# Bonnor-Melvin universe with a cosmological constant

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We generalize the well-known Bonnor-Melvin solution of the Einstein-Maxwell equations to the case of a nonvanishing cosmological constant. The spacetime is again cylindrically symmetric and static but, unlike the original solution, it truly represents a homogeneous magnetic field.

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## I. INTRODUCTION

Magnetic fields play an important role in many astrophysical phenomena across a range of distance scales, from stars and accretion disks to galactic nuclei and intergalactic regions. As they often occur in the vicinity of compact massive objects or in strong gravitational fields it is important to study them in the context of general relativity as well. One of the interesting exact solutions of the Einstein-Maxwell equations is the Bonnor-Melvin universe describing a static, cylindrically symmetric (electro)magnetic field immersed in its own gravitational field [1,2]. The magnetic field is aligned with the symmetry axis. This is one possible analogy of the classical homogeneous magnetic field. However, since the magnetic field contributes to the energy-momentum tensor, which curves the spacetime, the field needs to decrease away from the axis, so as not to collapse on itself, and the scalar invariant $F_{\mu\nu}F^{\mu\nu}$ is thus not constant (unlike in the classical case) and the field is not homogeneous.

Already the original paper by Melvin suggested several possible ways of generalizing the spacetime. If we wish to restore the balance for a homogeneous field, we need to incorporate an element countering the collapse of the field. It thus makes sense to search for such a solution with a nonvanishing—and, in fact, positive—cosmological constant. It is of interest then that the observed intergalactic magnetic fields [3,4] are also considered to be of cosmological origin and thus related to the large-scale structure of the Universe [5]. We first briefly review the Bonnor-Melvin case to contrast it with the homogeneous cosmological case.

## II. BONNOR-MELVIN

In cylindrical coordinates, one possible form of the metric reads

$$g_{\mu\nu} = \alpha^{-2}(-\mathrm{d}t^2 + \mathrm{d}z^2) + \alpha^{-5}\mathrm{d}r^2 + \alpha r^2 \mathrm{d}\varphi^2, \qquad (1)$$

where

$$\alpha = 1 - K^2 r^2, \qquad (2)$$

and $r \leq 1/K$, with the upper limit corresponding to proper radial infinity. The constant $K$ determines the strength of the magnetic field, the 4-potential of which is

$$A = K r^2 \mathrm{d}\varphi. \qquad (3)$$

The Maxwell tensor only has one nonzero component,

$$F = 2Kr\mathrm{d}r \wedge \mathrm{d}\varphi, \qquad (4)$$

and its invariant reads

$$F_{\mu\nu}F^{\mu\nu} = 8K^2\alpha^4, \qquad (5)$$

which is obviously nonconstant. In flat spacetime with a homogeneous magnetic test field along the $z$ direction with $\vec{B} = B\vec{e}_z$, we have $F = Br\mathrm{d}r \wedge \mathrm{d}\varphi$ and $F_{\mu\nu}F^{\mu\nu} = 2B^2$. We compare this to the Bonnor-Melvin solution on the axis of symmetry, where the field approaches the Minkowski spacetime. This yields an analogy between the two electromagnetic fields with $B = 2K$.

The solution is a Kundt class spacetime of algebraic type D that is not asymptotically flat far away from the axis and that was shown to be stable against radial perturbations [6]. It has been generalized for the case of nonlinear electrodynamics [7] and there are also cylindrically symmetric, magnetohydrodynamic cosmological models involving a generally nonstatic perfect fluid [8,9].

We now proceed to formulate the general Einstein-Maxwell equations for a cylindrically symmetric spacetime featuring a magnetic field aligned with the axis and a nonvanishing cosmological constant.

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## III. GENERAL EINSTEIN-MAXWELL EQUATIONS

Denoting the proper radius by $r$, we may write a static cylindrically symmetric metric as

$$ds^2 = -\exp A(r)dt^2 + dr^2 + \exp B(r)dz^2 + \exp C(r)d\varphi^2, \tag{6}$$

where $t, z \in \mathbb{R}$, $r \in \mathbb{R}^+$, and $\varphi \in [0, 2\pi)$. Next, we list the components of $G_{\mu\nu} - \Lambda g_{\mu\nu}$:

$$G_{tt} - \Lambda g_{tt} = \frac{1}{4}e^A(2B'' + (B')^2 + 2C'' + (C')^2 + C'B' + 4\Lambda), \tag{7}$$

$$G_{rr} - \Lambda g_{rr} = -\frac{1}{4}A'B' - \frac{1}{4}A'C' - \frac{1}{4}B'C' - \Lambda, \tag{8}$$

$$G_{zz} - \Lambda g_{zz} = -\frac{1}{4}e^B(2A'' + (A')^2 + 2C'' + (C')^2 + A'C' + 4\Lambda), \tag{9}$$

$$G_{\varphi\varphi} - \Lambda g_{\varphi\varphi} = -\frac{1}{4}e^C(2A'' + (A')^2 + 2B'' + (B')^2 + A'B' + 4\Lambda), \tag{10}$$

with primes denoting derivatives with respect to the radial coordinate. We assume a purely magnetic field aligned with the axis of symmetry, thus yielding a Maxwell tensor of the form

$$F = H(r)\mathrm{d}r \wedge \mathrm{d}\varphi. \tag{11}$$

The invariant of the field is

$$F_{\mu\nu}F^{\mu\nu} = 2H^2 e^{-C} \equiv 2f^2, \tag{12}$$

where we defined a new quantity, $f(r)$, while $\star F_{\mu\nu}F^{\mu\nu} = 0$. The stress-energy tensor reads

$$T_{tt} = \frac{1}{8\pi}e^{A-C}H^2, \tag{13}$$

$$T_{rr} = \frac{1}{8\pi}e^{-C}H^2, \tag{14}$$

$$T_{zz} = -\frac{1}{8\pi}e^{B-C}H^2, \tag{15}$$

$$T_{\varphi\varphi} = \frac{1}{8\pi}H^2. \tag{16}$$

Finally, the Einstein equations are equivalent to

$$0 = 2(B'' + C'') + (B')^2 + (C')^2 + B'C' + 4\Lambda + 4f^2, \tag{17}$$

$$0 = 2(A'' + C'') + (A')^2 + (C')^2 + A'C' + 4\Lambda + 4f^2, \tag{18}$$

$$0 = 2(A'' + B'') + (A')^2 + (B')^2 + A'B' + 4\Lambda - 4f^2, \tag{19}$$

$$0 = A'B' + A'C' + B'C' + 4\Lambda - 4f^2. \tag{20}$$

The Maxwell equations $\sqrt{-g}F^{\mu\nu}{}_{;\nu} = (\sqrt{-g}F^{\mu\nu})_{,\nu} = 0$ are identities apart from

$$\sqrt{-g}F^{\varphi\alpha}{}_{;\alpha} = (\sqrt{-g}F^{\varphi r})_{,r} = (e^{\frac{A+B-C}{2}}F_{\varphi r})_{,r} = -(e^{\frac{A+B-C}{2}}H)_{,r} = -(e^{\frac{A+B}{2}}f)_{,r} = 0, \tag{21}$$

which yields

$$e^{\frac{A+B}{2}}f = \mathrm{const}. \tag{22}$$

However, Eq. (22) is a consequence of the Einstein equations, which can be seen as follows: differentiate Eq. (20) and subtract from it $A'$.(17) + $B'$.(18) + $C'$.(19) to obtain

$$16ff' + 4f^2(A + B - C)' + (A + B + C)'(4\Lambda + A'B' + A'C' + B'C') = 0, \tag{23}$$

where we substitute for the second bracket in the last term from Eq. (20) to yield

$$2f' + f(A + B)' = 0, \tag{24}$$

which can be integrated to yield Eq. (22).

## IV. THE HOMOGENEOUS SOLUTION

We now specialize to a homogeneous magnetic field, which means that we require the invariant of the field $F_{\mu\nu}F^{\mu\nu}$ to be constant throughout the spacetime, and thus

$$f = \mathrm{const}. \tag{25}$$

It then follows immediately from Eq. (22) that $A + B =$ const. However, the sum of Eqs. (19) and (20) then implies

$$2(A + B)'' + ((A + B)')^2 + C'(A + B)' + 8(\Lambda - f^2) = 0 \Rightarrow f^2 = \Lambda. \tag{26}$$

We also note that we thus have $\Lambda > 0$, as expected. Equation (20) then immediately shows that $A'B' = 0$ while $A' + B' = 0$, which yields

$$A = \mathrm{const}, \qquad B = \mathrm{const}. \tag{27}$$

We can always rescale $t$ and $z$ to achieve $A = 0$ and $B = 0$. There is only a single Einstein equation remaining, namely,

$$C'' + \frac{1}{2}C'^2 + 4\Lambda = 0. \tag{28}$$

It thus follows that

$$C(r) = 2\ln\sigma + 2\ln\sin(\sqrt{2\Lambda}(r+R)), \tag{29}$$

2where $\sigma$ and $R$ are integration constants. We shift the radial coordinate (thus removing $R$) to finally express the metric as

$$ds^2 = -dt^2 + dr^2 + dz^2 + \sigma^2\sin^2(\sqrt{2\Lambda}r)\ d\varphi^2, \tag{30}$$

while the magnetic field reads

$$H(r) = \sqrt{\Lambda}\sigma\sin(\sqrt{2\Lambda}r). \tag{31}$$

As we approach $\sqrt{2\Lambda}r = \pi$ the circumference of the rings $r = \text{const}$ vanishes, which suggests that this is the location of an axis of some sort. Then we rescale both $r$ and $\varphi$ to bring the line element into the form

$$ds^2 = -dt^2 + dz^2 + \frac{1}{2\Lambda}(dr^2 + \sin^2 r d\varphi^2), \tag{32}$$

with

$$H(r) = \frac{1}{\sqrt{2}}\sin r. \tag{33}$$

This is locally a direct product of two-dimensional Minkowski spacetime and a 2-sphere of constant radius $1/\sqrt{2\Lambda}$. The curvature scalars are $\Phi_{11} = \frac{1}{2}\Lambda > 0$ and $\Psi_2 = -\frac{1}{3}\Lambda$, and thus they satisfy the condition $2\Phi_{11} + 3\Psi_2 = 0$. Hence, these spacetimes belong to the "exceptional electrovacuum type D metrics with cosmological constant" investigated by Plebański and Hacyan [10] (see also Ref. [11]). They admit a six-dimensional group of isometries $ISO(1,1) \times SO(3)$.

Generally, the solution admits a deficit angle due to the presence of $\sigma$ in Eq. (30) and the axis is thus singular, forming the spacetime's only singularity. Therefore, the spacetime is in fact a direct product of Minkowski spacetime and a squashed sphere at every point. The solution again belongs in the Kundt class. Since this is a homogeneous spacetime, both charged and uncharged test particles can remain static anywhere or just sail along the magnetic field direction at a constant speed. As expected, we also find helical paths.

Using the electromagnetic duality, $F \to \star F, \star F \to -F$, the Maxwell field can also be converted to a homogeneous electric field parallel to the cylindrical axis with $E_z = -\sqrt{\Lambda}$ and $F_{\mu\nu}F^{\mu\nu} = 2\Lambda$. It is of interest to look at possible shell sources: we identify cylindrical surfaces of the same circumference located symmetrically with respect to $\sqrt{2\Lambda}r = \pi/2$. We keep the region below the upper radius and as we leave it, heading outwards, we reappear at the lower radius, entering the same region again. We thus get an infinitely thin shell and, using the Israel formalism, we find that the 3-current induced on the shell vanishes while the induced three-dimensional stress-energy tensor only has two nonzero components, corresponding to a positive pressure along the axis equal in magnitude to a negative energy density. This is analogous to the shell source producing a cosmic-string spacetime with a surplus angle as opposed to a deficit one. The resulting electromagnetic field is due to sources at infinity along the axis, while the gravitational field is due both to the shell and the stress-energy tensor of the electromagnetic field.

To our knowledge, this is the closest general-relativistic analogue of the classical homogeneous magnetic field including particularly the motion of test particles. As an extension of this work we intend to study whether Eqs. (17)–(20) admit any other solution at all or whether, in fact, the Plebański-Hacyan spacetime is the only cylindrically symmetric static solution with the cosmological constant and an aligned magnetic field.

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