Conformal Killing Initial Data

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Abstract

We find necessary and sufficient conditions ensuring that the vacuum development of an initial data set of the Einstein's field equations admits a conformal Killing vector. We refer to these conditions as *conformal Killing initial data* (CKID) and they extend the well-known *Killing initial data* (KID) that have been known for a long time. The procedure used to find the CKID is a classical argument, which is reviewed and presented in a form that may have an independent interest, based on identifying a suitable *propagation* identity and checking the well-posedness of the corresponding initial value problem. As example applications, we review the derivation of the KID conditions, as well as give a more thorough treatment of the homothetic Killing initial data (HKID) conditions than was previously available in the literature.

1 Introduction

It is an interesting observation, first made in various forms in [8, 23, 24, 12], that there exists a set of linear partial differential equations (PDEs) defined on the background of an initial data surface for Einstein's vacuum equations such that the solutions of these PDEs are in bijection with the Killing vectors of the vacuum spacetime evolved from this surface. Fittingly, this system of PDEs is known as the *Killing initial data (KID)* equations, so named in [6]. Later, the KID equations have been generalized to cover Einstein's equations coupled to rather general kinds of matter [29, 30]. After [6], KID equations have received a fair amount of attention in the mathematical relativity literature.

Killing vectors are solutions of the Killing equation, a geometric PDE on a Lorentzian (more generally, pseudo-Riemannian) geometry. A natural question

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arises: what other geometric PDEs have analogous initial data systems? A solution of such an initial data system on an initial data surface (for Einstein's vacuum or other related equations) would give rise to a unique solution of the corresponding geometric PDE in the bulk geometry evolved from the initial data surface. It seems that this question has so far been considered in only a small number of cases. An early and somewhat neglected example is [9], which treated the initial data equations for homothetic Killing vectors and partially for conformal Killing vectors. Rather recently, the valence $(1,0)$, $(0,1)$, $(2,0)$ and $(0,2)$ Killing spinor equations in 4 dimensions were also treated [16]. In a follow-up work [17], some of them were adapted to Friedrich's conformal vacuum equations. Also, Killing $(2,0)$-spinor initial data equations have found applications to the characterization of initial data for the Kerr black hole family [1, 3, 2, 11], and in the study of the manifold structure of the infinite dimensional space of initial data for Einstein's equations [7].

Conceivably, such initial data equations (or at least the methods used to obtain them) could also find applications in the study of the uniqueness and rigidity of asymptotically flat black holes. For example, such rigidity results were discussed in [19] and, while they did not directly use KID equations, they did use a propagation equation (see below) analogous but not identical to the standard one we later give in Section 2.1.

In this work, we obtain for the first time a complete derivation of the *conformal Killing initial data (CKID)* equations, that is, whose solutions on an initial data surface are in bijection with conformal Killing vectors on the Einstein vacuum geometry (in any number of dimensions, $n > 2$) evolved from this surface. This question was first approached in the early work [9], which gave some necessary conditions on the initial data of a conformal Killing vector, but stopped short of giving a complete list and, a fortiori, did not prove the sufficiency of any such list. We make a more detailed comparison with our results in the introductions to Sections 3 and 4. Some of the ideas from [9] were picked up again in [13], and some classes of exact solutions to the corresponding initial data equations were studied in [34, 31]. But no progress on the CKID problem appears to have been been made since then. In [26], the author obtained the transformation of the KID system under a conformal transformation of the bulk geometry under the assumption that that the metric conformal equations of Friedrich are fulfilled in the bulk but did not obtain what we call the CKID system.

We expect our CKID equations to have applications in mathematical relativity analogous to the ones already mentioned for other initial data systems. The CKID equations may be particularly useful when coupled with Friedrich's conformal version of Einstein equations [14], or the equivalent system of conformally covariant nonlinear wave equations [25, 10]. When restricted to 4 dimensions, the conformal Killing and Killing $(1,1)$-spinor equations are equivalent. Hence the spinorial version of the CKIDs could have been extracted from the intermediate results of [16], but only in 4 spacetime dimensions.

Our method of proof follows the same basic strategy as the old work on the Killing equation [12]. It relies on a key identity, which we call a *propagation* *equation*. This strategy is summarized in Section 2, where the main observation is Lemma 1, with the generic form of the desired key identity expressed in Equation (1). In Section 2.1 we recall how the KID equations are derived, as well as introduce some notation that is heavily used in subsequent sections. In Section 3, we apply the same strategy to *homothetic Killing initial data* (HKID), confirming that the initial data conditions first obtained in [9] are in fact sufficient. Finally, in Section 4 we follow an analogous route to obtain the CKID equations (Theorem 3).

It is also worth noting that the propagation equation identities giving rise to KID, HKID and CKID systems are covariantly constructed and their form does not explicitly depend on the signature of the metric tensor. Thus, they would apply also in other signatures, like in Riemannian geometry. In Lorentzian signature, we expect the propagation equations to be hyperbolic and hence have a well-posed initial value problem. On the other hand, in Riemannian signature, we expect the propagation equations to be elliptic and hence have a well-posed boundary value problem. Then, the bulk Killing or conformal Killing vectors will still induce solutions of the KID or CKID equations on the boundary, but there may exist solutions on the boundary that do not correspond to bulk solutions when the elliptic propagation equations have non-trivial solutions for homogeneous boundary conditions. The uniqueness of the (trivial) solution for elliptic homogeneous boundary value problems may be guaranteed using the Bochner method [33], or some other technique. Under such hypotheses, then the existence of Killing or conformal Killing vectors on Ricci flat Riemannian manifolds with boundary could be predicted by the existence of solutions of KID, HKID or CKID equations with respect to the boundary data. It seems that such applications have not yet been considered in Riemannian geometry.

All the computations of this paper have been double-checked with the tensor computer algebra systems *Cadabra* and *xAct* [27, 28, 20, 21].

2 Propagation equations and initial data

From now on, all or our differential operators are presumed to be defined between vector bundles over a manifold M and have smooth coefficients.

We call a linear partial differential equation (PDE) $P[\psi] = 0$ a *propagation equation (of order $k \geq 1$)* if it has a well-posed initial value problem: given a Cauchy surface $\Sigma \subset M$ with unit normal n^a, the equation can be put into Cauchy-Kovalevskaya form (solved for the highest time derivative) and for each assignment of arbitrary smooth initial data $\psi|_\Sigma = \psi_0, \ldots, \nabla_n^{k-1}\psi|_\Sigma = \psi_{k-1}$ (where $\nabla_n = n^a\nabla_a$) there exists a unique solution of $P[\psi] = 0$ on all of M. In particular, due to the linearity of the propagation equation, if the initial data all vanish, $\psi_0 = \cdots = \psi_{k-1} = 0$, then $\psi = 0$ is the corresponding unique solution on M.

There are multiple examples of propagation equations: (a) Wave (a.k.a *normally-hyperbolic*) equations, $P[\psi] = \Box\psi + P'(\nabla\psi, \psi)$ [4]. (b) Transport equations, $P[\psi] = u^a\nabla_a\psi + P'(\psi)$, with u^a everywhere transverse to Σ [19].

(c) Special cases, like $P_{bcd}[\psi] = \nabla^a \psi_{abcd}$ for ψ_{abcd} satisfying the symmetry and tracelessness conditions of the Weyl tensor in 4 dimensions [19]. (d) At the end of this section (Lemma 3), we give the name *generalized normally-hyperbolic* to a class generalizing that in (a).

Lemma 1. *Consider a globally hyperbolic spacetime (M, g), satisfying the Einstein vacuum equations, $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 0$. Let $E[\phi] = 0$ be a PDE (system) defined on some (possibly multicomponent) field ϕ. Suppose that there exist propagation equations $P[\psi] = 0$, $Q[\phi] = 0$ (of respective orders k and l), where the differential operators P and Q satisfy the identity*

$$P[E[\phi]] = \sigma[Q[\phi]] + \tau[G], \tag{1}$$

for some linear differential operators σ and τ. Then, given a Cauchy surface $\Sigma \subset M$ with unit timelike normal n^a, the unique solution of $Q[\phi] = 0$ with initial data $\phi|_\Sigma = \phi_0, \ldots, \nabla_n^{l-1} \phi|_\Sigma = \phi_{l-1}$ satisfies the equation $E[\phi] = 0$ provided the initial data $\psi|_\Sigma = 0, \ldots, \nabla_n^{k-1} \psi|_\Sigma = 0$, for $\psi = E[\phi]$, vanish.

In addition, there exists a purely spatial linear PDE on Σ, $E^\Sigma[\phi_0, \ldots, \phi_{l-1}] = 0$ such that the conditions $Q[\phi] = 0$ and $E^\Sigma[\phi|_\Sigma, \ldots, \nabla_n^{l-1} \phi|_\Sigma] = 0$ imply the vanishing of the initial data $\psi|_\Sigma = 0, \ldots, \nabla_n^{k-1} \psi|_\Sigma = 0$ for $\psi = E[\phi]$.

Proof. Under the hypotheses on the metric g and ϕ, both $G = 0$ and $Q[\phi] = 0$ vanish. Then, letting $\psi = E[\phi]$, the identity (1) implies $P[\psi] = 0$. But, by the definition of a propagation equation, the vanishing of the initial data for ψ on Σ implies that $E[\phi] = \psi = 0$ on all of M.

For the second part, first notice that our notion of well-posedness for the propagation equation $Q[\phi] = 0$ implies that the values of the time derivatives $\nabla_n^N \phi$ for $N \geq l$ are given by local algebraic expressions in terms of the $\nabla_n^N \phi$ for $0 \leq N < l$. Setting $\psi = E[\phi]$, the vanishing of the initial data for ψ may a priori involve time derivatives $\nabla_n^N \phi$ of orders $N \geq l$. But replacing these higher order time derivatives by the above expressions, reduces the dependence on time derivatives $\nabla_n^N \phi$ of order at most $N < l$. Then, obviously, these reduced order conditions can be collected into a single equation, which we can denote by $E^\Sigma[\phi_0, \ldots, \phi_{l-1}] = 0$. $\square$

Intuitively, the original equation $E[\phi] = 0$ should be more restrictive than the propagation equation $Q[\phi] = 0$, thus requiring the $E^\Sigma[\phi|_\Sigma, \ldots] = 0$ initial data constraints to make up the difference. However, as written above, Lemma 1 does not exclude situations where the equation $Q[\phi] = 0$ is the more restrictive one (like the extreme example $Q[\phi] = \phi$). Thus, when we would like every solution of $E[\phi] = 0$ to also be a solution of $Q[\phi] = 0$, we will refer to the following obvious

Lemma 2. *Using the notation of Lemma 1, suppose there exists a linear differential operator ρ such that*

$$Q[\phi] = \rho[E[\phi]]. \tag{2}$$

Then any solution of $E[\phi] = 0$ is also a solution of $Q[\phi] = 0$ with vanishing initial data $\psi|_\Sigma = 0, \ldots, \nabla_n^{k-1} \psi|_\Sigma = 0$, for $\psi = E[\phi]$.

For an operator E^Σ satisfying the second part of Lemma 1, we call

$$E^\Sigma[\phi_0, \dots, \phi_{l-1}] = 0 \tag{3}$$

a set of *E-initial data conditions* or a *E-initial data system*. Clearly, the operator E^Σ is not uniquely fixed. For instance, its components may contain many redundant equations. Thus, in practice, once some P-initial data conditions have been obtained, they will be significantly simplified by eliminating as many higher order (in spatial derivatives) terms as possible. Also, when some of the components of $E^\Sigma[\phi_0, \dots, \phi_{l-1}] = 0$ can be used to directly solve for one of the arguments, say ϕ_{l-1}, in terms of the remaining ones, we can split the initial data system into (a) $\phi_{l-1} = \cdots$ and (b) a system involving only the remaining arguments, $E'^\Sigma[\phi_0, \dots, \phi_{l-2}] = 0$. When presenting an initial data system, we will omit from E^Σ those components that can be rewritten as type (a) and only write the remaining components of type (b), reduced to the smallest convenient set of arguments. Of course, the derivation of the initial data system will provide the information about how all type (a) components can be recovered.

There is a limited set of known examples of propagation identities for geometrically motivated equations $E[\phi] = 0$ in Lorentzian (or Riemannian) geometry. The most prominent example concerns the Killing equation in any spacetime dimension (examined in detail in Section 2.1) [6]. The list of known examples is then exhausted by the 4-spacetime dimensional Killing spinor equations of valences $(1,0)$, $(0,1)$, $(2,0)$ and $(0,2)$ [16, 17]. The propagation identity ostensibly obtained for the homothetic Killing vector equation in [9] was not not formally checked for well-posedness. We close this small gap in Section 3, where we check the well-posedness of our propagation identity for this equation.

Normally-hyperbolic equations [4], mentioned earlier in this section, are a large and easy to recognize class of propagation equations. One need only check that the principal symbol of $Q[\phi] = 0$ coincides with that of the wave operator $\Box$, possibly tensored with the identity endomorphism of the vector bundle where the field ϕ takes its values. However, there exist second order operators Q whose highest order terms consist of more than just $\Box$, yet are closely tied to the normally-hyperbolic class.

We will call *generalized normally-hyperbolic* any operator Q (of order l) that is determined (acts between vector bundles of equal rank) and for which there exists an operator Q' (or order $2m - l$, $m \geq 1$) such that

$$N[\phi] := Q'[Q[\phi]] = \Box^m \phi + \text{l.o.t}, \tag{4}$$

where l.o.t stands for term of differential order lower than $2m$. That is, the principal symbol of $N[\phi]$ is a power of the wave operator and hence N is normally-hyperbolic.[1] Generalized normally-hyperbolic operators will appear as propagation operators in the study of the conformal Killing equation. Thus,

[1] While [4] only treats *second order* normally-hyperbolic equations, any such *higher order* equation can be order reduced to a second order normally-hyperbolic *system* of equations, which *are* treated in [4].

for later convenience, imitating the treatment of the Dirac operator in [4, 5], we establish the following

Lemma 3. *Any generalized normally-hyperbolic operator has a well-posed initial value problem.*

Proof. Consider (M, g) to be a globally hyperbolic Lorentzian manifold, with a Cauchy surface $\Sigma \subset M$ with unit timelike normal n^a. Suppose Q is of order l and generalized normally-hyperbolic, with Q' of order $2m - l$ such that $N := Q' \circ Q = \Box^m + \text{l.o.t}$, as in the definition. As we have noted already, N is normally-hyperbolic and hence has a well-posed initial value problem of order $2m$.

There is an immediate consequence of the existence of such an operator Q'. Namely, because of the condition on the differential orders of all the operators, we know that $\sigma_p(N) = \sigma_p(Q')\sigma_p(Q)$, where $\sigma_p(-)$ denotes the principal symbol of an operator (a vector bundle morphism valued function on the cotangent bundle $T^*M \ni p$). Now, because the principal symbol of N coincides with that of $\Box^m$, we know that $\sigma_p(N)$ is invertible everywhere except at null covectors $p \in T^*M$. Therefore, $\sigma_p(Q)$ is invertible wherever $\sigma_p(N)$ is, in particular whenever p is non-null, because

$$\left[\sigma_p(N)^{-1}\sigma_p(Q')\right]\sigma_p(Q) = \text{id} \quad \text{and} \quad \sigma_p(Q)\left[\sigma_p(N)^{-1}\sigma_p(Q')\right] = \text{id}, \tag{5}$$

where the second equality holds because Q is determined (its principal symbol is a square matrix in components). Thus, the operator Q may be expanded as

$$Q[\phi] = \sigma_n(Q)(\nabla_n^l \phi) + \text{l.o.t}_n, \tag{6}$$

where l.o.t_n stands for terms of lower differential order in normal derivatives ∇_n, and the notation $\sigma_n(-)$ stands for the principal symbol evaluated specifically at the covector n_a, which is non-null, being orthogonal to Σ.

This means that, $Q[\phi] = 0$ may be put into Cauchy-Kovalevskaya form. In other words, whenever $Q[\phi] = 0$, we can write the normal derivative $\nabla_n^l \phi|_\Sigma$ as a *linear local* (meaning as a purely spatial differential operator on Σ) expression of the lower order normal derivatives, $\phi_0 = \phi|_\Sigma, \ldots, \phi_{l-1} = \nabla_n^{l-1}\phi|_\Sigma$. Which means that, after applying ∇_n multiple times to that relation, all normal derivatives also up to $\nabla_n^{2m-1}\phi$ can be linearly locally expressed in terms of $\phi_0, \ldots, \phi_{l-1}$ as well. In other words, the initial data for $Q[\phi] = 0$ uniquely determine the initial data for $N[\phi] = 0$, while the latter equation produces a unique solution ϕ with those initial data. It remains to check that $Q[\phi] = 0$ is actually satisfied by this ϕ. But to that end, we need only apply Lemma 1 to the obvious identity

$$Q[N[\phi]] = Q[Q'[Q[\phi]]] = N'[Q[\phi]], \tag{7}$$

where $N' := Q \circ Q'$ and we know that $N' = \Box^m + \text{l.o.t}$, as a consequence of the second equality in (5). As a technicality, we need to check that the solution of $N[\phi] = 0$ with the initial data constructed earlier gives $\psi = Q[\phi]$ with vanishing initial data for $N'[\psi] = 0$, namely $\nabla_n^{2m-1}\psi|_\Sigma = 0, \ldots, \psi|_\Sigma = 0$. For that to be true, we just need to show that the two ways of solving for the higher order derivatives $\nabla_n^{k+2m}\phi$ actually agree, namely from solving $\nabla_n^k N[\phi] = 0$ or $\nabla_n^{k+2m-l} Q[\phi] = 0$ on Σ. But applying to Q' the argument from the first part of the proof, we know that $Q'[\psi] = 0$ can also be put in Cauchy-Kovalevskaya form, so that by the identity

$$\nabla_n^k N[\phi] = \nabla_n^k Q'[Q[\phi]] = \sigma_n(Q')(\nabla_n^{k+2m-l} Q[\phi] + \mathrm{l.o.t}_n), \tag{8}$$

the two ways are indeed equivalent.

Hence, since arbitrary initial data of order l determine a unique solution to $Q[\phi] = 0$, this equation has a well-posed initial value problem of order l. $\square$

2.1 Example: Killing initial data

The canonical illustration of Lemma 1 is the case of the *Killing equation* [6],

$$\mathrm{K}_{ab}[v] = \nabla_a v_b + \nabla_b v_a = 0 \quad (E[\phi] = 0). \tag{9}$$

The corresponding propagation equations are

$$\Box v_a + R_a{}^b v_b = 0 \quad (Q[\phi] = 0), \tag{10}$$

$$\Box h_{ab} - 2R^c{}_{ab}{}^d h_{cd} = 0 \quad (P[\psi] = 0), \tag{11}$$

where h_{ab} is considered to be symmetric, while the propagation identities (1) and (2) take the form

$$\Box \mathrm{K}_{ab}[v] - 2R^c{}_{ab}{}^d \mathrm{K}_{cd}[v] = \mathrm{K}_{ab}[\Box v + R\cdot v] + 2R_{(a}{}^c \mathrm{K}_{b)c}[v] - 2\mathcal{L}_v R_{ab}, \quad (P[E[\phi]] = \sigma[Q[\phi]] + \tau[G]) \tag{12}$$

$$\Box v_a + R_a{}^b v_b = \nabla^b \mathrm{K}_{ab}[v] - \frac{1}{2}\nabla_a \mathrm{K}^b{}_b[v], \quad (Q[\phi] = \rho[E[\phi]]) \tag{13}$$

where we denoted $(R\cdot v)_a = R_a{}^b v_b$ and $\mathcal{L}_v R_{ab} = v^c \nabla_c R_{ab} + 2R_{c(a}\nabla_{b)} v^c$ is the Lie derivative of R_{ab} with respect to the vector field v.

To obtain the K-initial data conditions, or more commonly the *Killing initial data (KID)* conditions, we must first introduce a space-time split around a Cauchy surface $\Sigma \subset M$, $\dim M = n$ and $\dim \Sigma = n-1$. Let us use Gaussian normal coordinates to set up a codimension-1 foliation on an open neighborhood $U \supset \Sigma$ by level sets of a smooth temporal function $t\colon U \to \mathbb{R}$, of which $\Sigma = \{t = 0\}$ is the zero level set. Choose t such that $n_a = \nabla_a t$ is a unit normal to the level sets of t. Let us identify tensors on Σ by upper case Latin indices $A, B, C, \ldots$, denote the pullback of the ambient metric to Σ by g_{AB} and its inverse by g^{AB}, and also denote by h_A^a the injection $T_\Sigma \to TM$ induced by the foliation. Raising and lowering the respective indices on h_A^a with g_{ab} and g_{AB}, we get the corresponding injections and orthogonal projections between $T\Sigma$,

$T^*\Sigma$, TM and T^*M. In our notation, all covariant and contravariant tensors split according to

$$v_a = v_0 n_a + h_a^A v_A, \quad u^b = -u^0 n^b + h_B^b u^B, \tag{14}$$

which we also denote by

$$v_a \to \begin{bmatrix} v_0 \\ v_A \end{bmatrix}, \quad u^b \to \begin{bmatrix} u^0 \\ u^B \end{bmatrix}. \tag{15}$$

Thus, in our convention, the ambient metric splits as

$$g_{ab} \to \begin{bmatrix} -1 & 0 \\ 0 & g_{AB} \end{bmatrix}. \tag{16}$$

Let D_A denote the Levi-Civita connection on (Σ, g_{AB}), depending on the foliation time t of course, and let $\partial_t = \mathcal{L}_{-n}$ denote the Lie derivative with respect to the future-pointing normal vector $-n^a$. The action of ∂_t extends to t-dependent tensors on Σ in the natural way. The (t-dependent) extrinsic curvature on Σ is then defined by

$$\pi_{AB} = \frac{1}{2}\partial_t g_{AB} \tag{17}$$

and the ambient spacetime connection decomposes as

$$\nabla_a v_b \to \begin{bmatrix} \nabla_0 v_b \\ \nabla_A v_b \end{bmatrix}, \tag{18}$$

where

$$\nabla_0 v_a \to \begin{bmatrix} \nabla_0 v_0 \\ \nabla_0 v_A \end{bmatrix} = \begin{bmatrix} \partial_t & 0 \\ 0 & \partial_t \delta_A^B - \pi_A{}^B \end{bmatrix} \begin{bmatrix} v_0 \\ v_B \end{bmatrix}, \tag{19}$$

$$\nabla_A v_b \to \begin{bmatrix} \nabla_A v_0 \\ \nabla_A v_B \end{bmatrix} = \begin{bmatrix} D_A & -\pi_A{}^C \\ -\pi_{AB} & D_A \delta_B^C \end{bmatrix} \begin{bmatrix} v_0 \\ v_C \end{bmatrix}. \tag{20}$$

The ambient vacuum Einstein equations $R_{ab} = 0$ decompose as

$$R_{ab} \to \begin{bmatrix} -\nabla_0 \pi_C{}^C - \pi \cdot \pi & D^C \pi_{CB} - D_B \pi_C{}^C \\ D^C \pi_{CA} - D_A \pi & \nabla_0 \pi_{AB} + \pi \pi_{AB} + r_{AB} \end{bmatrix} = 0, \tag{21}$$

where now r_{AB} is the Ricci tensor of g_{AB} on Σ, $\pi = \pi_C{}^C$, $(\pi \cdot \pi)_{AB} = \pi_A{}^C \pi_{CB}$ and $\pi \cdot \pi = (\pi \cdot \pi)_C{}^C$. Note that we have found it convenient to use the ∇_0 operator instead of ∂_t, because of its preservation of both the orthogonal splitting with respect to the foliation and of the spatial metric, $\nabla_0 g_{AB} = \nabla_0 g^{AB} = 0$. For convenience, we note the commutator

$$(\nabla_0 D_A - D_A \nabla_0) \begin{bmatrix} v_0 \\ v_B \end{bmatrix} = -\pi_A{}^C D_C \begin{bmatrix} v_0 \\ v_B \end{bmatrix} + \begin{bmatrix} 0 \\ (D^C \pi_{AB} - D_B \pi_A{}^C) \end{bmatrix} v_C \,. \tag{22}$$

According to Lemma 1 and the specific identity (12), the Killing equation $\mathrm{K}_{ab}[v] = 0$ is satisfied when v_a is any solution of (10) where both

$$\mathrm{K}_{ab}[v]|_\Sigma \to \left.\begin{bmatrix} \mathrm{K}_{00}[v] & \mathrm{K}_{0B}[v] \\ \mathrm{K}_{0A}[v] & \mathrm{K}_{AB}[v] \end{bmatrix}\right|_\Sigma = 0 \tag{23a}$$

$$\text{and} \quad \nabla_0 \mathrm{K}_{ab}[v]|_\Sigma \to \left.\begin{bmatrix} \nabla_0\mathrm{K}_{00}[v] & \nabla_0\mathrm{K}_{0B}[v] \\ \nabla_0\mathrm{K}_{0A}[v] & \nabla_0\mathrm{K}_{AB}[v] \end{bmatrix}\right|_\Sigma = 0. \tag{23b}$$

In more detail, these components are

$$\mathrm{K}_{00}[v] = 2\nabla_0 v_0, \tag{24a}$$

$$\mathrm{K}_{0B}[v] = \nabla_0 v_B - \pi_{BC} v^C + D_B v_0, \tag{24b}$$

$$\mathrm{K}_{AB}[v] = D_A v_B + D_B v_A - 2\pi_{AB} v_0, \tag{24c}$$

$$\nabla_0 \mathrm{K}_{00}[v] = 2\nabla_0\nabla_0 v_0, \tag{24d}$$

$$\begin{aligned} \nabla_0 \mathrm{K}_{0B}[v] &= \nabla_0\nabla_0 v_B - (\nabla_0 \pi_{BC}) v^C \\ &\quad - \pi_{BC} \nabla_0 v_0 + D_B \nabla_0 v_0 - \pi_{BC} D^C v_0, \end{aligned} \tag{24e}$$

$$\begin{aligned} \nabla_0 \mathrm{K}_{AB}[v] &= D_A \nabla_0 v_B + D_B \nabla_0 v_A + 2(D^C \pi_{AB}) v_C - 2\pi_{C(A} D^C v_{B)} \\ &\quad - 2 D_{(A}\pi_{B)C} v^C - 2\pi_{AB} \nabla_0 v_0 - 2(\nabla_0 \pi_{AB}) v_0. \end{aligned} \tag{24f}$$

On the other hand, the propagation equation (10) splits (modulo $R_{ab} = 0$) as

$$\begin{aligned} -\nabla_0\nabla_0 v_0 + D^C D_C v_0 - 2\pi^{BC} D_B v_C - \pi \nabla_0 v_0 & \\ - (D_B \pi^{BC}) v_C + (\pi\cdot\pi) v_0 &= 0, \end{aligned} \tag{25a}$$

$$\begin{aligned} -\nabla_0\nabla_0 v_A + D^C D_C v_A - 2\pi_A{}^B D_B v_0 - \pi \nabla_0 v_A & \\ - (D^B \pi_{AB}) v_0 + (\pi\cdot\pi)_{AC} v^C &= 0. \end{aligned} \tag{25b}$$

Finally, eliminating the time derivatives of v_0 and v_A, while also eliminating the time derivatives of π_{AB} using the vacuum Einstein equations (21), we obtain the well-known Killing initial data (KID) conditions:

$$D_A v_B + D_B v_A - 2\pi_{AB} v_0 = 0, \tag{26a}$$

$$\begin{aligned} D_A D_B v_0 + (2(\pi\cdot\pi)_{AB} - \pi\pi_{AB} - r_{AB}) v_0 & \\ - 2\pi_{(B}{}^C D_{A)} v_C - (D^C \pi_{AB}) v_C &= 0. \end{aligned} \tag{26b}$$

For ease of comparison with the conformal and homothetic case in Sections 3 and 4, let us note the traces of the above KID conditions:

$$2(D_C v^C - \pi v_0) = 0, \tag{27a}$$

$$\begin{aligned} D^C D_C v_0 - (\pi\cdot\pi) v_0 - (D^C \pi) v_C & \\ - \pi^{EF}(D_E v_F + D_F v_E - 2\pi_{EF} v_0) &= 0, \end{aligned} \tag{27b}$$

where we have used the Hamiltonian constraint $r + \pi^2 - \pi\cdot\pi = 0$ from the vacuum Einstein equations to eliminate the spatial scalar curvature r.

The propagation identity and Killing initial data equations recalled in this section, coupled with Lemmas 1 and 2, allow us in particular to identify those initial data sets for the metric g that give rise to a solution of the Einstein vacuum equations equations with Killing symmetries. This was the original motivation under which the Killing initial data conditions (26) were first identified [12, 23, 24, 6]. Under such considerations, the propagation identity (12) could have been simplified, by dropping any terms that vanish in vacuum. However, similar condition have also been found to identify initial data sets some non-vacuum solutions of Einstein equations with Killing symmetries [29, 30], where the full propagation identity (12) plays a crucial role.

3 Homothetic Killing initial data

Recall that *homothetic Killing* vectors v_a are those that satisfy the *conformal Killing* equation in addition to having a constant divergence (together giving $E[\phi] = 0$ in the notation of Section 2), namely

$$\mathrm{CK}_{ab}[v] := \nabla_a v_b + \nabla_b v_a - \frac{2}{n} g_{ab} \nabla^c v_c \tag{28}$$

$$\nabla_a(\nabla^b v_b) = 0. \tag{29}$$

The $\mathrm{CK}_{ab}[v]$ operator is simply the trace free part of the Killing operator $\mathrm{K}_{ab}[v]$ from (9),

$$\mathrm{CK}_{ab}[v] = \mathrm{K}_{ab}[v] - \frac{1}{n} g_{ab} \mathrm{K}_c{}^c[v]. \tag{30}$$

The homothetic Killing vector equation is less restrictive than the Killing equation itself, but is more restrictive than just the conformal Killing vector equation. Homothetic Killing vectors are important in relativity because of the interesting observation [13, Thm.3] that a conformal Killing vector on a 4-dimensional Einstein vacuum spacetime (where $R_{ab} = 0$) is almost always homothetic. This is a purely local result, independent of the global features of a spacetime. The only exceptions that admit proper conformal Killing vectors (non-homothetic ones) are locally (a) flat spacetime or (b) a subclass of type N spacetimes. It is likely that analogous restrictions exist in other dimensions and non-Lorentzian signatures.

Homothetic Killing vectors also play an essential role in the definition of *(asymptotically) self-similar solutions* and therefore their characterization from an initial data set point of view could be relevant in the study of these solutions with applications to *critical phenomena* in gravity (see [18] for more details about the relation between self-similar solutions and critical phenomena).

As was discussed in the Introduction, the existence of initial data equations for homothetic Killing vector fields is a natural question. It was first treated and essentially solved in the early work [9], quickly following the seminal work [8, 23] work on Killing initial data. Though they did not exactly use the same language as we did in Section 2, the early references [8, 23, 9] did derive propagation identities, but did not provide sufficient conditions to show that the corresponding equations have a well-posed initial value problem. They did write the propagation equations in Cauchy-Kovalevskaya form (solved for the highest order time derivatives), but that is sufficient for well-posedness only for analytic initial data. In this section, we reobtain the *homothetic Killing initial data* (HKID) equations from [9, Prop.2]. However, the way that we derive propagation allows us to write them directly using wave operators, making their well-posedness manifest (since they belong to the well-studied normally-hyperbolic class mentioned in Section 2). For Killing initial data this was first done in [12], but to date does not appear to have been done explicitly for the homothetic case.

The most straightforward way to obtain the relevant propagation identity is to start with the analogous identity (12) for the Killing equation and first take its trace,

$$\Box \mathrm{K}_c{}^c[v] = \mathrm{K}_c{}^c[\Box v + R \cdot v] - 2v^c \nabla_c R - 4R^{cd}\nabla_c v_d, \tag{31}$$

and then the take gradient of that, noting the relation $\frac{1}{2}\mathrm{K}_c{}^c[v] = \nabla^c v_c =: \delta v$ and commuting ∇_a with $\Box$,

$$\begin{aligned}\Box(\nabla_a \delta v) = \nabla_a \delta(\Box v + R \cdot v)& \\ - \nabla_a(v^c \nabla_c R) - 2\nabla_a(R^{cd}\nabla_c v_d) &+ R_a{}^d \nabla_d \delta v.\end{aligned} \tag{32}$$

Next, using the identity $\mathrm{K}_{ab}[v] = \mathrm{CK}_{ab}[v] + \frac{1}{n} g_{ab}\mathrm{K}_c{}^c[v]$ in (12) and simplifying the result with the help of (31), we obtain the following propagation identity for the $\mathrm{CK}_{ab}[v]$ operator

$$\begin{aligned}\Box\mathrm{CK}_{ab}[v] - 2R^c{}_{ab}{}^d\mathrm{CK}_{cd}[v] - 2R_{(a}{}^c\mathrm{CK}_{b)c}[v] = \mathrm{CK}_{ab}[\Box v + R\cdot v]& \\ + 2\mathcal{L}_v\left(\frac{R}{n}g_{ab} - R_{ab}\right) + \frac{2}{n}\left(g_{ab}R^{cd} - R\delta_a{}^{(c}\delta_b{}^{d)}\right)\mathrm{K}_{cd}[v],&\end{aligned} \tag{33}$$

which is manifestly traceless. Together, Equations (33) and (32) make up the propagation identity ($P[E[\phi]] = \sigma[Q[\phi]] + \tau[G]$) for the homothetic Killing vector equations. The corresponding propagation equations are

$$\Box v_a + R_a{}^b v_b = 0 \quad (Q[\phi] = 0), \tag{34}$$

$$\begin{bmatrix}\Box h_{ab} - 2R^c{}_{ab}{}^d h_{cd} - 2R_{(a}{}^c h_{b)c} \\ \Box w_a\end{bmatrix} = 0 \quad (P[\psi] = 0), \tag{35}$$

where h_{ab} is considered to be symmetric and traceless, and we must note that

$$\Box v_a + R_a{}^b v_b = \nabla^b \mathrm{CK}_{ab}[v] - \frac{n-2}{n}\nabla_a \delta v \quad (Q[\phi] = \rho[E[\phi]]). \tag{36}$$

By performing an analysis similar to that of Section 2.1, that is, relying on Lemmas 1 and 2 as well as the propagation identities (33), (32) and (36), we can compute the necessary and sufficient conditions which yield a homothetic Killing initial data set (HKID) on Σ, namely any set of equations equivalent to the following:

$$\mathrm{CK}_{ab}[v]|_\Sigma \to \begin{bmatrix} \mathrm{CK}_{00}[v] & \mathrm{CK}_{0B}[v] \\ \mathrm{CK}_{0A}[v] & \mathrm{CK}_{AB}[v] \end{bmatrix}\Bigg|_\Sigma = 0, \tag{37a}$$

$$\nabla_0\mathrm{CK}_{ab}[v]|_\Sigma \to \begin{bmatrix} \nabla_0\mathrm{CK}_{00}[v] & \nabla_0\mathrm{CK}_{0B}[v] \\ \nabla_0\mathrm{CK}_{0A}[v] & \nabla_0\mathrm{CK}_{AB}[v] \end{bmatrix}\Bigg|_\Sigma = 0, \tag{37b}$$

$$\nabla_a\delta v|_\Sigma \to \begin{bmatrix} \nabla_0\delta v \\ D_A\delta v \end{bmatrix}\Bigg|_\Sigma = 0, \tag{37c}$$

$$\text{and} \quad \nabla_0\nabla_a\delta v|_\Sigma \to \begin{bmatrix} \nabla_0\nabla_0\delta v \\ \nabla_0 D_A\delta v \end{bmatrix}\Bigg|_\Sigma = 0. \tag{37d}$$

For reference, some of the explicit components of the above operators are

$$\mathrm{CK}_{00}[v] = \frac{2}{n}\left[(n-1)\nabla_0 v_0 - \pi v_0 + D_A v^A\right], \tag{38a}$$

$$\mathrm{CK}_{0B}[v] = \nabla_0 v_B + D_B v_0 - \pi_B{}^A v_A, \tag{38b}$$

$$\begin{aligned}\mathrm{CK}_{AB}[v] &= D_A v_B + D_B v_A - 2\pi_{AB}v_0 \\ &\quad - \frac{2}{n}g_{AB}\left(-\nabla_0 v_0 - \pi v_0 + D_C v^C\right),\end{aligned} \tag{38c}$$

$$\begin{aligned}\nabla_0\delta v &= -\nabla_0\nabla_0 v_0 - \nabla_0(\pi v_0) \\ &\quad + D^B\nabla_0 v_B - \pi^{BC}D_B v_C - R_{0B}v^B,\end{aligned} \tag{38d}$$

$$D_A\delta v = D_A(-\nabla_0 v_0 - \pi v_0 + D_B v^B), \tag{38e}$$

where we recall that $R_{0B} = D^C\pi_{CB} - D_B\pi$.

We are now ready to formulate the main result of this section:

Theorem 1. *Consider an n-dimensional globally hyperbolic Einstein vacuum Lorentzian manifold, (M, g) with $R_{ab} = 0$, and a Cauchy surface $\Sigma \subset M$. For $n > 2$, the necessary and sufficient conditions yielding a set of* homothetic Killing initial data *(HKID) for v_a on Σ are given by the following equations:*

$$D_A v_B + D_B v_A - 2\pi_{AB}v_0 - \frac{2g_{AB}}{n-1}(D_C v^C - \pi v_0) = 0\ , \tag{39a}$$

$$\begin{aligned}&D_A D_B v_0 + (2(\pi\cdot\pi)_{AB} - \pi\pi_{AB} - r_{AB})\, v_0 \\ &\quad - 2\pi_{C(A}D_{B)}v^C - v^C D_C\pi_{AB} + \frac{\pi_{AB}}{n-1}(D_C v^C - v_0\pi) = 0\ ,\end{aligned} \tag{39b}$$

$$D_B(D_A v^A - \pi v_0) = 0\ . \tag{39c}$$

Proof. The proof is straightforward by direct calculation. The derivatives $\nabla_0 v_0$, $\nabla_0 v_B$, $\nabla_0\nabla_0 v_0$ and $\nabla_0\nabla_0 v_B$ are eliminated respectively by $\mathrm{CK}_{00}[v]$, $\mathrm{CK}_{0B}[v]$, $\nabla_0\mathrm{CK}_{00}[v]$ and $\nabla_0\mathrm{CK}_{0B}[v]$. Thus, the components $\mathrm{CK}_{AB}[v]$ and $D_B\delta v$ respectively lead to the desired initial data equations (39a) and (39c). Note that equation (39a) is manifestly traceless. Thus, we expect the vanishing $\nabla_0\mathrm{CK}_{AB}[v]$ to contribute another traceless equation. It turns out to be convenient to add to it a trace component proportional to $\nabla_0 \delta v$, thus leading to the last independent initial data equation (39b). The remaining initial data equations are not independent because of the identities

$$\nabla_0 D_A \delta v = D_A(\nabla_0 \delta v) - \pi_A{}^B (D_B \delta v), \tag{40a}$$

$$\nabla_0 \nabla_0 \delta v = D^A (D_A \delta v) - \pi(\nabla_0 \delta v) + \nabla_0 (\Box v)_0 - D^A (\Box v)_A + \pi(\Box v)_0, \tag{40b}$$

where the last identity follows from the splitting of (31), with $(\Box v)_0$ and $(\Box v)_A$ themselves expressible as

$$(\Box v)_0 = -\nabla_0 \mathrm{CK}_{00}[v] + D^B \mathrm{CK}_{0B}[v] - \pi \mathrm{CK}_{00}[v] - \frac{n-2}{n}(\nabla_0 \delta v), \tag{41a}$$

$$\begin{aligned}(\Box v)_A = {} & -\nabla_0 \mathrm{CK}_{0A}[v] + D^B \mathrm{CK}_{AB}[v] - \pi \mathrm{CK}_{0A}[v] \\ & - \pi_A{}^B \mathrm{CK}_{0B}[v] - \frac{n-2}{n}(\nabla_A \delta v),\end{aligned} \tag{41b}$$

due to the splitting of (36). $\Box$

Note that a homothetic Killing vector v_a is also a normal Killing vector exactly when it is divergence free, $\delta v = 0$. Eliminating ∇_0 derivatives, as in the proof of the theorem, the divergence can be written as

$$\begin{aligned}\delta v = \nabla_a v^a &= -\nabla_0 v_0 - \pi v_0 + D_A v^A \\ &= \frac{n}{n-1}\left(D_A v^A - \pi v_0\right) + \frac{n}{2(n-1)} \mathrm{CK}_{00}[v].\end{aligned} \tag{42}$$

Note that we have written the HKID equations (39) in such a way that when the spatial divergence free condition

$$D_A v^A - \pi v_0 = 0 \tag{43}$$

is satisfied, the HKID equations manifestly reduce to the KID equations (26).

4 Conformal Killing initial data

A *conformal Killing* vector v_a satisfies the equation $\mathrm{CK}_{ab}[v] = 0$, defined in (28). It is less restrictive than either the Killing or the homothetic Killing equations, discussed in Sections 2.1 and 3. As we have seen, heuristically, the less restrictive the equation, the more complicated the corresponding propagation identity and the corresponding initial data equations (if they exist). That pattern will repeat in this section, where we for the first time both prove the existence of a propagation identity (53) and explicitly construct the *conformal Killing initial data* (CKID) equations (Theorem 3). The structure of this section is modeled on and uses notation from Sections 2.1 and 3.

The first attempt to construct the CKID equations (though without using that terminology) was in the early paper [9], which quickly followed the original work on the KID equations [8, 23]. However, the construction was not complete and only obtained some necessary conditions on the initial data of v_a [9, Sec.V], but without deriving a set of initial data conditions that could be sufficient. Unfortunately, the CKID problem does not appear to have been seriously revisited since then. In Theorem 3, we finally and for the first time give a complete construction of the CKID equations (which are both necessary and sufficient) on Einstein vacuum spacetimes. In retrospect, we can also answer the following question: why was a full set of initial data conditions not discovered already in [9]? The answer is simple. The strategy in [9] was to split the components of the conformal Killing equation into evolution equations and spatial constraint equations, and then take time derivatives of the latter generating further spatial constraints (modulo the evolution equations), until hopefully after a certain number of time derivatives no new spatial constraint equations would be generated, giving an analog of what we called a propagation identity. The existence of our fourth-order propagation identity (53) implies that this strategy would have succeeded after four time differentiations (cf. the discussion of Equations (61) in the proof of Theorem 3). Unfortunately, the calculations in [9] stopped at the third time derivative, just short of the necessary differential order.

There are a couple of points at which the discussion below must deviate from the parallel Killing case of 2.1. In particular, to simplify the various identities to appear below, we make the blanket assumption that for the rest of this section we are dealing with an Einstein vacuum background, satisfying $R_{ab} = 0$.

The first problem is that the candidate propagation operator

$$Q_a[v] := \Box v_a + \frac{n-2}{n}\nabla_a\nabla^b v_b = \nabla^b \mathrm{CK}_{ab}[v] \quad (Q[\phi] = \rho[E[\phi]]) \tag{44}$$

is no longer normally-hyperbolic, as was the case for the Killing equation, because in addition to the wave term $\Box v_a$ also the $\nabla_a\nabla^b v_b$ term contributes to the principal symbol. Fortunately, Q does belongs to the generalized normally-hyperbolic class that we discussed at the end of Section 2, and therefore $Q[\phi] = 0$ is a propagation equation.

Lemma 4. *Any operator of the form*

$$Q_a[v] = x\Box v_a + (z-x)\nabla_a\nabla^b v_b, \tag{45}$$

where $x \neq 0$ and $z \neq 0$ is generalized normally-hyperbolic.

Proof. The identity

$$\frac{1}{x}\Box Q_a[v] + \left(\frac{1}{z} - \frac{1}{x}\right)\nabla_a\nabla^b Q_b[v] = \Box^2 v_a + \text{l.o.t} \tag{46}$$

is all that is needed, which holds precisely when $x \neq 0$ and $z \neq 0$. □

Clearly, our Q from (44) is a special case of the operator in Lemma 4 with $x = 1 \neq 0$, $z = \frac{2(n-1)}{n} \neq 0$. Indeed, for these values of x and z a computation shows that the identity (46) adopts the form

$$\Box Q_a[v] - \frac{(n-2)}{2(n-1)}\nabla_a\nabla^b Q_b[v] = \Box^2 v_a. \tag{47}$$

The second problem is that the actual propagation identity for the CK operator is more convenient to express by coupling it to one of its integrability conditions, which is propagated separately. Namely, due to the identity (with the notation $(\delta\mathrm{CK}[v])_a = \nabla^b \mathrm{CK}_{ab}[v]$)

$$\begin{aligned}\nabla_a \nabla_b (\nabla^c v_c) &= S_{ab}[\mathrm{CK}[v]] \\ &:= -\frac{n}{2(n-2)} \Box \mathrm{CK}_{ab}[v] + \frac{n}{2(n-2)} \mathrm{CK}_{ab}[\delta \mathrm{CK}[v]] \\ &\quad + \frac{1}{2(n-1)} g_{ab} \nabla^c \nabla^d \mathrm{CK}_{cd}[v] - \frac{n}{(n-2)} R_a{}^c{}_b{}^d \mathrm{CK}_{cd}[v],\end{aligned} \tag{48}$$

when $\mathrm{CK}_{ab}[v] = 0$, the divergence $u = \nabla^c v_c$ satisfies the *covariant affine* equation

$$\mathrm{Af}_{ab}[u] := \nabla_a \nabla_b u = 0. \tag{49}$$

It satisfies (modulo $R_{ab} = 0$) the propagation identities

$$\Box u = \mathrm{Af}^c{}_c[u], \quad \Box \, \mathrm{Af}_{ab}[u] + 2 R_a{}^c{}_b{}^d \, \mathrm{Af}_{cd}[u] = \nabla_a \nabla_b \Box u. \tag{50}$$

Importantly, the propagation equation for u follows from the propagation equation for v_a because

$$\Box(\delta v) = \frac{n}{2(n-1)} \nabla^c Q_c[v]. \tag{51}$$

The coupled propagation identity for the CK operator then takes the form

$$\begin{aligned}&\Box \mathrm{CK}_{ab}[v] + 2 R_a{}^c{}_b{}^d \mathrm{CK}_{cd}[v] \\ &\quad + \frac{2(n-2)}{n} \left(\mathrm{Af}_{ab}[\delta v] - \frac{1}{n} g_{ab} \, \mathrm{Af}^c{}_c[\delta v] \right) = \mathrm{CK}_{ab}[Q[v]].\end{aligned} \tag{52}$$

The coupled propagation system (52) and (50) can be combined into a single propagation identity for the CK operator, at the expense of making it higher order:

$$\begin{aligned}&\Box^2 \mathrm{CK}_{ab}[v] + 2\Box(R_a{}^p{}_b{}^q \mathrm{CK}_{pq}[v]) - \frac{4(n-2)}{n} R_a{}^c{}_b{}^d S_{cd}[\mathrm{CK}[v]] \\ &\quad = \Box \mathrm{CK}_{ab}[Q[v]] - \frac{(n-2)}{(n-1)} \nabla_a \nabla_b \nabla^c Q_c[v] + \frac{(n-2)}{n(n-1)} g_{ab} \Box \nabla^c Q_c[v] \\ &\qquad\qquad (P[E[\phi]] = \sigma[Q[\phi]]).\end{aligned} \tag{53}$$

Introducing the trace-free operator $\overline{\mathrm{Af}}_{ab}[u] = \mathrm{Af}_{ab}[u] - \frac{g_{ab}}{n} \mathrm{Af}_c{}^c[u]$, expanding the definition of S_{ab} from (48) and using basic simplifications, the propagation identity becomes

$$\begin{aligned}&\Box^2 \mathrm{CK}_{ab}[v] + 2\Box(R_a{}^c{}_b{}^d \mathrm{CK}_{cd}[v]) + 2 R_a{}^c{}_b{}^d \Box \mathrm{CK}_{cd}[v] + 4 R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f \mathrm{CK}_{ef}[v] \\ &\quad = \Box \mathrm{CK}_{ab}[Q[v]] - \frac{(n-2)}{(n-1)} \overline{\mathrm{Af}}_{ab}[\delta Q[v]] \, .\end{aligned} \tag{54}$$

The advantage of the higher order propagation identity (53) is that it, together with (44), fits directly into the hypotheses of our Lemmas 1 and 2, implying that there exists a system of *conformal Killing initial data* (CKID) conditions $\mathrm{CK}^{\Sigma}[v|_\Sigma, \nabla_0 v|_\Sigma] = 0$, whose solutions on a Cauchy surface $\Sigma \subset M$ are are in bijection with solutions of $\mathrm{CK}[v] = 0$ on M. It remains only to compute it.

It is more practical to carry out this calculation starting from the coupled second order system of propagation identities (52) and (50). For that purpose, let us fix a Cauchy surface $\Sigma \subset M$ and follow the notational conventions introduced in Section 2.1. The components of the conformal Killing and covariant affine operators

$$\mathrm{CK}_{ab}[v]|_\Sigma \to \left.\begin{bmatrix} \mathrm{CK}_{00}[v] & \mathrm{CK}_{0B}[v] \\ \mathrm{CK}_{0A}[v] & \mathrm{CK}_{AB}[v] \end{bmatrix}\right|_\Sigma = 0 \tag{55a}$$

$$\text{and} \quad \mathrm{Af}_{ab}[u]|_\Sigma \to \left.\begin{bmatrix} \mathrm{Af}_{00}[u] & \mathrm{Af}_{0B}[u] \\ \mathrm{Af}_{0A}[u] & \mathrm{Af}_{AB}[u] \end{bmatrix}\right|_\Sigma = 0 \tag{55b}$$

take the explicit form

$$\mathrm{CK}_{00}[v] = \frac{2}{n}\left[(n-1)\nabla_0 v_0 - \pi v_0 + D_A v^A\right], \tag{56a}$$

$$\mathrm{CK}_{0B}[v] = \nabla_0 v_B + D_B v_0 - \pi_B{}^A v_A, \tag{56b}$$

$$\begin{aligned}\mathrm{CK}_{AB}[v] &= D_A v_B + D_B v_A - 2\pi_{AB} v_0 \\ &\quad - \frac{2}{n} g_{AB}\left(D_C v^C - \pi v_0 - \nabla_0 v_0\right),\end{aligned} \tag{56c}$$

$$\mathrm{Af}_{00}[u] = \nabla_0 \nabla_0 u, \tag{56d}$$

$$\mathrm{Af}_{0B}[u] = D_B(\nabla_0 u) - \pi_{BC} D^C u, \tag{56e}$$

$$\mathrm{Af}_{AB}[u] = D_A D_B u - \pi_{AB}(\nabla_0 u). \tag{56f}$$

As a first step, completely analogous to the review of the Killing vector case in Section 2.1, we have the following

Theorem 2. *Consider a globally hyperbolic Einstein vacuum Lorentzian manifold, (M, g) of dimension $n > 0$ with $R_{ab} = 0$, and a Cauchy surface $\Sigma \subset M$. The necessary and sufficient conditions yielding a set of* covariant affine initial data *(AfID) for u on Σ are given by the following equations:*

$$D_B(\nabla_0 u) - \pi_{BC} D^C u = 0\,, \tag{57a}$$

$$D_A D_B u - \pi_{AB}(\nabla_0 u) = 0\,, \tag{57b}$$

$$\begin{aligned}&(r_{AB} + \pi\pi_{AB} - (\pi\cdot\pi)_{AB})(\nabla_0 u) \\ &\quad - (D_{(A}\pi_{B)C} - D_C \pi_{AB}) D^C u = 0\,.\end{aligned} \tag{57c}$$

Proof. The components $\mathrm{Af}_{0B}[u]$ and $\mathrm{Af}_{AB}[u]$ directly give (57a) and (57b). The derivative $\nabla_0 \mathrm{Af}_{AB}[u]$ gives (57c), after using $\mathrm{Af}_{00}[u]$ to eliminate $\nabla_0\nabla_0 u$, and further simplifications from the first two initial data conditions. Lastly, the condition $\nabla_0 \operatorname{Af}_{0B}[u] = 0$ turns out not to be independent from the other ones due to the identity

$$\nabla_0 \operatorname{Af}_{0B}[u] = 2g^{CA} D_{[C} \left(D_{B]} D_A u - \pi_{B]A} (\nabla_0 u) \right), \tag{58}$$

after simplifications from $\operatorname{Af}_{00}[u] = 0$ and the vacuum Einstein equations. □

So, now what we need to do is start with the conditions $\mathrm{CK}[v]|_\Sigma = 0$, ..., $\nabla_0^3 \mathrm{CK}[v]|_\Sigma = 0$, and extract from them a (hopefully small) subset of components whose vanishing ensures the vanishing of the remaining components as well.

Theorem 3. *Consider a globally hyperbolic Einstein vacuum Lorentzian manifold, (M, g) of dimension $n > 2$ with $R_{ab} = 0$, and a Cauchy surface $\Sigma \subset M$. For a conformal Killing vector v_a, $\mathrm{CK}_{ab}[v] = 0$, its rescaled divergence $u = \frac{(n-1)}{n} \nabla^a v_a$ and its derivative $\nabla_0 u$ take the following form when restricted to Σ, after eliminating the $\nabla_0 v_a$ derivatives,*

$$u = \left(D_C v^C - \pi v_0 \right) , \tag{59a}$$

$$\nabla_0 u = \frac{1}{n-1} \pi u + \left(-D^A D_A v_0 + (\pi \cdot \pi) v_0 + (D^A \pi) v_A \right) . \tag{59b}$$

Using the above notation, the necessary and sufficient conditions yielding a set of conformal Killing initial data *(CKID) for v_a on Σ are given by the following equations:*

$$D_A v_B + D_B v_A - 2\pi_{AB} v_0 - \frac{2}{n-1} g_{AB} u = 0 , \tag{60a}$$

$$\begin{aligned} &D_B D_A v_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - r_{AB}) v_0 - 2\pi_{C(A} D_{B)} v^C \\ &\quad + v^C (D_C \pi_{AB}) + \frac{1}{n-1} (u \pi_{AB} + g_{AB} \nabla_0 u) = 0 , \end{aligned} \tag{60b}$$

$$D_A D_B u - \pi_{AB} (\nabla_0 u) = 0, \tag{60c}$$

$$\begin{aligned} &(r_{AB} + \pi \pi_{AB} - (\pi \cdot \pi)_{AB})(\nabla_0 u) \\ &\quad - (D_{(A} \pi_{B)C} - D_C \pi_{AB}) D^C u = 0 . \end{aligned} \tag{60d}$$

Proof. Schematically (modulo the $Q_a[v] = 0$ propagation equation), the propagation identities (52) and (50) imply, respectively, the initial data identities

$$\nabla_0 \nabla_0 \mathrm{CK} = O(\mathrm{Af}) + O(\nabla_0 \mathrm{CK}) + O(\mathrm{CK}), \tag{61a}$$

$$\nabla_0 \nabla_0 \mathrm{Af} = O(\nabla_0 \mathrm{Af}) + O(\mathrm{Af}), \tag{61b}$$

where CK and Af stand respectively for the components of $\mathrm{CK}[v]$ and $\mathrm{Af}[\delta v]$, while the notation $O(-)$ indicates proportionality to the argument or any spatial derivative thereof. Taking ∇_0 derivatives of (61a) and using (61b) to eliminating as many higher order ∇_0 derivatives of CK and Af as possible, we obtain the further initial data identities

$$\nabla_0^3 \mathrm{CK} = O(\nabla_0 \mathrm{Af}) + O(\mathrm{Af}) + O(\nabla_0 \mathrm{CK}) + O(\mathrm{CK}), \tag{61c}$$

$$\nabla_0^4 \mathrm{CK} = O(\nabla_0 \mathrm{Af}) + O(\mathrm{Af}) + O(\nabla_0 \mathrm{CK}) + O(\mathrm{CK}). \tag{61d}$$

Thus, it is sufficient to keep only the CK, ∇_0CK, Af, ∇_0 Af components for the CKID, and by Lemmas 1 and 2 the solutions of the resulting CKID conditions on $\Sigma \subset M$ would be in bijection with the solutions of the conformal Killing equation on M.

The $\mathrm{CK}_{00}[v]$ and $\mathrm{CK}_{0B}[v]$ components can be used to eliminate any ∇_0 derivatives of v_0 and v_B, respectively, and thus they and their ∇_0 derivatives need not appear in the final CKID system. The $\mathrm{CK}_{AB}[v]$ and $\nabla_0\mathrm{CK}_{AB}[v]$ components, after eliminating the ∇_0 derivatives, give respectively (60a) and (60b), where also (60a) was used to simplify the form of (60b).

Theorem 2 has already shown that the vanishing of Af and ∇_0 Af are equivalent to the vanishing of Af_{00} and the AfID system (57). It remains only to plug in the following expressions, with

$$u = \frac{(n-1)}{n}\delta v = \frac{(n-1)}{n}\left(-\nabla_0 v_0 - \pi v_0 + D_C v^C\right), \tag{62}$$

which by direct calculation, after eliminating the ∇_0 derivatives of v_0 and v_B using $\mathrm{CK}_{00}[v]$ and $\mathrm{CK}_{0B}[v]$, leads to the expressions in (59). The resulting $\mathrm{Af}_{00}[\delta v]$ and $\mathrm{Af}_{0B}[\delta v]$ expressions are not independent, due to the identities

$$\begin{aligned}\mathrm{Af}_{00}[\delta v] = \frac{n}{2(n-1)(n-2)}\big[&-(2n-3)\pi^{AC}\nabla_0\mathrm{CK}_{AC}[v] \\ &+ D^A D^C \mathrm{CK}_{AC}[v] - r^{AC}\mathrm{CK}_{AC}[v]\big] ,\end{aligned} \tag{63}$$

$$\begin{aligned}\mathrm{Af}_{0B}[\delta v] = \frac{n}{2(n-2)}\big[&D^A\nabla_0\mathrm{CK}_{AB}[v] + \pi^{AC}D_A\mathrm{CK}_{CB}[v] - \pi_B{}^C D^A\mathrm{CK}_{AC}[v] \\ &- \pi^{AC}D_B\mathrm{CK}_{AC}[v] + (D^C\pi)\mathrm{CK}_{CB}[v] - (D^A\pi^C{}_B)\mathrm{CK}_{AC}[v]\big] ,\end{aligned} \tag{64}$$

again modulo $\mathrm{CK}_{00}[v] = 0$ and $\mathrm{CK}_{0B}[v] = 0$, which are obtained by splitting the spacetime identity (48). The conditions (57b) and (57c), after the u and $\nabla_0 u$ substitution, directly give respectively the remaining CKID conditions (60c) and (60d), which completes the proof. □

Obviously, a Killing vector v, is a conformal Killing vector satisfying the extra divergence condition $\delta v = 0$. As we have seen in the above proof, according to (59), the vanishing of the divergence δv and its derivative $\nabla_0\delta v$ are equivalent to the initial data conditions

$$D_C v^C - \pi v_0 = 0 , \tag{65a}$$

$$-D^A D_A v_0 + (\pi\cdot\pi)v_0 + (D^A\pi)v_A = 0 , \tag{65b}$$

once ∇_0 derivatives have been eliminated using $\mathrm{CK}_{00}[v] = 0$ and $\mathrm{CK}_{0B}[v] = 0$. Thus, when the initial data for v_a is divergence free in the above sense, it is obvious that the CKID conditions (60c) and (60d) are tautological, while the conditions (60a) and (60a) recover the KID system (26), as was to be expected.

5 Discussion

We have presented for the first time in the literature a set of necessary and sufficient conditions (the CKID equations) ensuring that a vacuum initial data set of the Einstein's equations in any dimension ($n > 2$) admits a conformal Killing vector in any globally hyperbolic development of this initial data. In addition to the standard quantities required for the construction of vacuum initial data (the first and the second fundamental forms, given respectively by g_{AB}, π_{AB} in our notation) we need the *conformal Killing lapse* v_0 and *conformal Killing shift* v_A. The CKID conditions are given by (60) of Theorem 3 and they are a set of linear PDEs for v_0, v_A on the Riemannian manifold with extrinsic curvature $(\Sigma,\, g_{AB},\, \pi_{AB})$. Along the way, we have reviewed construction of the Killing initial data (KID) and gave a new derivation of the homothetic Killing initial data (HKID) equations. Just as in the KID case, the HKID and CKID equations likely constitute an overdetermined elliptic system for v_0, v_A, but the true extent of this assertion requires a separate investigation.

A natural continuation of this work would be to try to construct initial data systems for other geometric PDEs, like for instance Killing-Yano equations, higher rank Killing tensor equations, and their conformal and/or closed versions. For instance, the existence of a principal (closed and non-degenerate) conformal Killing-Yano 2-form is known to characterize the Kerr-NUT-(A)dS family of higher dimensional black holes and related solutions [15]. So it is reasonable to suppose that the knowledge of the corresponding initial data system could be of use in the study of the stability and rigidity of this family. In 4 spacetime dimensions, the conformal Killing-Yano 2-form equation is equivalent to the Killing $(2,0)$-spinor equation [22], whose initial data system was already constructed in [16]. A tensorial version of this initial data system will appear in future work, along with an extension to the closed conformal Killing-Yano 2-form case in higher dimensions. The question of which other variations of the Killing equations have initial data systems appears to be completely open.

Since, in 4 spacetime dimensions, the conformal Killing equation is equivalent to the Killing $(1,1)$-spinor equation [32], it would be interesting to translate our CKID system into the initial data conditions for Killing $(1,1)$-spinors. Alternatively, such initial data conditions in 4 dimensions could be rederived from the relevant spinorial propagation identity used as an intermediate result in [16].

In all the known cases where initial data systems have been found, a certain amount of trial and error has been necessary for success. It would be an interesting problem to find a systematic way to identify those cases where no initial data system can exist.

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