Aspects of stability of hairy black holes

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We analyze spherical and odd-parity linear perturbations of hairy black holes with a minimally coupled scalar field.

1. Spherical Modes

In order to have an asymptotically flat hairy black hole with a minimally coupled scalar field a necessary condition is a scalar field potential with a negative region. For a recent review see, for example.[1]

As is well-known the stability problem can be mapped to the analysis of the spectrum of a Schrödinger operator, which appears in the master equation for perturbations. An everywhere positive spectrum implies there are no modes which exponentially grow in time. Using the method of Ref. 2 we study the equations of motion for the radial perturbations of the form

$$-\frac{d^2u}{d\rho^2} + V_{\rm eff}u = E^2u \qquad \text{with} \qquad \delta\phi \propto e^{\mathrm{i}Et}u(\rho)\,, \tag{1}$$

where $V_{\rm eff}(\rho)$ (explicitly given below) is an effective potential in which the modes $u(\rho)$ propagate and ρ is a "tortoise" radial coordinate sending the horizon at minus infinity; it always exhibits a negative region. A sufficient condition for the existence of bound states with negative E^2 (for bounded $V_{\rm eff}$ that fall-off faster than $|\rho|^{-2}$) is the Simon criteria, which states that if $S \equiv \int_{-\infty}^{+\infty} V_{\rm eff}\, d\rho$ is negative there will always be at least one bound state with negative E^2; hence, we only study positive Simon integrals. Using "shooting" techniques to solve (1), we do indeed find unstable modes in Ref. 1. However, there is only a finite number of unstable modes and, moreover, their characteristic time of growth can be made arbitrarily large for certain values of the black holes parameters, as is the case if the size of the black hole is small enough.

We are interested in studying the linearized dynamics around a background solution. Hence, starting from the metric in the form

$$ds^2 = -\left[A(r) + \epsilon A_1(r,t)\right]dt^2 + \left[B(r) + \epsilon B_1(r,t)\right]dr^2 + C(r)d\Omega^2\ , \tag{2}$$

where r is a general radial coordinate, not necessarily ρ, and the scalar field is assumed in the form $\phi = \phi_0(r) + \epsilon\phi_1(r,t)$. We expand the scalar field potential as $V(\phi) = V_0 + \epsilon V_1\phi_1(r,t)$ where $V_0 = V(\phi_0)$, $V_n = \left.\frac{d^nV}{d\phi^n}\right|_{\phi=\phi_0}$. As a consequence of spherical symmetry all the dynamics is driven by the scalar field. Indeed, it is possible to write the metric perturbations in terms of the $\phi_1(r,t)$ by using the Einstein field equations. We introduce the master variable

$$\psi(\rho,t) = \phi_1(r,t)C(r)^{1/2}\ , \tag{3}$$

where ρ is the tortoise coordinate

$$d\rho = \left(\frac{B}{A}\right)^{1/2} dr \ . \tag{4}$$

The master equation is

$$-\partial_\rho^2\psi + V_{\rm eff}\psi = -\partial_t^2\psi \ , \tag{5}$$

with the effective potential

$$\frac{V_{\rm eff}}{A} = 4\kappa C\left[(\kappa V_0 C - 1)\left(\frac{d\phi_0}{dC}\right)^2 + V_1\left(\frac{d\phi_0}{dC}\right)\right] - \kappa V_0 + V_2 + \frac{1}{C} - \frac{1}{4B}\left(\frac{C'}{C}\right)^2 \ . \tag{6}$$

If ρ takes its values in the whole real line and $V_{\rm eff}$ is non-negative, the operator (5) is essentially self-adjoint and its spectrum is positive which implies that the background is mode stable under spherically symmetric perturbations.

As mentioned above, using the shooting method, we found the asymptotically flat black holes unstable with respect to spherically symmetric perturbations. However, if their mass M and size r_+ are small compared with the coupling constant appearing in the scalar field potential, the time for the instability to develop is long compared to the scale set by M. More details are given in Ref. 3, see also Ref. 4.

For asymptotically anti-de Sitter boundary conditions, the tortoise coordinate takes its values in the half real line $\rho \in \,]-\infty, 0]$. If $V_{\rm eff}$ is non-negative, the spectrum can still contain a negative eigenvalue which depends on the details of the theory and the boundary conditions of the scalar field. This was recently discussed in more detail by one of us in Ref. 5.

2. Odd-Parity Modes

In the second part we analyze the stability of hairy black holes under odd-parity perturbations following our work in Ref. 6. In contrast to the radial perturbations in asymptotically flat spacetimes we show that independently of the scalar field potential and of specific asymptotic properties of spacetime (asymptotically flat, de Sitter or anti-de Sitter), any static, spherically symmetric or planar, black hole solution of the Einstein theory minimally coupled to a real scalar field with a general potential is mode stable under linear odd-parity perturbations. We analyze their odd-parity perturbations following the general treatment of the "axial" perturbations of spherically symmetric spacetime which are *not* necessarily vacuum, by Chandrasekhar. The perturbed metric reads

$$ds^2 = -Adt^2 + Bdr^2 + C\left[\frac{dz^2}{(1-kz^2)} + \left(1-kz^2\right)(d\varphi + k_1 dt + k_2 dr + k_3 dz)^2\right] \ , \tag{7}$$

where k_1, k_2 and k_3 are functions of (t, r, z), $A(r)$, $B(r)$ and $C(r)$ are the metric functions parameterizing the most general static background solution of a scalar-tensor theory. For asymptotically locally AdS solutions, $k = \pm 1$ or 0. Asymptotically flat or de Sitter solutions have $k = 1$. The scalar field is taken to be of the form $\phi = \phi_0(r) + \epsilon\Phi(t,r,z)$, where ϕ_0 is the background field. The metric perturbations (k_1, k_2, k_3) are all taken to be first order in ϵ. Since any surface of constant (t,r) is of constant curvature, we consider only axisymmetric perturbations, without any loss of generality. The Einstein field equations are truncated at first order in ϵ. This yields the vanishing of Φ. Introducing the variable $Q = CA^{1/2}B^{-1/2}(1-kz^2)^2(\partial_z k_2 - \partial_r k_3)$ and assuming $Q = q(r,t)D(z)$, we find that a combination of the field equations yield

$$\frac{C^2}{\sqrt{AB}}\frac{\partial}{\partial r}\left[\frac{A^{1/2}}{CB^{1/2}}\frac{\partial q}{\partial r}\right] - \lambda q = \frac{C}{A}\partial_t^2 q \ , \tag{8}$$

$$\left(1-kz^2\right)^2\frac{\partial}{\partial z}\left[\frac{1}{(1-kz^2)}\frac{\partial D}{\partial z}\right] = -\lambda D \ , \tag{9}$$

where λ is a separation constant. Let us put $k=1$ and set $z = \cos\theta$ in equation (9); then $C_{l+2}^{-3/2}(\theta) = D(z)$ is the Gegenbauer polynomial with $\lambda = (l-1)(l+2)$, $l \geq 1$ holds. The master variable in this case is $\Psi(\rho,t) = q(r,t)C^{-1/2}$ where $\frac{\partial}{\partial r} = \frac{B^{1/2}}{A^{1/2}}\frac{\partial}{\partial \rho}$. Fourier decomposing the master variable, $\Psi(\rho) = \int \Psi_\omega e^{i\omega t}dt$, yields the master equation

$$\mathcal{H}\Psi_\omega \equiv -\frac{d^2\Psi_\omega}{d\rho^2} + \left(\lambda\frac{A}{C} + \frac{3}{4C^2}\left(\frac{dC}{d\rho}\right)^2 - \frac{1}{2C}\frac{d^2C}{d\rho^2}\right)\Psi_\omega = \omega^2\Psi_\omega \ . \tag{10}$$

The scalar field perturbation vanishes, however equation (10) depends on the background scalar field through its influence on the background metric (in vacuum, the equation (10) becomes the Regge-Wheeler equation). The operator $\mathcal{H}$ is not manifestly positive, but its spectrum is positively defined as has been shown by finding a suitable $S-$deformation.

3. Slowly Rotating Hairy Black Holes

Consider a stationary perturbations with $k_2 = k_3 = 0$ and $k_1 = \omega(r)$. In this case we find

$$\omega = -c_1\int\frac{\sqrt{AB}}{C^2}dr + c_2 \ , \tag{11}$$

where c_1 and c_2 are two integration constants. Let us first consider the case of the Schwarzschild black hole. We have $\sqrt{AB} = 1$ and $C = r^2$, so it follows that

$$\omega = \frac{c_1}{3r^3} + c_2 \ . \tag{12}$$

Hence, choosing $c_2 = 0$ and $c_1 = 3Ma$, we find the slowly rotating Kerr black hole.

Now let us consider the hairy black hole family reviewed in Ref. 1. In analogy with the Kerr solution, the slowly rotating hairy black hole is a deformation of the static one plus $g_{t\varphi} = \omega_\nu(1-z^2)C(r)$, $C(r)$ is the areal function. The metric

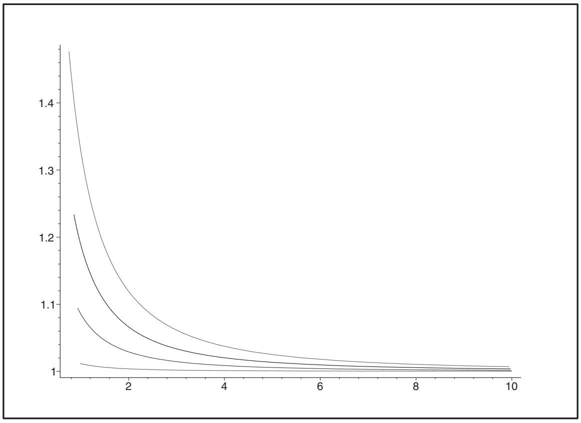

Fig. 1. The ratio $\omega/\omega_{\nu=1}$ versus the square root of the areal function, $\sqrt{C(r)}$, for different values of ν. The plots are for $\nu = 1.2, \nu = 2.1, \nu = 3$ and $\nu = 4$ (from down up).

component $g_{t\varphi}$ determines the frame dragging potential. We find that

$$\omega_\nu = \bar{c}_1 \left(\frac{r^{2-\nu}}{\nu^2(\nu^2-4)} \left((\nu-2)\, r^{2\nu} + \left(4-\nu^2\right) r^\nu - 2 - \nu\right) + \frac{1}{\nu^2-4} \right) . \tag{13}$$

To measure the deviation of the dragging effects from those from the slowly rotating Kerr solution we plot the ratio $\omega/\omega_{\nu=1}$ versus the square root of the areal function $\sqrt{C(r)}$. In Figure 1 it can be seen that there is a smooth departure from the Kerr frame dragging as both coincide when ν approaches 1 or asymptotically for large $\sqrt{C(r)}$. It should be noticed that the departure from Kerr dragging can be important and that the horizon can be located at any point in the graph. Indeed, the location of the horizon is defined by the equation $A(r_+) = 0$, which has a solution for any r_+ by adjusting the value of the other parameters in the metric, see Ref. 6.

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