Algebraic aspects of general non-twisting and shear-free spacetimes

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The algebraic structure, given by a null alignment of the Weyl tensor, of expanding Robinson-Trautman and non-expanding Kundt geometries is analyzed in an arbitrary dimension. Conditions for all possible algebraic types are identified in closed form. Since the expansion parameter Θ is explicitly kept in all expressions, it can be simply set to zero to obtain results for the Kundt class. Usefulness of these general results obtained for all non-twisting and shear-free geometries in any metric theory of gravitation are demonstrated on specific vacuum solutions to the Einstein field equations.

Keywords: Optical scalars; Robinson–Trautman and Kundt spacetimes; algebraic types.

1. Introduction

This contribution is based primarily on our recent works[1,2] in which we investigated the algebraic structure of a fully general class on non-twisting and shear-free geometries in any dimension D, i.e., the *Robinson–Trautman* and *Kundt* family.[3–8]

The line element of the most general non-twisting and shear-free geometry[1,5,6] can be written as

$$\mathrm{d}s^2 = g_{pq}(r,u,x)\,\mathrm{d}x^p\,\mathrm{d}x^q + 2\,g_{up}(r,u,x)\,\mathrm{d}u\,\mathrm{d}x^p - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \quad (1)$$

where $g_{pq} = R^2(r,u,x)\,h_{pq}(u,x)$ with $R = \exp\left(\int\Theta(r,u,x)\,\mathrm{d}r\right)$. The coordinates employed correspond to the non-twisting character of the spacetime, which is equivalent to the existence of its global *null foliation*. Coordinate u thus labels null hypersurfaces ($u = \text{const.}$) with a tangent *non-twisting* null vector field $\mathbf{k}$. The affine parameter r along a null geodesic congruence generated by $\mathbf{k}$ is taken as the second coordinate, i.e., $\mathbf{k} = \partial_r$. Finally, at any fixed u and r we are left with a Riemannian manifold covered[a] by $D-2$ spatial coordinates x^p.

The *Kundt class* is defined by having the vanishing expansion, $\Theta = 0$, in which case g_{pq} is r-independent and R reduces to $R = 1$, i.e. $g_{pq} = h_{pq}(u,x)$. The case $\Theta \neq 0$ gives the expanding *Robinson–Trautman class* for which R is a non-trivial function of r.

In particular, by projecting the Weyl tensor onto the natural null frame we evaluated the corresponding scalars and proved in Ref. 1 that all these geometries are of type I(b), or more special, with the *Weyl aligned null direction* (WAND) corresponding to the optically privileged non-twisting and shear-free null direction $\mathbf{k}$. We were able to explicitly derive the necessary and sufficient conditions of all principal

[a]Throughout this contribution, the indices m, n, p, q (ranging from 2 to $D-1$) label the *spatial coordinates*.

alignment types[9] with $\mathbf{k}$ being a multiple WAND. No field equations were employed in these calculations, so that all results are "purely geometrical". They can be applied in *any metric theory of gravity* that admits Robinson–Trautman and Kundt geometries. The nonexpanding Kundt family can be obtained by setting $\Theta = 0$.

Of course, there are specific constraints on the spacetime metric imposed by the field equations. To illustrate the utility of our results, we investigated the Robinson–Trautman vacuum solutions in *Einstein's theory.*[3,5,6] We proved that in $D > 4$ there only exist types D(a)≡D(abd), D(c)≡D(bcd) and O. This is in striking contrast to the classical $D = 4$ case, which is much richer (see Table 2 below).

2. Algebraic structure of the Weyl tensor

The most natural null frame, satisfying $\boldsymbol{k}\cdot\boldsymbol{l} = -1$, $\boldsymbol{m}_i\cdot\boldsymbol{m}_j = \delta_{ij}$,[b] for the metric (1) is given by

$$\boldsymbol{k} = \mathbf{k} = \partial_r\,, \qquad \boldsymbol{l} = \tfrac{1}{2} g_{uu}\,\partial_r + \partial_u\,, \qquad \boldsymbol{m}_i = m_i^p\,(g_{up}\,\partial_r + \partial_p)\,. \tag{2}$$

All the Weyl tensor components with respect to such a null frame are

$$\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}\, k^a\, m_i^b\, k^c\, m_j^d\,, & & \\
\Psi_{1^{ijk}} &= C_{abcd}\, k^a\, m_i^b\, m_j^c\, m_k^d\,, & \Psi_{1T^i} &= C_{abcd}\, k^a\, l^b\, k^c\, m_i^d \\
\Psi_{2^{ijkl}} &= C_{abcd}\, m_i^a\, m_j^b\, m_k^c\, m_l^d\,, & \Psi_{2S} &= C_{abcd}\, k^a\, l^b\, l^c\, k^d\,, \\
\Psi_{2^{ij}} &= C_{abcd}\, k^a\, l^b\, m_i^c\, m_j^d\,, & \Psi_{2T^{ij}} &= C_{abcd}\, k^a\, m_i^b\, l^c\, m_j^d\,, \\
\Psi_{3^{ijk}} &= C_{abcd}\, l^a\, m_i^b\, m_j^c\, m_k^d\,, & \Psi_{3T^i} &= C_{abcd}\, l^a\, k^b\, l^c\, m_i^d\,, \\
\Psi_{4^{ij}} &= C_{abcd}\, l^a\, m_i^b\, l^c\, m_j^d\,, & &
\end{aligned} \tag{3}$$

see e.g. Refs. 1, 2, 9, ordered according to their boost weight, and their irreducible components[c] are

$$\begin{aligned}
\tilde{\Psi}_{1^{ijk}} &= \Psi_{1^{ijk}} - \tfrac{2}{D-3}\delta_{i[j}\Psi_{1T^{k]}}\,, \\
\tilde{\Psi}_{2T^{(ij)}} &= \Psi_{2T^{(ij)}} - \tfrac{1}{D-2}\delta_{ij}\Psi_{2S}\,, \\
\tilde{\Psi}_{2^{ijkl}} &= \Psi_{2^{ijkl}} - \tfrac{2}{D-4}\big(\delta_{ik}\tilde{\Psi}_{2T^{(jl)}} + \delta_{jl}\tilde{\Psi}_{2T^{(ik)}} - \delta_{il}\tilde{\Psi}_{2T^{(jk)}} - \delta_{jk}\tilde{\Psi}_{2T^{(il)}}\big) \\
&\quad - \tfrac{4\,\delta_{i[k}\delta_{l]j}}{(D-2)(D-3)}\Psi_{2S}\,, \\
\tilde{\Psi}_{3^{ijk}} &= \Psi_{3^{ijk}} - \tfrac{2}{D-3}\delta_{i[j}\Psi_{3T^{k]}}\,.
\end{aligned} \tag{4}$$

A rather long calculation reveals that $\Psi_{0^{ij}} = 0 = \tilde{\Psi}_{1^{ijk}}$ so that a *general non-twisting and shear-free geometry* (1) is at least of *algebraic type* I(b) with $\boldsymbol{k} = \mathbf{k} = \partial_r$ being a WAND. The remaining Weyl scalars take the following explicit and

[b]The indices i, j, k, l label the transverse *spatial directions* of the frame.
[c]See Ref. 2 for relations of these Newman–Penrose-like quantities to other equivalent notations employed in Ref. 9.

surprisingly simple form:

$$\Psi_{1T^i} = m_i^p \,\tfrac{D-3}{D-2}\left[(-\tfrac{1}{2}g_{up,r} + \Theta g_{up})_{,r} + \Theta_{,p}\right], \tag{5}$$

$$\Psi_{2S} = \tfrac{D-3}{D-1}\,P\,, \tag{6}$$

$$\tilde{\Psi}_{2T^{(ij)}} = m_i^p m_j^q \,\tfrac{1}{D-2}\left(Q_{pq} - \tfrac{1}{D-2}\,g_{pq}\,Q\right), \tag{7}$$

$$\tilde{\Psi}_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q \;{}^S C_{mpnq}\,, \tag{8}$$

$$\Psi_{2^{ij}} = m_i^p m_j^q \,F_{pq}\,, \tag{9}$$

$$\Psi_{3T^i} = m_i^p \,\tfrac{D-3}{D-2}\,V_p\,, \tag{10}$$

$$\tilde{\Psi}_{3^{ijk}} = m_i^p m_j^m m_k^q \left(X_{pmq} - \tfrac{2}{D-3}\,g_{p[m}X_{q]}\right), \tag{11}$$

$$\Psi_{4^{ij}} = m_i^p m_j^q \left(W_{pq} - \tfrac{1}{D-2}\,g_{pq}W\right), \tag{12}$$

where

$$\begin{aligned}
P = {} & \left(\tfrac{1}{2}g_{uu,r} - \Theta g_{uu}\right)_{,r} + \tfrac{1}{(D-2)(D-3)}\,{}^S R - \tfrac{1}{4}\tfrac{D-4}{D-2}\,g^{mn}g_{um,r}g_{un,r} \\
& + \tfrac{1}{D-2}\left(g^{rn}g_{un,rr} + g^{mn}g_{um,r||n}\right) - \tfrac{2}{D-2}\,g^{rn}g_{un}\Theta_{,r} - 2\Theta_{,u} - \tfrac{4}{D-2}\,g^{rn}\Theta_{,n} \\
& - \Theta^2\tfrac{D-4}{D-2}\,g^{rn}g_{un} + \Theta\left(\tfrac{D-6}{D-2}\,g^{rn}g_{un,r} - \tfrac{2}{D-2}\,g^{mn}g_{um||n}\right),
\end{aligned} \tag{13}$$

$$\begin{aligned}
Q_{pq} = {} & {}^S R_{pq} + (D-4)\left[\tfrac{1}{2}\left(f_{pq} + g_{u(p}g_{q)u,rr}\right) - \left(\Theta_{,r} - \Theta^2\right)g_{up}g_{uq}\right. \\
& \left. -2g_{u(p}\Theta_{,q)} - \Theta\left(g_{u(p||q)} + 2g_{u(p}g_{q)u,r}\right)\right],
\end{aligned} \tag{14}$$

$$F_{pq} = g_{u[p,q],r} - g_{u[p}g_{q]u,rr} + 2\Theta(g_{u[p}g_{q]u,r} - g_{u[p,q]})\,, \tag{15}$$

$$\begin{aligned}
V_p = {} & \tfrac{1}{2}\left[\tfrac{1}{2}g_{uu}g_{up,rr} - g_{uu,rp} + g_{up,ru} - \tfrac{1}{2}g^{rn}g_{un,r}g_{up,r}\right. \\
& \left. + g^{mn}g_{um,r}E_{np} - g_{up}\left(g_{uu,rr} - \tfrac{1}{2}g^{mn}g_{um,r}g_{un,r}\right)\right] \\
& + \tfrac{1}{D-3}\left[\tfrac{1}{2}g^{rn}g_{un}g_{up,rr} + g^{mn}e_{m[n}g_{p]u,r} - g^{rn}g_{u[n,p],r} + \tfrac{1}{2}g^{rn}\left(g_{u[p,r||n]} + f_{pn}\right)\right. \\
& \left. - g^{mn}\left(g_{m[p,u||n]} + g_{u[m,p]||n}\right) - \tfrac{1}{2}g_{up}\left(g^{rn}g_{un,rr} + g^{mn}f_{mn}\right)\right] \\
& + \tfrac{1}{2}g_{up}g_{uu}\Theta_{,r} + g_{up}\Theta_{,u} + \tfrac{1}{2}g_{uu}\Theta_{,p} \\
& - \Theta\left[\tfrac{1}{2}g_{uu}g_{up,r} - g_{uu,p} + g_{up,u} - g^{rn}g_{u[n}g_{p]u,r} + g^{rn}E_{np} - g_{up}g_{uu,r}\right. \\
& \left. + \tfrac{1}{D-3}(3g^{rn}g_{u[n}g_{p]u,r} - 3g^{rn}g_{u[n,p]} - \tfrac{1}{2}g_{up}g^{mn}g_{mn,u} + \tfrac{1}{2}g^{rn}g_{np,u})\right],
\end{aligned} \tag{16}$$

$$\begin{aligned}
X_{pmq} = {} & g_{p[m,u||q]} + g_{u[q,m]||p} + g_{up}g_{u[m}g_{q]u,rr} + e_{p[m}g_{q]u,r} \\
& - g_{u[q}g_{m]u,r||p} - g_{up}g_{u[m,r||q]} - \tfrac{1}{2}g_{u[q}g_{m]u,r}g_{up,r} + \Theta\left(3g_{u[q}g_{m]u,r}g_{up}\right. \\
& \left. + g_{u[q}g_{m]p,u} + g_{u[q}g_{m]u||p} - g_{up||[m}g_{q]u} - 2g_{u[q,m]}g_{up}\right),
\end{aligned} \tag{17}$$

$$\begin{aligned}
W_{pq} = {} & -\tfrac{1}{2}g_{uu||p||q} - \tfrac{1}{2}g_{pq,uu} + g_{u(p,u||q)} - \tfrac{1}{2}g_{uu,r}e_{pq} + \tfrac{1}{2}g_{uu,(p}g_{q)u,r} - g_{uu,r(p}g_{q)u} \\
& + \tfrac{1}{2}g_{uu}g_{u(p,r||q)} + \tfrac{1}{2}g_{uu}g_{u(q}g_{p)u,rr} - \tfrac{1}{2}g_{uu,rr}g_{up}g_{uq} + g_{u(q}g_{p)u,ru} \\
& + \tfrac{1}{4}g^{mn}\left(g_{um}g_{un}g_{up,r}g_{uq,r} + g_{um,r}g_{un,r}g_{up}g_{uq}\right) - \tfrac{1}{2}g^{mn}g_{um}g_{un,r}g_{u(q}g_{p)u,r} \\
& + g^{mn}\left(E_{mp}E_{nq} + g_{um,r}E_{n(p}g_{q)u} - g_{um}E_{n(p}g_{q)u,r}\right) \\
& + \Theta\left(g_{up}g_{uq}g_{uu,r} + g_{uu,(p}g_{q)u} - g_{uu}g_{u(p}g_{q)u,r} - 2g_{u(p}g_{q)u,u} - \tfrac{1}{2}g_{uu}g_{pq,u}\right),
\end{aligned} \tag{18}$$

with contractions $Q \equiv g^{pq}Q_{pq}$, $X_q \equiv g^{pm}X_{pmq}$, $W \equiv g^{pq}W_{pq}$, and auxiliary quantities defined as $e_{pq} = g_{u(p||q)} - \frac{1}{2}g_{pq,u}$, $E_{pq} = g_{u[p,q]} + \frac{1}{2}g_{pq,u}$, and $f_{pq} = g_{u(p,r||q)} + \frac{1}{2}g_{up,r}g_{uq,r}$ (the symbol $_{||}$ stands for the covariant derivative with respect to transverse metric g_{pq}), and ${}^S C_{mpnq}$, ${}^S R_{pq}$ and ${}^S R$ are the Weyl tensor, Ricci tensor and Ricci scalar for g_{pq}, respectively.

Expressions (5)–(12) with (13)–(18) give the explicit conditions under which $\boldsymbol{k} = \partial_r$ becomes a *multiple WAND* and Weyl tensor will thus be algebraicaly special with respect to $\boldsymbol{k}$ (for explicit form of the conditions see Ref. 1). The corresponding classification scheme is summarized in Table 1.

Table 1. The principal Weyl alignment types and subtypes defined with respect to (multiple) WAND $\boldsymbol{k} = \partial_r$.

type	vanishing components of the Weyl tensor							
II ≡ I(a) = I(ab)	Ψ_{1T^i}							
II(a)	Ψ_{1T^i}	Ψ_{2S}						
II(b)	Ψ_{1T^i}	$\tilde{\Psi}_{2T^{(ij)}}$						
II(c)	Ψ_{1T^i}	$\tilde{\Psi}_{2ijkl}$						
II(d)	Ψ_{1T^i}	Ψ_{2ij}						
III ≡ II(abcd)	Ψ_{1T^i}	Ψ_{2S}	$\tilde{\Psi}_{2T(ij)}$	$\tilde{\Psi}_{2ijkl}$	Ψ_{2ij}			
III(a)	Ψ_{1T^i}	Ψ_{2S}	$\tilde{\Psi}_{2T(ij)}$	$\tilde{\Psi}_{2ijkl}$	Ψ_{2ij}	Ψ_{3T^i}		
III(b)	Ψ_{1T^i}	Ψ_{2S}	$\tilde{\Psi}_{2T(ij)}$	$\tilde{\Psi}_{2ijkl}$	Ψ_{2ij}	$\tilde{\Psi}_{3ijk}$		
N ≡ III(ab)	Ψ_{1T^i}	Ψ_{2S}	$\tilde{\Psi}_{2T(ij)}$	$\tilde{\Psi}_{2ijkl}$	Ψ_{2ij}	Ψ_{3T^i}	$\tilde{\Psi}_{3ijk}$	
O	Ψ_{1T^i}	Ψ_{2S}	$\tilde{\Psi}_{2T(ij)}$	$\tilde{\Psi}_{2ijkl}$	Ψ_{2ij}	Ψ_{3T^i}	$\tilde{\Psi}_{3ijk}$	Ψ_{4ij}

3. Illustration: Robinson–Trautman vacuum spacetimes

A fully general Robinson–Trautman vacuum solution in the Einstein theory[6] is given by the metric (1) with

$$g_{pq} = r^2 h_{pq}(u,x)\,, \quad g_{up} = r^2 e_p(u,x)\,, \quad g_{uu} = -a - b(u)\, r^{3-D} - c\, r + \gamma\, r^2\,, \tag{19}$$

and

$$a \equiv \frac{\mathcal{R}(u,x)}{(D-2)(D-3)}\,, \quad c \equiv -\frac{2}{D-2}\left(e^n{}_{||n} - \tfrac{1}{2}h^{mn}h_{mn,u}\right), \quad \gamma \equiv e^n e_n + \frac{2\Lambda}{(D-1)(D-2)}\,, \tag{20}$$

where the metric functions are restricted by constraints following from Einstein's equations:[d]

$$\Theta = r^{-1}\,, \qquad \mathcal{R}_{pq} = \frac{1}{D-2}\, h_{pq}\, \mathcal{R}\,, \qquad h_{pq,u} = 2\, e_{(p||q)} + c\, h_{pq}\,,$$
$$(D-4)\, a_{,p} = 0\,, \qquad h^{mn}\, a_{||m||n} + \tfrac{1}{2}(D-1)(D-2)\, b\, c + (D-2)\, b_{,u} = 0\,. \tag{21}$$

[d]The symbols $\mathcal{R}$, $\mathcal{R}_{pq}$ and $\mathcal{C}_{mpnq}$ stand for quantities calculated with respect to the spatial r-independent metric h_{pq}.

Expression (5) implies $\Psi_{1T^i} = 0$, so that these spacetimes are *at least of algebraic type* II. It is now convenient to perform a *null rotation* of the frame (2) with $\boldsymbol{k}$ fixed, see Appendix C of Ref. 10, to obatin an *alternative* null frame $\boldsymbol{k}' = \partial_r$, $\boldsymbol{l}' = -\frac{1}{2}g^{rr}\,\partial_r + \partial_u - e^p\,\partial_p$ and $\boldsymbol{m}'_i = m_i^p\,\partial_p$, in which the nonvanishing irreducible Weyl scalars (6)–(12) become

$$\Psi'_{2S} = -\tfrac{1}{2}(D-2)(D-3)\,b\,r^{1-D}\,, \tag{22}$$

$$\tilde{\Psi}'_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q\,\mathcal{C}_{mpnq}\,r^2\,, \tag{23}$$

$$\Psi'_{3T^i} = -m_i^p\,\tfrac{D-3}{D-2}\,a_{,p}\,r^{-1}\,, \tag{24}$$

$$\Psi'_{4^{ij}} = \tfrac{1}{2}m_i^p m_j^q\left[\left(a_{||p||q} + c_{||p||q}\,r\right) - \tfrac{1}{D-2}\,h_{pq}\,h^{mn}\left(a_{||m||n} + c_{||m||n}\,r\right)\right]. \tag{25}$$

Using the scheme given in Table 1, together with scalars (22)–(25) and constraints (21), we can explicitly determine the algebraic structure of all vacuum Robinson–Trautman spacetimes in any dimension D, see Table 2.

Table 2. The necessary and sufficient conditions for all possible algebraic (sub)types of the Robinson–Trautman vacuum solutions in Einstein's theory. The algebraic structure differs significantly in the case $D = 4$ and in $D > 4$.

type	$D = 4$	$D > 4$		
II(a)	$b = 0$	$b = 0$	⇔	D(a)
II(b)	always	always	⇔	D(b)
II(c)	always	$\mathcal{C}_{mpnq} = 0$	⇔	D(c)
II(d)	always	always	⇔	D(d)
III	II(abcd)			
III(a)	$b = 0 = \mathcal{R}_{,p}$	equivalent to		O
III(b)	always for $b = 0$	equivalent to		O
N	III(ab)			
O	$b = 0 = \mathcal{R}_{,p}$ and $c_{\vert\vert p\vert\vert q} = \frac{1}{D-2}\,h_{pq}\,h^{mn}c_{\vert\vert m\vert\vert n}$	equivalent to		D(ac)
D	$\mathcal{R}_{,p} = 0$ and $c_{\vert\vert p\vert\vert q} = \frac{1}{D-2}\,h_{pq}\,h^{mn}c_{\vert\vert m\vert\vert n}$	always		D(bd)

To prove the *non-existence of* type N *and* type II *vacuum solutions in* $D > 4$ it is crucial to employ the non-trivial identity $(D-4)\left(c_{||p||q} - \frac{1}{D-2}\,h_{pq}\,h^{mn}c_{||m||n}\right) = 0$, see Ref. 1 for more details.

In the case of Robinson–Trautman family the Einstein gravity is thus more restrictive in higher dimensions $D > 4$. There only type D Schwarzschild-like black hole solutions are allowed, while type N radiative spacetimes are prohibited. This is in striking contrast to the classical $D = 4$ case.

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