Exact Black Holes in Quadratic Gravity with any Cosmological Constant

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We present a new explicit class of black holes in general quadratic gravity with a cosmological constant. These spherically symmetric Schwarzschild–Bach–(anti-)de Sitter geometries, derived under the assumption of constant scalar curvature, form a three-parameter family determined by the black-hole horizon position, the value of the Bach invariant on the horizon, and the cosmological constant. Using a conformal to Kundt metric ansatz, the fourth-order field equations simplify to a compact autonomous system. Its solutions are found as power series, enabling us to directly set the Bach parameter and/or cosmological constant equal to zero. To interpret these spacetimes, we analyze the metric functions. In particular, we demonstrate that for a certain range of positive cosmological constant there are both black-hole and cosmological horizons, with a static region between them. The tidal effects on free test particles and basic thermodynamic quantities are also determined.

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Introduction.—Black holes, regions with very strong gravity from which not even light can escape, are one of the most fascinating theoretical predictions of Einstein's general relativity [1]. The first exact solution to this theory was almost immediately found by Schwarzschild [2], describing a static spherically symmetric spacetime. However, it took several decades to fully understand its black-hole nature. This initiated the "golden age" of black hole studies, epitomized by the discovery of astrophysically more relevant Kerr rotating solution [3]. Studies of various aspects of these "collapsed objects," such as influence on matter and fields, the no-hair conjecture, thermodynamic properties, or quantum evaporation, followed soon. Moreover, a great observational effort brought the direct evidence of their existence in our universe when the Cygnus X-1 source was identified as a black hole. Also, now it seems that supermassive black holes reside in the nuclei of almost all galaxies. Mergers of black-hole binaries have been recently detected as the first gravitational wave signals.

Another remarkable interplay between Einstein's theory and observed astronomical phenomena is a concept of the cosmological constant. The famous Λ term was introduced by Einstein into his field equations to allow a static cosmological model [4]. However, it was soon demonstrated by de Sitter [5] that the cosmological constant causes even an empty space to expand exponentially fast [6]. Nowadays, this is employed for a phenomenological description of the observed accelerated expansion of our universe caused by "dark energy." The de Sitter solution also captures the main features of the inflationary epoch in the very early Universe.

Despite all the great successes of Einstein's gravity theory, it also has its limits, in particular, the impossibility to quantize it in the same way as other fundamental interactions, and perhaps some open cosmological issues. Various extensions of general relativity have thus been considered, see Refs. [7–10] for reviews. In these modified theories, the black hole solutions play a prominent role, providing natural test beds for their comparison [11–14].

Assuming a constant scalar curvature, we derive a new class of static spherically symmetric black hole solutions with a cosmological constant Λ in quadratic gravity [15], which includes the Einstein–Weyl theory [16,17]. It generalizes [18] to include higher-order gravity corrections, and [14,19] to admit any Λ. In contrast with the black holes of Refs. [14,19], the second (cosmological) horizon may appear due to $\Lambda > 0$. On large scales, the higher-order corrections considerably affect the asymptotic behavior of the geometry, which, even in the case of $\Lambda = 0$, is not asymptotically flat (except for finely tuned parameters). This additional freedom thus opens completely new and more involved possibilities. Moreover, both the cosmological constant and higher-order corrections are of key importance in quantum gravity models, e.g., Ref. [20].

Within this setting, the vacuum action of quadratic gravity contains Λ, the Ricci scalar R, and a contraction of the Weyl tensor C_{abcd}, namely,

$$S = \int d^4x\sqrt{-g}\,[\gamma(R-2\Lambda) + \beta R^2 - \alpha\, C_{abcd}C^{abcd}], \quad (1)$$

where $\alpha, \beta, \gamma = G^{-1}$ are constants. The corresponding field equations read

$$\gamma\left(R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda g_{ab}\right)-4\alpha B_{ab} + 2\beta\left(R_{ab}-\frac{1}{4}Rg_{ab}+g_{ab}\Box-\nabla_b\nabla_a\right)R=0, \quad (2)$$

where $B_{ab}\equiv(\nabla^c\nabla^d+\frac{1}{2}R^{cd})C_{acbd}$ is the traceless, symmetric, and conserved Bach tensor. Assuming $R=\text{const}$, the last term in (2) simplifies and the trace of the field equations implies $R=4\Lambda$, so that they become

$$R_{ab}-\Lambda g_{ab}=4kB_{ab}, \quad \text{with} \quad k\equiv\frac{\alpha}{\gamma+8\beta\Lambda}; \quad (3)$$

see Ref. [21]. For $k=0$ vacuum Einstein's equations with a cosmological constant are obtained. For $\beta=0$ we get Einstein–Weyl gravity. For $\gamma+8\beta\Lambda=0$ the conformal Weyl theory is restored, in which the rotational curves of galaxies were studied [22] within the spherically symmetric setting. Our solution, as a unifying model, may enable the analysis of relations between these theories in such astrophysical situations.

The geometry.—A spherically symmetric metric is usually written as

$$ds^2=-h(\bar r)dt^2+\frac{d\bar r^2}{f(\bar r)}+\bar r^2(d\theta^2+\sin^2\theta d\phi^2). \quad (4)$$

However, in Refs. [19,23] it was shown that for investigation of such geometries in quadratic gravity, an alternative form is more convenient,

$$ds^2=\Omega^2(r)[d\theta^2+\sin^2\theta d\phi^2-2du dr+\mathcal{H}(r)du^2]. \quad (5)$$

This is related to (4) via

$$\bar r=\Omega(r), \qquad t=u-\int\mathcal{H}(r)^{-1}dr, \quad (6)$$

and the metric functions Ω, $\mathcal{H}$ give f, h using

$$h(\bar r)=-\Omega^2\mathcal{H}, \qquad f(\bar r)=-\left(\frac{\Omega'}{\Omega}\right)^2\mathcal{H} \quad (7)$$

(prime denotes the derivative with respect to r). The new metric (5) is *conformal* to a simple direct-product Kundt "seed," $ds^2=\Omega^2 ds^2_{\text{Kundt}}$, which is of the algebraic type D, see Refs. [21,24,25].

In the metric (5), the Killing horizons corresponding to $\partial_u=\partial_t$ are located at specific radii r_h satisfying

$$\mathcal{H}|_{r=r_h}=0. \quad (8)$$

Of course, via (7) this gives $h(\bar r_h)=0=f(\bar r_h)$. There is a time-scaling freedom $t\to\sigma^{-1}t$ of the metric (4) implying $h\to\sigma^2 h$, which can be used, e.g., to adjust appropriate value of h at a chosen radius.

To uniquely characterize the geometries (5), we need the Weyl and Bach scalar curvature invariants,

$$C_{abcd}C^{abcd}=\frac{1}{3}\Omega^{-4}(\mathcal{H}''+2)^2, \quad (9)$$

$$B_{ab}B^{ab}=\frac{1}{72}\Omega^{-8}[(\mathcal{B}_1)^2+2(\mathcal{B}_1+\mathcal{B}_2)^2], \quad (10)$$

where two independent Bach components are

$$\mathcal{B}_1\equiv\mathcal{H}\mathcal{H}'''', \qquad \mathcal{B}_2\equiv\mathcal{H}'\mathcal{H}'''-\frac{1}{2}\mathcal{H}''^2+2. \quad (11)$$

Interestingly, $B_{ab}=0\Leftrightarrow B_{ab}B^{ab}=0$. Thus, we distinguish two geometrically different types of solutions in quadratic gravity defined by $B_{ab}=0$ and $B_{ab}\neq0$, respectively.

The field equations.—Under conformal transformations, the Bach tensor simply scales as $B_{ab}=\Omega^{-2}B_{ab}^{\text{Kundt}}$ and since higher-order corrections in (3) are represented by the Bach tensor, using the metric (5) leads to a remarkable simplification of the field equations. Explicit evaluation of the field equations (3) for (5), using the Bianchi identities, yields two simple ODEs for the metric functions $\Omega(r)$ and $\mathcal{H}(r)$, namely,

$$\Omega\Omega''-2\Omega'^2=\frac{1}{3}k\mathcal{B}_1\mathcal{H}^{-1}, \quad (12)$$

$$\Omega\Omega'\mathcal{H}'+3\Omega'^2\mathcal{H}+\Omega^2-\Lambda\Omega^4=\frac{1}{3}k\mathcal{B}_2; \quad (13)$$

see Ref. [26] for more details. It is also convenient to express the trace of (3), namely, $R=4\Lambda$,

$$\mathcal{H}\Omega''+\mathcal{H}'\Omega'+\frac{1}{6}(\mathcal{H}''+2)\Omega=\frac{2}{3}\Lambda\Omega^3. \quad (14)$$

In fact, it is the derivative of (13) minus $\mathcal{H}'$ times (12). The crucial point for further investigations is that Eqs. (12), (13) do not explicitly depend on r. Solutions to such an autonomous system can thus be found as a power series in r expanded around *any* point r_0

$$\Omega(r)=\Delta^n\sum_{i=0}^{\infty}a_i\Delta^i, \qquad \mathcal{H}(r)=\Delta^p\sum_{i=0}^{\infty}c_i\Delta^i, \quad (15)$$

where $\Delta\equiv r-r_0$, $n, p\in\mathbb{R}$, and a_0, $c_0\neq0$.

Vanishing Bach tensor. For $\mathcal{B}_1=0=\mathcal{B}_2$, we deal with Einstein's theory, and Eqs. (12), (13) can be directly integrated. Using the gauge freedom $r\to\lambda r+\nu$, $u\to\lambda^{-1}u$ of the metric (5), this immediately implies

TABLE I. The only admitted parameters $[n, p]$ in (15), and the cosmological constant Λ, restricted by dominant powers of Δ in the field equations (12), (13), and the trace (14). Note that in the last column, $n \neq -1, -1/2$.

n	0	0	1	-1	-1	0	0	<0
p	1	0	0	2	0	2	≥ 2	$2n+2$
Λ	any	any	any	0	$\neq 0$	$\neq 0$	$3/8k$	$(3/8k)(11n^2+6n+1)/(1-4n^2)$

$$\Omega(r) = \bar{r} = -\frac{1}{r}, \qquad \mathcal{H}(r) = \frac{\Lambda}{3} - r^2 - 2mr^3, \tag{16}$$

where the mass parameter m is fixed by (8); see Eq. (19). These functions represent the Schwarzschild–(anti-)de Sitter spacetime [18,24,25] which, expressed in the form (4) using (7), reads $f = h = 1–2m\bar{r}^{-1} - \frac{1}{3}\Lambda\bar{r}^2$.

It is well known [25] that for $0 < 9\Lambda m^2 < 1$ there are two horizons determined by Eq. (8), namely, the black-hole event horizon at r_h and the cosmological horizon at $r_c > r_h$ (they degenerate to $\bar{r}_h = \bar{r}_c = 3m = 1/\sqrt{\Lambda}$ when $9\Lambda m^2 = 1$; $\Lambda < 0$ admits only the black hole horizon).

Nonvanishing Bach tensor. In a generic case ($\mathcal{B}_1, \mathcal{B}_2 \neq 0$), the system (12), (13) becomes nontrivially coupled but its solutions can be found in the form (15). Substituting these series into the field equations, we obtain polynomial expressions where the dominant (lowest) powers of Δ immediately put specific restrictions on the parameters $[n, p]$ and the possible value of Λ; see Table I and Ref. [26]. In the next section, we will discuss the most interesting case [0, 1] corresponding to a *single* root r_0 of (8).

Explicit black holes.—In the case $n = 0$, $p = 1$, the root of $\mathcal{H}$ representing the nondegenerate Killing horizon (8) is explicitly given by $r_0 \equiv r_h$. The field equations (12), (13), with (14), then restrict the coefficients in the expansions (15) as

$$\begin{aligned}
a_1 &= \frac{1}{3c_0}[2\Lambda a_0^3 - a_0(1+c_1)],\\
c_2 &= \frac{1}{6kc_0}[a_0^2(2-c_1-\Lambda a_0^2) + 2k(c_1^2-1)],\\
a_l &= \frac{1}{l^2 c_0}\left\{\frac{2}{3}\Lambda \sum_{j=0}^{l-1}\sum_{i=0}^{j} a_i a_{j-i} a_{l-1-j} - \frac{1}{3}a_{l-1}\right.\\
&\qquad \left. - \sum_{i=1}^{l} c_i a_{l-i}\left[l(l-i) + \frac{1}{6}i(i+1)\right]\right\},\\
c_{l+1} &= \frac{3}{k(l+2)(l+1)l(l-1)}\\
&\qquad \times \sum_{i=0}^{l-1} a_i a_{l-i}(l-i)(l-1-3i), \quad \text{for } l \geq 2,
\end{aligned} \tag{17}$$

with three free parameters a_0, c_0, c_1.

To identify the Schwarzschild–(anti-)de Sitter spacetime (16) in the form (15) with (17), first we evaluate the Bach tensor (11) on the horizon, yielding $\mathcal{B}_1(r_h) = 0$, $\mathcal{B}_2(r_h) = -(3/k)a_0^2 b$, where $b \equiv \frac{1}{3}(c_1 - 2 + \Lambda a_0^2)$. Interestingly, by setting $b = 0$ (i.e., for $c_1 = 2 - \Lambda a_0^2$), the Bach tensor vanishes *everywhere*. Employing the gauge freedom of (5), we may also set

$$a_0 = -\frac{1}{r_h}, \qquad c_0 = r_h - \frac{\Lambda}{r_h}. \tag{18}$$

The explicit solution (15), (17) for $b = 0$ then becomes

$$\Omega(r) = -\frac{1}{r}, \qquad \mathcal{H}(r) = \frac{\Lambda}{3} - r^2 - \left(\frac{\Lambda}{3} - r_h^2\right)\frac{r^3}{r_h^3}, \tag{19}$$

where the expansions (15) were summed up as geometric series. This is exactly the Schwarzschild–(anti-)de Sitter black hole (16) since $(\Lambda/3) - r_h^2 = 2mr_h^3$.

In the case $b \neq 0$, we may now separate the "Bach contribution" in the coefficients (17) proportional to b by introducing α_i, γ_i. With the same gauge choice (18), we obtain a one-parameter *extension* of the Schwarzschild–(anti-)de Sitter spacetime in quadratic gravity,

$$\Omega(r) = -\frac{1}{r} - \frac{b}{r_h}\sum_{i=1}^{\infty}\alpha_i\left(\frac{r_h - r}{\rho r_h}\right)^i, \tag{20}$$

$$\begin{aligned}
\mathcal{H}(r) = (r - r_h)\Bigg[&\frac{r^2}{r_h} - \frac{\Lambda}{3r_h^3}(r^2 + rr_h + r_h^2)\\
&+ 3b\rho r_h \sum_{i=1}^{\infty}\gamma_i\left(\frac{r - r_h}{\rho r_h}\right)^i\Bigg],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\rho \equiv 1 - \frac{\Lambda}{r_h^2}, \qquad \alpha_1 \equiv 1, \qquad \gamma_1 = 1,\\
&\gamma_2 = \frac{1}{3}\left[4 - \frac{1}{r_h^2}\left(2\Lambda + \frac{1}{2k}\right) + 3b\right],
\end{aligned} \tag{22}$$

and α_l, γ_{l+1} are (with $\alpha_0 \equiv 0$) *recursively* given by

$$\alpha_l = \frac{1}{l^2}\left(-\frac{2\Lambda}{3r_h^2}\sum_{j=0}^{l-1}\sum_{i=0}^{j}\{\alpha_{l-1-j}\,\rho^j + (\rho^{l-1-j} + b\alpha_{l-1-j})[\alpha_i\,\rho^{j-i} + \alpha_{j-i}(\rho^i + b\alpha_i)]\} - \frac{1}{3}\alpha_{l-2}(2+\rho)\,\rho(l-1)^2\right.$$
$$\left. + \alpha_{l-1}\left\{\frac{1}{3} + (1+\rho)\left[l(l-1) + \frac{1}{3}\right]\right\} - 3\sum_{i=1}^{l}(-1)^i\gamma_i(\rho^{l-i} + b\alpha_{l-i})\left[l(l-i) + \frac{1}{6}i(i+1)\right]\right),$$
$$\gamma_{l+1} = \frac{(-1)^l}{kr_h^2(l+2)(l+1)l(l-1)}\sum_{i=0}^{l-1}[\alpha_i\,\rho^{l-i} + \alpha_{l-i}(\rho^i + b\alpha_i)](l-i)(l-1-3i), \quad \text{for } l \geq 2. \tag{23}$$

All these solutions form a three-parameter family of spherically symmetric black holes (with static regions). In particular: (i) The radius $r = r_h$ determines the Killing horizon since $\mathcal{H}(r_h) = 0$; see Eqs. (21), (8). (ii) The parameter $\Lambda = R/4$ is the cosmological constant. It can be zero, recovering the results of Ref. [19]. (iii) The Bach parameter b determines the Bach tensor contribution. For $b = 0$, this Schwarzschild–Bach–(anti-)de Sitter black hole (20), (21) reduces to (19).

In terms of these three physical parameters, the scalar invariants (9), (10) on the horizon are

$$C_{abcd}C^{abcd}(r_h) = 12\left[(1+b)r_h^2 - \frac{\Lambda}{3}\right]^2, \tag{24}$$

$$B_{ab}B^{ab}(r_h) = \frac{r_h^4}{4k^2}b^2. \tag{25}$$

In Fig. 1, convergence of the series in (20), (21) is examined using the d'Alembert ratio test for two different sets of parameters. It clearly indicates that, with n growing, the ratio between two subsequent terms approaches a specific constant. The series thus asymptotically behave as geometric series. This enables us to estimate the radius of convergence.

Typical behavior of the metric function $\mathcal{H}(r)$ outside the black-hole horizon is plotted in Fig. 2. There is a significant qualitative difference between $\Lambda < 0$ and $\Lambda > 0$. In both cases, the black-hole horizon separates static ($r > r_h$) and non-static ($r < r_h$) regions of the spacetime. However, for $\Lambda > 0$ an outer boundary of this static region appears, which corresponds to the cosmological horizon given by the second root of $\mathcal{H}$ (as in the classic Schwarzschild–de Sitter black hole). This is also demonstrated in Fig. 3 by plotting the function $f(\bar{r})$ of the common metric (4).

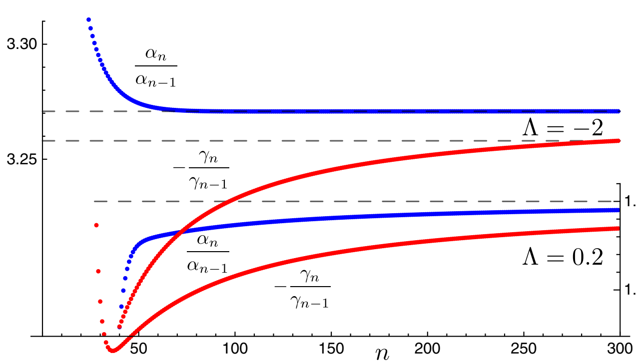

FIG. 1. The convergence radius can be estimated from the ratio convergence test for solutions (20), (21), here given by $r_h = -1, k = 0.5$ with $b = 0.3, \Lambda = 0.2$ (bottom) and $b = 0.2$, $\Lambda = -2$ (top).

Specific tidal effects.—The two independent parts (11) of the Bach tensor $\mathcal{B}_1$, $\mathcal{B}_2$ can be observed via a specific relative motion of free test particles described by the equation of geodesic deviation [27]. For an invariant description, we employ an orthonormal frame associated with initially static observer ($\dot{r} = \dot{\theta} = \dot{\phi} = 0$) with velocity $\mathbf{u} = \dot{u}\partial_u \equiv \mathbf{e}_{(0)}$, namely, $\mathbf{e}_{(1)} = -\dot{u}(\partial_u + \mathcal{H}\partial_r)$, $\mathbf{e}_{(2)} = \Omega^{-1}\partial_\theta$, and $\mathbf{e}_{(3)} = (\Omega\sin\theta)^{-1}\partial_\phi$. Projection of the equation of geodesic deviation onto this frame gives

$$\ddot{Z}^{(1)} = \frac{\Lambda}{3}Z^{(1)} + \frac{1}{6}\frac{\mathcal{H}'' + 2}{\Omega^2}Z^{(1)} - \frac{k}{3}\frac{\mathcal{B}_1 + \mathcal{B}_2}{\Omega^4}Z^{(1)}, \tag{26}$$

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3}Z^{(i)} - \frac{1}{12}\frac{\mathcal{H}'' + 2}{\Omega^2}Z^{(i)} - \frac{k}{6}\frac{\mathcal{B}_1}{\Omega^4}Z^{(i)}, \tag{27}$$

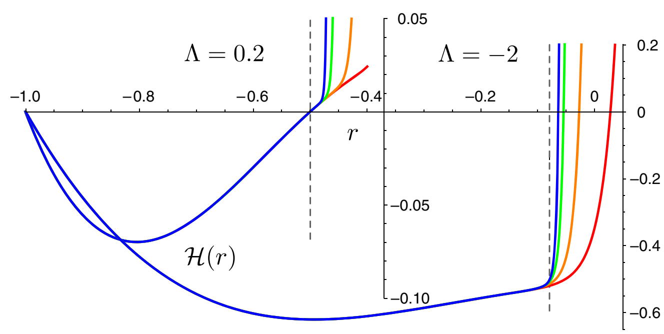

FIG. 2. The function $\mathcal{H}(r)$ given by Eq. (21) for two values of the cosmological constant Λ (with the same parameters as in Fig. 1). Both plots start on the black-hole horizon $r_h = -1$ and are reliable up to the vertical dashed lines indicating the radii of the convergence. For $\Lambda > 0$ the function $\mathcal{H}(r)$ seems to have another root corresponding to the cosmological horizon, while for $\Lambda < 0$ it remains nonvanishing. First 50 (red), 100 (orange), 200 (green), 300 (blue) terms in the expansions are used. The results fully agree with the numerical solutions up to the dashed lines, where such simulations also fail.

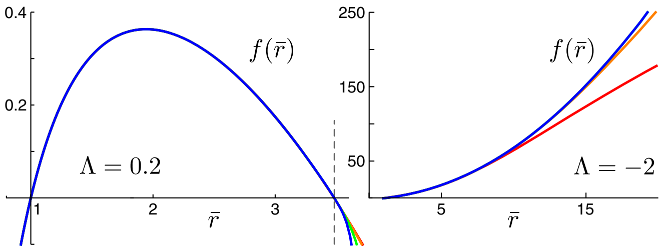

FIG. 3. The function $f(\bar{r})$ of standard line element (4) corresponding to the solution (20), (21), via (7) (with the same parameters as in Figs. 1, 2). The $\Lambda > 0$ case (left) indicates the presence of the cosmological horizon at the boundary of the convergence interval (the dashed line). For $\Lambda < 0$ (right), the series converge in the whole plotted range, indicating a static region everywhere above the black-hole horizon.

where $i = 2, 3$, $Z^{(a)} \equiv e^{(a)}{}_{\mu} Z^{\mu}$ denotes the relative position of two particles, and $\ddot{Z}^{(a)} \equiv e^{(a)}{}_{\mu}(\mathrm{D}^2 Z^{\mu}/\mathrm{d}\tau^2)$ their mutual acceleration. In (26), (27), we easily identify classic parts corresponding to the isotropic influence of the cosmological constant Λ and the Newtonian tidal effect caused by the Weyl tensor proportional to the square root of (9). Moreover, the theory satisfying (3) admits two additional effects encoded in the nontrivial Bach tensor components $\mathcal{B}_1$, $\mathcal{B}_2$. The first of them affects particles in the transverse directions ∂_θ, ∂_ϕ, see Eq. (27), while the second one induces their *radial* acceleration along $\partial_{\bar{r}}$ via (26). Since $\mathcal{B}_1(r_h) = 0$, on any horizon there is only the radial effect caused by $\mathcal{B}_2(r_h)$.

Thermodynamic quantities: horizon area, temperature, entropy.—Let us also determine main thermodynamic properties of this explicit family of spherically symmetric Schwarzschild–Bach–(anti-)de Sitter black holes. The horizon is generated by the (rescaled) null Killing vector $\ell \equiv \sigma\partial_u = \sigma\partial_t$ and thus is located at $r = r_h$ where $\mathcal{H} = 0$, cf. (8), (21). Its *area* is, using (5), (20),

$$\mathcal{A} = 4\pi r_h^{-2} = 4\pi \bar{r}_h^2, \tag{28}$$

while its surface gravity ($\kappa^2 \equiv -\frac{1}{2}\ell_{\mu;\nu}\ell^{\mu;\nu}$) reads

$$\kappa/\sigma = -\frac{1}{2}\mathcal{H}'(r_h) = -\frac{1}{2}\rho r_h = \frac{1}{2}\bar{r}_h^{-1}(1 - \Lambda\bar{r}_h^2). \tag{29}$$

It is the same expression as in the Schwarzschild–(anti-)de Sitter case, independent of the Bach parameter b. The black-hole horizon *temperature* $T = (1/2\pi)\kappa$ is thus

$$T/\sigma = -\frac{1}{4\pi}\rho r_h = \frac{1}{4\pi}\bar{r}_h^{-1}(1 - \Lambda\bar{r}_h^2). \tag{30}$$

This is zero for $\bar{r}_h = 1/\sqrt{\Lambda}$ corresponding to the case of extreme Schwarzschild–de Sitter black hole for which the black-hole and cosmological horizons coincide at $\bar{r}_h = \bar{r}_c$.

However, in higher-derivative theories, we must apply the generalized definition of *entropy* $S = (2\pi/\kappa)\oint \mathbf{Q}$, see Ref. [28], where the Noether charge 2-form on the horizon is

$$\mathbf{Q} = -\frac{\Omega^2\mathcal{H}'}{16\pi}\left[\gamma + \frac{4}{3}\Lambda(\alpha + 6\beta) + \frac{4}{3}k\alpha\frac{\mathcal{B}_1 + \mathcal{B}_2}{\Omega^4}\right]\Bigg|_{r=r_h} \times \sin\theta\, d\theta \wedge d\phi. \tag{31}$$

Evaluating the integral, using (28), (29), (25), and $r_h = -1/\bar{r}_h$, we get

$$S = \frac{1}{4}\mathcal{A}\left[\gamma + \frac{4}{3}\Lambda(\alpha + 6\beta) - 4\alpha\frac{b}{\bar{r}_h^2}\right]. \tag{32}$$

For the Schwarzschild black hole ($b = 0, \Lambda = 0$) or in the Einstein theory ($\alpha = 0$, $\beta = 0$), this reduces to the standard expression $S = (1/4G)\mathcal{A}$. For $\Lambda = 0$, the results of Refs. [14,19] are recovered. For the Schwarzschild–(anti-)de Sitter black hole ($b = 0$) in Einstein–Weyl gravity ($\beta = 0$), we obtain $S = (1/4G)\mathcal{A}(1 + \frac{4}{3}k\Lambda)$, which agrees with the results of [29]. In critical gravity, defined by $\beta = 0$, $\alpha = k\gamma$, $\Lambda = -(3/4k) < 0$, the entropy is zero. Our formula (32) for entropy generalizes all these expressions to the case of Schwarzschild–Bach–(anti-)de Sitter black holes when the Bach tensor is nonvanishing, parametrized by $b \neq 0$. In this case, the entropy is nonzero even in critical gravity. For smaller black holes, the deviations from $S = \frac{1}{4}\mathcal{A}[\gamma + \frac{4}{3}\Lambda(\alpha + 6\beta)]$ are larger.

By replacing the root r_h by r_c in (20), (21), the solution is expanded around the cosmological horizon. Its temperature and entropy are thus given by (30) and (32), respectively, in which $\bar{r}_h$ is simply replaced by $\bar{r}_c$.

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