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Exact solutions to quadratic gravity

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Since all Einstein spacetimes are vacuum solutions to quadratic gravity in four dimensions, in this paper we study various aspects of non-Einstein vacuum solutions to this theory. Most such known solutions are of traceless Ricci and Petrov type N with a constant Ricci scalar. Thus we assume the Ricci scalar to be constant which leads to a substantial simplification of the field equations. We prove that a vacuum solution to quadratic gravity with traceless Ricci tensor of type N and aligned Weyl tensor of any Petrov type is necessarily a Kundt spacetime. This will considerably simplify the search for new non-Einstein solutions. Similarly, a vacuum solution to quadratic gravity with traceless Ricci type III and aligned Weyl tensor of Petrov type II or more special is again necessarily a Kundt spacetime. Then we study the general role of conformal transformations in constructing vacuum solutions to quadratic gravity. We find that such solutions can be obtained by solving one nonlinear partial differential equation for a conformal factor on any Einstein spacetime or, more generally, on any background with vanishing Bach tensor. In particular, we show that all geometries conformal to Kundt are either Kundt or Robinson–Trautman, and we provide some explicit Kundt and Robinson–Trautman solutions to quadratic gravity by solving the above mentioned equation on certain Kundt backgrounds.

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I. INTRODUCTION AND SUMMARY

An extension of the Einstein–Hilbert action by adding higher-order terms in curvature is a natural generalization of Einstein's gravity theory. In the first approximation, these corrections give quadratic terms, and in four dimensions, they admit a general form[1]

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\gamma' (R - 2\Lambda) - \alpha' C_{abcd} C^{abcd} + \beta' R^2 \right). \quad (1)$$

Various four and higher-dimensional theories with such quadratic terms in the action and their exact solutions were studied in the literature (see e.g., [1–6]), with first particular theories proposed already shortly after the introduction of Einstein's general relativity [7,8].

The *vacuum* field equations of *quadratic gravity* following from the action (1) read

$$\gamma' \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right) - 4\alpha' B_{ab} + 2\beta' \left(R_{ab} - \frac{1}{4} R g_{ab} + g_{ab} \Box - \nabla_b \nabla_a \right) R = 0, \quad (2)$$

where $\Box \equiv g^{ab} \nabla_a \nabla_b$, and B_{ab} is the Bach tensor

$$B_{ab} \equiv \left(\nabla^c \nabla^d + \frac{1}{2} R^{cd} \right) C_{acbd}, \quad (3)$$

which is traceless, symmetric, and conserved (i.e., $B^{ab}{}_{;b} = 0$). It can be also equivalently written as

$$B_{ab} = \frac{1}{2} \Box R_{ab} - \frac{1}{6} \left(\nabla_a \nabla_b + \frac{1}{2} g_{ab} \Box \right) R - \frac{1}{3} R R_{ab} + R_{acbd} R^{cd} + \frac{1}{4} \left(\frac{1}{3} R^2 - R_{cd} R^{cd} \right) g_{ab}. \quad (4)$$

From this expression, it can be seen that the Bach tensor vanishes for all Einstein spacetimes.[2] The last term in the field equations (2) vanishes for Einstein spacetimes as well. This leads to the well-known observation that in four dimensions, *all vacuum solutions to the Einstein theory* (including possibly a cosmological constant Λ) *solve also vacuum equations of the quadratic gravity* (2). Note that this result does not extend to dimensions $n > 4$. In this sense, Einstein spacetimes are trivial vacuum solutions to the four-dimensional quadratic gravity. The main objective of this paper is to study general properties of nontrivial solutions to quadratic gravity, i.e., *non-Einstein* spacetimes obeying the vacuum field equations (2).

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[1]The constant γ' is often denoted by $1/\kappa$.

[2]Einstein spacetimes in four dimensions are defined by $R_{ab} = \frac{1}{4} R g_{ab}$, where the Ricci scalar R is necessarily constant.

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Due to the complexity of these fourth-order nonlinear field equations, only few non-Einstein exact solutions are known. In 1990, non-Einstein plane wave vacuum solutions to quadratic gravity were found [9]. Recently, AdS waves admitting a cosmological constant have been constructed using the Kerr–Schild ansatz [1]. Note that such AdS waves, which are in fact conformal to pp-waves, solve quadratic gravity in any dimension. Additional non-Einstein Kundt solutions to quadratic gravity have been found in [2], see Sec. II A for the definition of Kundt spacetimes.

In fact, all these explicit solutions to quadratic gravity are Kundt (i.e., spacetimes admitting a nonexpanding, shear-free, twistfree, geodetic null congruence, see Sec. II A). Moreover, their Ricci scalar is constant, which leads to a simplification of the field equations. In this paper, *we also focus on solutions with* $R = \text{const}$. Then the trace of (2), which is

$$\gamma'(4\Lambda - R) + 6\beta' \Box R = 0, \tag{5}$$

implies (for $\gamma' \neq 0$)[3] that

$$R = 4\Lambda, \tag{6}$$

and the field equations (2) reduce considerably to

$$(\gamma' + 8\beta'\Lambda)(R_{ab} - \Lambda g_{ab}) = 4\alpha' B_{ab}. \tag{7}$$

A. Kundt solutions to quadratic gravity

Since for all the above mentioned solutions to quadratic gravity their Weyl and traceless Ricci tensors are of type N in the algebraic classification [10,11], we begin with vacuum solutions to quadratic gravity with traceless Ricci tensor of type N. In Sec. II, we will prove

Proposition 1.1. *A vacuum solution to quadratic gravity* (2) *with the Ricci tensor of the form*

$$R_{ab} = \Lambda g_{ab} + \omega' \ell_a \ell_b, \qquad \omega' \neq 0, \qquad \ell^a \ell_a = 0,$$

and aligned Weyl tensor of any Petrov type is necessarily Kundt.

For traceless Ricci type N and aligned Weyl type N, this result has been already obtained in [2]. Note that in contrast, for $\omega' = 0$ (Einstein spacetimes), expansion and twist (and for Petrov type I also shear) can be nonvanishing.

Then, we will proceed with a generalization of this result to the case of traceless Ricci type III:

Proposition 1.2. *A vacuum solution to quadratic gravity* (2) *with the Ricci tensor of the form*

$$R_{ab} = \Lambda g_{ab} + \psi'_i(\ell_a m^{(i)}_b + m^{(i)}_a \ell_b) + \omega' \ell_a \ell_b, \quad \psi'_i \psi'_i \neq 0,$$

and aligned Weyl tensor of Petrov type II, or more special, is necessarily Kundt.

Note that the Ricci tensor is expressed using a null frame introduced in Sec. II.

Since the Kundt spacetimes have been extensively studied (see [12,13]), propositions 1.1 and 1.2 will allow for a systematic search of vacuum solutions of quadratic gravity (2) of the above Ricci types. Particular examples of such Kundt solutions [1,2,9] were mentioned above.

B. Conformally Kundt solutions to quadratic gravity

Non-Kundt (and non-Einstein) solutions to quadratic gravity also exist. Remarkably, a non-Schwarzschild static spherically symmetric black hole solution in Einstein–Weyl gravity (with $\Lambda = 0$) has been found very recently in [6], where its two metric functions are given in terms of two ordinary differential equations (ODE). We will point out that this solution belongs to the Robinson–Trautman (RT) class and in fact due to (the part of) proposition 3.1 of Sec. III it is necessarily conformal to Kundt:

Proposition 1.3. *All Robinson–Trautman spacetimes are conformal to Kundt.*

Under the conformal transformation

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{8}$$

the Bach tensor transforms as

$$\tilde{B}_{ab} = \Omega^{-2} B_{ab}. \tag{9}$$

Thus obviously, the Bach tensor vanishes not only for all Einstein spacetimes but also for all spacetimes *conformal* to Einstein spacetimes. However, vanishing of the Bach tensor is not a sufficient condition for a spacetime to be conformally related to an Einstein spacetime [14]. Indeed, explicit examples of spacetimes with vanishing Bach tensor which are not conformal to Einstein spacetimes are known [15,16].

One can employ (9) to construct new exact solutions to quadratic gravity with arbitrary nonzero parameters α', β', γ' but a special value of the cosmological constant

$$\Lambda = -\frac{\gamma'}{8\beta'} \tag{10}$$

with (6). The case $\Lambda \neq -\frac{\gamma'}{8\beta'}$ will be discussed elsewhere. Under the assumption (10), the equations of quadratic gravity (7) reduce to

[3] Note that for certain values of the parameters α', β', γ', the quadratic gravity reduces to more special theories. In particular, quadratic gravity with $\beta' = 0$ is *Einstein–Weyl gravity*. As follows from (5), for this theory the Ricci scalar is constant by default. Another important subcase of the quadratic gravity is *conformal gravity* given by $\beta' = 0 = \gamma'$, for which the field equations (2) reduce to $B_{ab} = 0$.

$$B_{ab} = 0. \tag{11}$$

Specifically, we will use "seed" geometries g_{ab} with vanishing Bach tensor *to generate solutions* $\tilde{g}_{ab}$ *to quadratic gravity using the conformal transformation* (8), implying $\tilde{B}_{ab} = 0$ due to (9).[4] It remains to satisfy

$$\tilde{R} = 4\Lambda, \tag{12}$$

where $\tilde{R}$ is the Ricci scalar of the conformally transformed metric $\tilde{g}_{ab}$. It is well known that the Ricci scalars of conformally related metrics obey

$$\tilde{R} = R\Omega^{-2} - 6\Omega^{-3}\Box\Omega. \tag{13}$$

Thus we can satisfy (12) by choosing the conformal factor Ω that solves the equation

$$6\Box\Omega - R\Omega + 4\Lambda\Omega^3 = 0, \tag{14}$$

where Λ is constrained by (10). Thus, in this approach, the problem of constructing new solutions to quadratic gravity reduces to solving one nonlinear partial differential equation (PDE) (14) with a cubic nonlinearity for one unknown function Ω on a curved background spacetime with vanishing Bach tensor.

In Sec. IV, we will use this generating technique to derive several explicit new solutions to quadratic gravity.

To conclude, let us note that there are solutions to quadratic gravity that are neither Kundt nor Robinson–Trautman. One such Petrov type N twisting solution will be briefly discussed in Sec. IV. For this solution, the Ricci tensor is more general than the form given in proposition 1.2.

II. KUNDT SPACETIMES IN QUADRATIC GRAVITY

In this section, we study spacetimes with certain algebraically special forms of the Ricci tensor and aligned Weyl tensor. We show that the vacuum field equations of quadratic gravity imply that these spacetimes are necessarily Kundt.

In the case of constant Ricci scalar, the Bach tensor (4) can be expressed as

$$\begin{aligned} B_{ab} &= \frac{1}{2}\Box R_{ab} - \frac{1}{3}RR_{ab} + R_{acbd}R^{cd} + \frac{1}{4}\left(\frac{1}{3}R^2 - R_{cd}R^{cd}\right)g_{ab} \\ &= B^R_{ab} + B^C_{ab}, \end{aligned} \tag{15}$$

where

$$B^R_{ab} = \frac{1}{2}\Box R_{ab} + \frac{1}{3}RR_{ab} - R_{ac}R^c{}_b + \frac{1}{4}\left(R_{cd}R^{cd} - \frac{1}{3}R^2\right)g_{ab}, \tag{16}$$

$$B^C_{ab} = C_{acbd}R^{cd} \tag{17}$$

are parts of the Bach tensor depending only on the Ricci tensor and also on the Weyl tensor, respectively. Let us employ a real null frame with two null vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ and two spacelike vectors $\boldsymbol{m}^{(i)}$ $(i,\ j = 2, 3)$ obeying

$$\ell^a\ell_a = n^a n_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a}m^{(j)}_a = \delta_{ij}. \tag{18}$$

For the algebraic classification of tensors, the crucial concept is a *boost weight* (b.w.). A quantity q has the boost weight b if it transforms as

$$\hat{q} = \lambda^{\mathrm{b}} q \tag{19}$$

under a boost

$$\hat{\boldsymbol{\ell}} = \lambda\boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1}\boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}. \tag{20}$$

Various frame components of a tensor will have in general different integer boost weights and we define boost order of a tensor $\boldsymbol{T}$ with respect to a given frame as the maximum b.w. of its frame components. It can be shown that the boost order of $\boldsymbol{T}$ in fact depends only on the frame vector $\boldsymbol{\ell}$ and thus we will denote it as $\mathrm{b}_{\boldsymbol{\ell}}(\boldsymbol{T})$ (see, e.g., [11]). Obviously $\mathrm{b}_{\boldsymbol{\ell}}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) = \mathrm{b}_{\boldsymbol{\ell}}(\boldsymbol{T}_1) + \mathrm{b}_{\boldsymbol{\ell}}(\boldsymbol{T}_2)$. Note also that boost order of a tensor does not increase under a contraction of indices.

Let us study spacetimes with the Ricci tensor of the form

$$R_{ab} = \Lambda g_{ab} + \psi'_i(\ell_a m^{(i)}_b + m^{(i)}_a\ell_b) + \omega'\ell_a\ell_b, \tag{21}$$

which clearly obeys (6). Boost order of the traceless Ricci tensor is thus -1 (for $\psi'_i\psi'_i \neq 0$) or -2 (for $\psi'_i\psi'_i = 0$, $\omega' \neq 0$) and thus the Ricci tensor is of type III or N, respectively (see [11]). From (16), we obtain

$$\begin{aligned} B^R_{ab} = {}& \frac{1}{2}\Box R_{ab} - \frac{2}{3}\Lambda\psi'_i(\ell_a m^{(i)}_b + m^{(i)}_a\ell_b) \\ & - \left(\frac{2}{3}\Lambda\omega' + \psi'_i\psi'_i\right)\ell_a\ell_b. \end{aligned} \tag{22}$$

A. Traceless Ricci type N

First, let us focus on the traceless Ricci type N for which

$$R_{ab} = \Lambda g_{ab} + \omega'\ell_a\ell_b, \qquad \omega' \neq 0. \tag{23}$$

From the contracted Bianchi equations $\nabla^b R_{ab} = \frac{1}{2}\nabla_a R = 0$ and (23), it follows that $\boldsymbol{\ell}$ is geodetic and without loss of

[4]Note that the seed metrics themselves *need not* to solve the field equations (2) of quadratic gravity.

generality, one can choose ℓ to be affinely parametrized and a frame to be parallelly transported along ℓ. Then, the covariant derivatives of the frame vectors in terms of spin coefficients read [11]

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m_b^{(i)} + \tau_i m_a^{(i)}\ell_b + \rho_{ij} m_a^{(i)} m_b^{(j)}, \tag{24}$$

$$n_{a;b} = -L_{11}n_a\ell_b - L_{1i}n_a m_b^{(i)} + \kappa'_i m_a^{(i)}\ell_b + \rho'_{ij} m_a^{(i)} m_b^{(j)}, \tag{25}$$

$$m^{(i)}_{a;b} = -\kappa'_i\ell_a\ell_b - \tau_i n_a\ell_b - \rho'_{ij}\ell_a m_b^{(j)} + \overset{i}{M}_{j1} m_a^{(j)}\ell_b - \rho_{ij} n_a m_b^{(j)} + \overset{i}{M}_{kl} m_a^{(k)} m_b^{(l)}. \tag{26}$$

Here, the optical matrix

$$\rho_{ij} \equiv \ell_{a;b} m^a_{(i)} m^b_{(j)} \tag{27}$$

can be decomposed into its trace θ *(expansion)*, trace-free symmetric part σ_{ij} and antisymmetric part A_{ij}, namely

$$\rho_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij},$$
$$\sigma_{ij} \equiv \rho_{(ij)} - \frac{1}{2}\rho_{kk}\delta_{ij}, \qquad \theta \equiv \frac{1}{2}\rho_{kk}, \quad A_{ij} \equiv \rho_{[ij]}. \tag{28}$$

Optical scalars *shear* and *twist* of ℓ are traces $\sigma^2 \equiv \sigma^2_{ii} = \sigma_{ij}\sigma_{ji}$ and $\omega^2 \equiv -A^2_{ii} = -A_{ij}A_{ji}$, respectively. Kundt spacetimes are defined as spacetimes with vanishing ρ_{ij}.

Using (23) and (24), we express $\nabla_c R_{ab}$

$$\nabla_c R_{ab} = D\omega'\ell_a\ell_b n_c + \omega'\rho_{ij}(m_a^{(i)}\ell_b + \ell_a m_b^{(i)}) m_c^{(j)} + \text{terms of b.w.} \le -2, \tag{29}$$

where $D \equiv \ell^a\nabla_a$. Employing (24)–(26), a further differentiation of (29) leads to

$$\Box R_{ab} = [-\omega'\rho_{ij}\rho_{ij}(\ell_a n_b + n_a\ell_b) + 2\omega'\rho_{ik}\rho_{jk} m_a^{(i)} m_b^{(j)}] + \text{terms of b.w.} \le -1, \tag{30}$$

i.e., from (22) and (30)

$$B^R_{ab} = \frac{1}{2}[-\omega'\rho_{ij}\rho_{ij}(\ell_a n_b + n_a\ell_b) + 2\omega'\rho_{ik}\rho_{jk} m_a^{(i)} m_b^{(j)}] + \text{terms of b.w.} \le -1. \tag{31}$$

For the Ricci tensor of the form (23), the left-hand side of the field equations of quadratic gravity (7) contains b.w. −2 terms only, while in general, the right-hand side contains terms of b.w. 0 due to the presence of the term $\Box R_{ab}$. Recall that we assume that the Weyl tensor is aligned with the Ricci tensor (i.e., $\mathrm{b}_{\ell}(C_{abcd}) \le 1$)[5] and thus $\mathrm{b}_{\ell}(B^C_{ab}) = \mathrm{b}_{\ell}(C_{abcd}) + \mathrm{b}_{\ell}(R_{ab} - \Lambda g_{ab}) \le -1$. Consequently, the leading term in (31) has to vanish, i.e.,

$$\rho_{ij}\rho_{ij} = 0. \tag{32}$$

Thus the optical matrix vanishes, $\rho_{ij} = 0$, obviously implying also $\rho_{ik}\rho_{jk} = 0$ for all i, j, which concludes the proof of proposition 1.1.

B. Traceless Ricci type III

Let us proceed with a more general form of the Ricci tensor (21) with $\psi'_i\psi'_i \ne 0$. First, we prove that ℓ is geodetic using the standard four-dimensional Newman–Penrose (NP) formalism. For the Ricci tensor of the form (21), the relevant NP components are Φ_{22} and $\Phi_{12} = \bar{\Phi}_{21} \ne 0$. For the Petrov types III/N/O, the Bianchi equation (7.32b) of [12] reduces to

$$\kappa\Phi_{12} = 0, \tag{33}$$

which implies that ℓ is geodetic. Similarly, for the Petrov type II, Eq. (7.32a) of [12] gives $\kappa\Psi_2 = 0$ and thus ℓ is also geodetic.

The first derivative of the Ricci tensor (21) reads

$$\nabla_c R_{ab} = m_a^{(i)} m_b^{(j)} m_c^{(k)}(\psi'_j\rho_{ik} + \psi'_i\rho_{jk}) + (m_a^{(i)}\ell_b + \ell_a m_b^{(i)}) n_c D\psi'_i + (n_a\ell_b + \ell_a n_b) m_c^{(i)}(-\psi'_s\rho_{si}) + \text{b.w.} \le -1 \text{ terms.} \tag{34}$$

Further differentiation gives

$$\Box R_{ab} = -(m_a^{(i)} n_b + n_a m_b^{(i)})(2\psi'_s\rho_{sk}\rho_{ik} + \psi'_i\rho_{sk}\rho_{sk}) + \text{terms of b.w.} \le 0, \tag{35}$$

i.e., from (22) and (35)

$$B^R_{ab} = -\frac{1}{2}(m_a^{(i)} n_b + n_a m_b^{(i)})(2\psi'_s\rho_{sk}\rho_{ik} + \psi'_i\rho_{sk}\rho_{sk}) + \text{terms of b.w.} \le 0. \tag{36}$$

If the Weyl tensor of any Petrov type and the Ricci tensor (21) are aligned then $\mathrm{b}_{\ell}(B^C_{ab}) \le 0$. Thus the b.w. +1 terms in (36) are the only b.w. > 0 terms in (7) and therefore they have to vanish. By multiplying (36) by ψ'_i, we get

[5] Note that for the Weyl types III and N this is not an assumption since from the Bianchi equations it follows that Weyl type III/N traceless Ricci type N spacetimes are aligned (see [17]).

$$(\psi'_i\psi'_i)(\rho_{sk}\rho_{sk}) + 2(\psi'_s\rho_{sk})(\psi'_i\rho_{ik}) = 0. \tag{37}$$

For $\psi'_i\psi'_i \neq 0$, the expression (37) clearly vanishes iff $\rho_{ij} = 0$ which concludes the proof of proposition 1.2.

Note that proposition 1.2 is valid also for aligned Petrov type I spacetimes with the Ricci tensor of the form (21) and a geodetic $\boldsymbol{\ell}$, however, in this case we did not prove geodecity of $\boldsymbol{\ell}$.

III. CONFORMAL RELATIONS OF KUNDT AND ROBINSON–TRAUTMAN SPACETIMES

To our knowledge, all exact solutions to quadratic gravity discussed in the literature so far are either Kundt, or conformal to it. It is thus important to identify the class of all spacetimes that are conformal to Kundt geometries, which admit a null geodetic congruence $\boldsymbol{\ell}$ with vanishing shear, twist, and expansion.[6] It has a canonical metric form [18–22]

$$\mathrm{d}s^2_{\mathrm{Kundt}} = 2H(u,r,x)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2W_i(u,r,x)\mathrm{d}u\mathrm{d}x^i + g_{ij}(u,x)\mathrm{d}x^i\mathrm{d}x^j, \tag{38}$$

where $\boldsymbol{\ell} = \partial_r$ (with a dual $-\mathrm{d}u$). A generic Kundt metric is of the Riemann type I or more special [22,23], and of the Weyl subtype I(b) in $n > 4$ [24], with $\boldsymbol{\ell}$ being both principal null direction (PND) and an aligned null direction of the Ricci tensor (since R_{rr} vanishes identically).

Now, a conformally transformed metric

$$\mathrm{d}\tilde{s}^2 = \Omega^2(u,r,x)\,\mathrm{d}s^2_{\mathrm{Kundt}} \tag{39}$$

is obviously of the *same Weyl type*, while the *Ricci type is in general distinct* from the Ricci type of (38). The vector $\boldsymbol{\ell}$ is also a null geodetic PND of the new metric (39) with vanishing shear and twist, while its nontrivial *expansion* reads

$$\tilde{\theta} = \frac{1}{n-2}(\tilde{g}^{ab}\ell_b)_{;a} = \frac{\Omega_{,r}}{\Omega^3}. \tag{40}$$

Under the conformal transformation (39), the Ricci tensor and scalar transform as [25]

$$\tilde{R}_{ab} = R_{ab} - \Omega^{-1}[(n-2)\delta^c_a\delta^d_b + g_{ab}g^{cd}]\nabla_c\nabla_d\Omega + \Omega^{-2}[2(n-2)\delta^c_a\delta^d_b - (n-3)g_{ab}g^{cd}](\nabla_c\Omega)(\nabla_d\Omega), \tag{41}$$

$$\tilde{R} = R\Omega^{-2} - 2(n-1)\Omega^{-3}\Box\Omega - (n-1)(n-4)\Omega^{-4}(\nabla_a\Omega)(\nabla^a\Omega). \tag{42}$$

[6]Since the results of this section are dimension-independent, here we work in a general dimension n.

Thus, in contrast with the Kundt metric (38), for the new metric (39), *the highest boost weight component of the Ricci tensor is in general nonvanishing*[7]:

$$\tilde{R}_{ab}\tilde{\ell}^a\tilde{\ell}^b = \tilde{g}^{ac}\tilde{g}^{bd}\tilde{R}_{ab}\ell_c\ell_d = \Omega^{-4}\tilde{R}_{rr} = -(n-2)\Omega^{-6}(\Omega\Omega_{,rr} - 2\Omega^2_{,r}). \tag{43}$$

Since $\boldsymbol{\ell} = \partial_r$ is geodetic, shearfree and twistfree null direction in the conformally related metric (39), this new metric is a *Robinson–Trautman* metric (as long as $\Omega_{,r} \neq 0$) or *Kundt* for $\Omega_{,r} = 0 \Leftrightarrow \tilde{\theta} = 0$, see (40). This can be explicitly seen by transforming (39) into the canonical Robinson–Trautman form [21,26,27]

$$\mathrm{d}\tilde{s}^2_{\mathrm{RT}} = 2\tilde{H}(u,\tilde{r},x)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}\tilde{r} + 2\tilde{W}_i(u,\tilde{r},x)\mathrm{d}u\mathrm{d}x^i + \mathcal{R}^2(u,\tilde{r},x)g_{ij}(u,x)\mathrm{d}x^i\mathrm{d}x^j, \tag{44}$$

where

$$\begin{aligned} \tilde{r} &= \rho(u,r,x), \qquad \text{such that} \quad \rho_{,r} = \Omega^2(u,r,x),\\ \mathrm{d}\tilde{r} &= \Omega^2\mathrm{d}r + \rho_{,u}\mathrm{d}u + \rho_{,i}\mathrm{d}x^i,\\ \tilde{H} &= \Omega^2 H + \rho_{,u},\\ \tilde{W}_i &= \Omega^2 W_i + \rho_{,i},\\ \mathcal{R} &= \Omega. \end{aligned} \tag{45}$$

In fact, by comparing the expansion, $\theta = \frac{1}{n-2}\ell^a{}_{;a}$, shear $\sigma^2 = \ell_{(a;b)}\ell^{(a;b)} - \frac{1}{n-2}(\ell^a{}_{;a})^2$, and twist $\omega^2 = \ell_{[a;b]}\ell^{a;b}$ of the geodetic affinely parametrized null vector ℓ_a expressed in the original and conformally transformed metrics, we arrive at (see also [28] for conformal properties of flows in arbitrary dimension)

$$\tilde{\theta} = \frac{\theta}{\Omega^2} + \frac{\boldsymbol{\ell}(\Omega)}{\Omega^3}, \qquad \tilde{\sigma}^2 = \frac{\sigma^2}{\Omega^4}, \qquad \tilde{\omega}^2 = \frac{\omega^2}{\Omega^4}, \tag{46}$$

where $\boldsymbol{\ell}(\Omega) \equiv \Omega_{,r}$. Together with the above results, this leads to

Proposition 3.1. *Spacetimes conformal to shearfree or twistfree spacetimes are shearfree or twistfree, respectively. In particular*:

(1) *All spacetimes conformal to Robinson–Trautman are Robinson–Trautman or Kundt.*
(2) *All Robinson–Trautman spacetimes are conformal to Kundt.*
(3) *All spacetimes conformal to Kundt are Robinson–Trautman (when $\boldsymbol{\ell}(\Omega) \equiv \Omega_{,r} \neq 0$) or Kundt (when $\boldsymbol{\ell}(\Omega) \equiv \Omega_{,r} = 0$).*[8]

[7]We set $\tilde{\ell}_a = \ell_a$. Note that this choice preserves geodeticity and the affine parametrization.

[8]Since Kundt and Robinson–Trautman spacetimes are defined by the existence of a geodetic shear-free and twist-free null congruence with $\theta = 0$ and $\theta \neq 0$, respectively, interestingly there are exceptional spacetimes that belong to *both* of these classes admitting two distinct congruences with these properties, see, e.g., the metric (71), (73).

Note that in the case of four dimensions, an extension of the Goldberg–Sachs theorem to conformally Einstein spacetimes immediately follows [12] and thus algebraically special solutions to quadratic gravity obtained by a conformal transformation of Einstein spacetimes are shearfree.

It has been shown [27,29] that in contrast to the four-dimensional case, for $n > 4$ *Einstein* Robinson–Trautman spacetimes of types III and N do not exist. From the above results, it follows that *non-Einstein Robinson–Trautman geometries of types N and III can be clearly constructed by a conformal transformation from their Kundt counterparts in any dimension*. Furthermore, starting with universal Kundt metrics of types N and III [30], one obtains type N and III Robinson–Trautman vacuum solutions to $n > 4$ conformal gravities (theories of gravity invariant under conformal transformations). The strong constraints on the optical matrix of higher-dimensional type N and III spacetimes, implying the nonexistence of Einstein Robinson–Trautman solutions within these classes, is thus connected to the Einstein field equations rather than to the geometric properties of Robinson–Trautman spacetimes in higher dimensions.

A. Static spherically symmetric spacetimes

As an important illustration, let us investigate static spherically symmetric spacetimes

$$\mathrm{d}s^2 = -h(\bar r)\mathrm{d}t^2 + \frac{\mathrm{d}\bar r^2}{f(\bar r)} + \bar r^2 \mathrm{d}\omega^2_{n-2},$$
$$\mathrm{d}\omega^2_{n-2} = \left(1 + \frac{1}{4}\delta_{kl}x^k x^l\right)^{-2} \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j. \qquad (47)$$

These spacetimes are of the Weyl type D in any dimension [31] and include many black hole solutions of various theories. They belong to the Robinson–Trautman class. Indeed, by performing a coordinate transformation

$$\mathrm{d}t = \mathrm{d}u + \frac{\mathrm{d}\bar r}{\sqrt{hf}}, \qquad \mathrm{d}\tilde r = \sqrt{h/f}\,\mathrm{d}\bar r \qquad (48)$$

we arrive at the *canonical Robinson–Trautman form* (44) with $\tilde W_i = 0$

$$\mathrm{d}\tilde s^2_{\mathrm{RT}} = -h(\bar r(\tilde r))\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}\tilde r + \bar r^2(\tilde r)\mathrm{d}\omega^2_{n-2}. \qquad (49)$$

A further coordinate transformation (45) for $\tilde r = \rho(r)$ such that $\mathrm{d}\tilde r = \Omega^2(r)\mathrm{d}r$ brings the metric (49) to the *form manifestly conformal to Kundt*

$$\mathrm{d}\tilde s^2_{\mathrm{RT}} = \Omega^2(r)\mathrm{d}s^2_{\mathrm{Kundt}}$$
$$= \Omega^2(r)\left(\mathcal{H}(r)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}\omega^2_{n-2}\right), \qquad (50)$$

cf. (38) with $\mathcal{H} = 2H$ and $W_i = 0$, where

$$\Omega^2\mathcal{H} = -h,$$
$$\Omega(r) = \bar r(\tilde r(r)),$$
$$\sqrt{\frac{f}{h}} = \frac{\mathrm{d}\bar r}{\mathrm{d}\tilde r} = \frac{\mathrm{d}\Omega}{\mathrm{d}\tilde r} = \Omega_{,r}\frac{\mathrm{d}r}{\mathrm{d}\tilde r} = \frac{\Omega_{,r}}{\Omega^2}. \qquad (51)$$

Note that the "seed" Kundt metric appearing in (50) is a *direct-product geometry*, containing as special cases, e.g., Bertotti–Robinson or Nariai space [13,32].

For the Schwarzschild–Tangherlini solution,

$$f(\bar r) = h(\bar r) = 1 - \mu\bar r^{3-n} \qquad (52)$$

in (47), and thus

$$\Omega = \bar r = \tilde r = -1/r,$$
$$\mathcal{H} = (-1 + \mu\bar r^{3-n})/\bar r^2 = -r^2 + (-1)^{n-1}\mu r^{n-1}. \qquad (53)$$

IV. EXACT SOLUTIONS TO QUADRATIC GRAVITY

Now let us discuss solutions to quadratic gravity (11) obtained via solving Eq. (14) on an appropriate seed spacetime. In principle, one can solve this nonlinear equation numerically on any background Einstein spacetime or, more generally, on any spacetime with vanishing Bach tensor. Here we will focus on cases where solutions can be obtained explicitly.

First, let us note that [16] gives a list of several metrics with vanishing Bach tensor that are not conformally Einstein. Many of them have constant Ricci scalar. Thus, one can obtain exact solutions of quadratic gravity (11) from these seeds by appropriate constant rescaling of these metrics to set $\tilde R = 4\Lambda = -\frac{\gamma'}{2\beta'}$. Interestingly, apart from Kundt metrics (solutions 2–5 of [16]), this rescaling leads also to solutions of quadratic gravity outside the Kundt and Robinson–Trautman classes. For example for a type N twisting, shearfree, expansion-free metric 6 of [16][9]

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + 10\mathrm{d}u\left(\frac{\mathrm{d}v}{r^2} - \frac{2\mathrm{d}x}{r}\right) + 2r\mathrm{d}x\mathrm{d}v$$
$$+ \frac{10\mathrm{d}u^2}{r^4} + r^2\mathrm{d}x^2, \qquad (54)$$

[9]The multiple PND, $\boldsymbol{\ell} = \partial_v$, of the metric (54) is twisting and thus it is not Kundt, nor RT. Since for Petrov type N the PND is unique this metric also does not admit a Kundt or RT congruence distinct from $\boldsymbol{\ell}$ (it follows from purely geometric considerations that Kundt or RT congruences always coincide with PNDs [11,22,33]).

the Ricci tensor is constant, $R = -3$, and thus an appropriate constant rescaling leads to a type N twisting solution of quadratic gravity with $\gamma'/\beta' > 0$.

In the following, we focus on solving Eq. (14) on the Kundt backgrounds (38). This approach leads to Kundt and Robinson–Trautman solutions of quadratic gravity, cf. proposition 3.1. The d'Alembert operator applied to a function $\Omega(u, r, x^i)$ then reads explicitly

$$\begin{aligned}\Box\Omega \equiv{}& g^{ab}\Omega_{;ab}\\ ={}& (-2H + W^iW_i)\Omega_{,rr} - 2\Omega_{,ru} + 2W^i\Omega_{,ri} + g^{ij}\Omega_{||ij}\\ &+ [-2H_{,r} + 2W^iW_{i,r} + g^{ij}(W_{(i||j)} - \tfrac{1}{2}g_{ij,u})]\Omega_{,r}\\ &+ g^{ij}W_{i,r}\Omega_{,j}\,, \end{aligned} \tag{55}$$

where $||$ denotes the covariant derivative associated with the spatial metric g_{ij}. In case of Kundt metrics without off-diagonal terms ($g_{ui} = W_i = 0$), this simplifies considerably to

$$\begin{aligned}\Box\Omega ={}& -2H\Omega_{,rr} - 2\Omega_{,ru} - 2H_{,r}\Omega_{,r}\\ &+ g^{ij}(\Omega_{||ij} - \tfrac{1}{2}g_{ij,u}\Omega_{,r}). \end{aligned} \tag{56}$$

A. Solutions generated by a *pp*-wave seed

On a pp-wave background,

$$\mathrm{d}s^2_{\text{seed}} = 2H(u,x,y)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{57}$$

Eq. (14) with (56), assuming Ω to be function of z, where

$$z \equiv p_a x^a = p_r r + p_u u + p_x x + p_y y, \qquad p_a = \text{const.} \tag{58}$$

and using the fact that $R = 0$, reduces to

$$\Omega'' + \frac{\tilde{R}}{6p_a p^a}\Omega^3 = 0, \tag{59}$$

where prime denotes the derivative with respect to z and the contravariant vector p^a is obtained using the metric (57). If either $p_r = 0$ or $H = \text{const.}$, this is a constant-coefficients ODE for $\Omega(z)$, with the first integral

$$\begin{aligned}&\Omega'^2 = -L\Omega^4 + K,\\ &K = \text{const.}, \qquad L = \frac{\tilde{R}}{12p_a p^a} = \frac{\Lambda}{3p_a p^a}. \end{aligned} \tag{60}$$

Equation (60) can be integrated:

(i) for the case with a vanishing integration constant $K = 0$ and $L < 0$,

$$\Omega(z) = \pm\sqrt{-\frac{1}{L}\frac{1}{z-c}}, \qquad c = \text{const.}, \tag{61}$$

(ii) while for a nonvanishing K it can be solved in terms of elliptic Jacobi function[10]

$$\begin{aligned}&\Omega(z) = K^{\frac{1}{4}}L^{-\frac{1}{4}}\,\mathrm{sn}\,(K^{\frac{1}{4}}L^{\frac{1}{4}}(z-c), -1),\\ &\quad c = \text{const.}, \qquad K > 0, \qquad L \neq 0. \end{aligned} \tag{62}$$

In the $L \to 0$ limit, this reduces to

$$\Omega(z) = \sqrt{K}(z-c), \qquad c = \text{const.}, \qquad K > 0, \tag{63}$$

which solves (59) with $\tilde{R} = 0$. However, note that (63) corresponds to a solution of a special subcase of quadratic gravity with $\gamma' = 0$ only.

In general, the Weyl, Ricci, and Bach tensors of (57) possess only boost weight -2 components. In particular

$$R_{uu} = -\Delta H = -(H_{,xx} + H_{,yy}), \tag{64}$$

$$B_{uu} = -\frac{1}{2}\Delta\Delta H = -\frac{1}{2}(H_{,xxxx} + 2H_{,xxyy} + H_{,yyyy}), \tag{65}$$

so that R_{uu} and B_{uu} vanish if, and only if,

$$R_{uu} = 0 \Leftrightarrow H = F(\zeta) + \bar{F}(\bar{\zeta}), \tag{66}$$

$$B_{uu} = 0 \Leftrightarrow H = F(\zeta) + \bar{F}(\bar{\zeta}) + \bar{\zeta}G(\zeta) + \zeta\bar{G}(\bar{\zeta}), \tag{67}$$

respectively, where F and G are arbitrary holomorphic functions of $\zeta \equiv x + \mathrm{i}y$. Note that for $G \neq 0$, the Bach tensor vanishes, while the Ricci tensor does not.

(1) Case $p_r = 0$: All the seed metrics (57) with (67) thus generate explicit solutions to quadratic gravity (2) with (10), using the conformal transformations (61) and (62) with $p_r = 0$, i.e., $z = p_u u + p_x x + p_y y = p_u u + p_\zeta\zeta + p_{\bar{\zeta}}\bar{\zeta}$:

(a) The solutions corresponding to (61) are of the Weyl and traceless Ricci type N and represent AdS waves, cf. [1], which have the Siklos geometry [13]

$$\begin{aligned}\mathrm{d}\tilde{s}^2 = -\frac{1}{L}\frac{1}{(z-c)^2}[&2(F(\zeta) + \bar{F}(\bar{\zeta}) + \bar{\zeta}G(\zeta)\\ &+ \zeta\bar{G}(\bar{\zeta}))\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2]. \end{aligned} \tag{68}$$

(b) The solutions corresponding to (62) are of the Weyl type N and traceless Ricci type II

[10] Note that sometimes this elliptic function is denoted as sn(, i).

$$
\mathrm{d}\tilde{s}^2 = \sqrt{\frac{K}{L}}\mathrm{sn}^2(K^{\frac{1}{4}}L^{\frac{1}{4}}(z-c),-1) \times [2(F(\zeta)+\bar{F}(\bar{\zeta})+\bar{\zeta}G(\zeta)+\zeta\bar{G}(\bar{\zeta}))\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2]. \quad (69)
$$

They involve curvature singularities at zeroes of the function sn (that is at $z = c$ and $z = c + P$ where P is the period) since

$$
\tilde{R}_{ab}\tilde{R}^{ab} = \frac{1}{12}\tilde{R}^2\left(3 + \mathrm{sn}^{-8}\left(K^{\frac{1}{4}}L^{\frac{1}{4}}(z-c),-1\right)\right). \quad (70)
$$

(2) Case $p_r \neq 0$: In the exceptional case of a flat background ($2H = 1$), it is possible to employ the conformal transformation (62) with $p_r \neq 0$. The resulting solution of quadratic gravity (2) with (10)

$$
\mathrm{d}\tilde{s}^2 = \sqrt{\frac{K}{L}}\mathrm{sn}^2(K^{\frac{1}{4}}L^{\frac{1}{4}}(z-c),-1) \times [\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2] \quad (71)
$$

is a conformally flat Robinson–Trautman metric. The generic case is of general Ricci type. However, in the frame

$$
\boldsymbol{\ell} = -\mathrm{d}u, \qquad \boldsymbol{n} = \Omega(z)^2\left(\mathrm{d}r - \frac{1}{2}\mathrm{d}u\right),
$$
$$
\boldsymbol{m}^{(2)} = \Omega(z)\mathrm{d}x, \qquad \boldsymbol{m}^{(3)} = \Omega(z)\mathrm{d}y, \quad (72)
$$

components of the Ricci tensor $R_{ab}n^a n^b$ and $R_{ab}n^a m^b_{(i)}$ vanish for

$$
p_r + 2p_u = 0. \quad (73)
$$

Thus, in this case the vector $\boldsymbol{n}$ is a multiply aligned null direction of the Ricci tensor and the spacetime is therefore of (traceless) Ricci type II. In fact, $\boldsymbol{n}$ is geodetic, shearfree, twistfree, and nonexpanding and thus this spacetime belongs to *both* the Kundt (with respect to $\boldsymbol{n}$) and Robinson–Trautman (with respect to $\boldsymbol{\ell}$) classes.

Obviously, a coordinate freedom can be used to simplify the above metrics. This is left for future work, including their physical and geometrical study.

B. Robinson–Trautman solution of a general Ricci type

As another seed metric, let us consider direct-product Kundt metrics of the form

$$
\mathrm{d}s^2_{\mathrm{seed}} = 2H(r)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2, \quad (74)
$$

cf. (50). For a particular function

$$
H(r) = -\frac{1}{16}c^3r^3 \pm \frac{1}{4}c\sqrt{3cd}\,r^2 - dr + a, \quad cd \geq 0, \quad (75)
$$

the Bach tensor vanishes while the Ricci scalar is $R = 2H_{,rr}$. One can show that for $a = 0 = d$, the metric is conformal to a Ricci-flat spacetime (with $\Omega = r^{-1}$), i.e., the metric is conformally Einstein, and from now on we study this case.

Equation (14) then reads

$$
\Omega'' + \frac{3}{r}\Omega' + \frac{1}{r^2}\Omega + \frac{4\tilde{R}}{3c^3r^3}\Omega^3 = 0. \quad (76)
$$

An exact solution of this equation of the form

$$
\Omega^2(r) = \frac{1}{2}c_1 r \quad \text{with} \quad c_1 = -\frac{27c^3}{8\tilde{R}} > 0 \quad (77)
$$

leads, introducing $\tilde{r} = c_1(r/2)^2$ and rescaling both x and y by a constant factor $(c_1/c)^{\frac{3}{4}}$, to a Robinson–Trautman solution of quadratic gravity in the form

$$
\mathrm{d}\tilde{s}^2_{\mathrm{RT}} = -b^2\tilde{r}^2\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}\tilde{r} + \sqrt{b\tilde{r}}(\mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2), \quad (78)
$$

where $b^2 = c^3/c_1 > 0$, i.e., $\tilde{R} < 0$. This metric was discussed in different coordinates in the context of conformal gravity (i.e., for $\beta' = 0 = \gamma'$) in [16], see Eq. (12) therein with $b = z_3 = 4z_1 = 4z_2$). Note that this spacetime is of *Weyl type D* while it is of the *general Ricci type* (with respect to $\ell_a \mathrm{d}x^a = -\mathrm{d}u$).

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