit. Thus, one cannot tell safely whether the orbit is chaotic or not. In fact, there are many Lyapunov-like chaotic indicators (see, e.g., Ref. [11] for a review), but no indicator can safely reveal the chaotic nature of an orbit at time scales comparable with the Lyapunov time. All the indicators show the nature of the orbit, much after the magnitude of the Lyapunov time has been reached.

In order to investigate the dependence of the chaoticity of the MP equations on the spin's value in Ref. [3], the energy, the angular momentum, the initial radius r of the orbit and the orientation of the spin are kept constant, while the spin's value varies (Figs. 7–13 and 15 in Ref. [3]). This might seem reasonable since the investigation depends only on one varying parameter, but this approach is misleading. The phase space of the system is mixed in the sense that chaotic and regular orbits coexist in the phase space. When we change a parameter of the system, the phase space changes, and as a consequence, the position of the orbit we suppose to follow changes as well. For example, if we start with an initial setup at which the orbit we examine is chaotic, by changing the spin parameter, the orbit with the otherwise same setup will correspond to another trajectory, which might be chaotic or not. Even if we assume that the method of estimating the Lyapunov numbers followed in Ref. [3] was correct, then what we see in Figs. 7–13 and 15 of Ref. [3] is not correlated with the chaoticity of one single orbit; i.e., it cannot provide qualitative information about the development of the system.

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