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ABSTRACT

The properties of light rays around compact objects surrounded by a plasma are affected by both strong gravitational fields described by a general-relativistic spacetime and by a dispersive and refractive medium, characterized by the density distribution of the plasma. We study these effects employing the relativistic Hamiltonian formalism under the assumption of stationarity and axisymmetry. The necessary and sufficient conditions on the metric and on the plasma frequency are formulated such that the rays can be analytically determined from a fully separated Hamilton–Jacobi equation. We demonstrate how these results allow us to analytically calculate the photon region and the shadow if they exist. Several specific examples are discussed in detail: the "hairy" Kerr black holes, the Hartle–Thorne spacetime metrics, the Melvin universe, and the Teo rotating traversable wormhole. In all of these cases, a plasma medium is present as well.

I. INTRODUCTION

It was 110 years ago, as early as 1912, when Einstein, during his stay in Prague (April 1911–July 1912), sketched the basic properties of a gravitational lens in one of his notebooks, presumably on the occasion of a visit to Berlin in April to meet astronomer Freundlich.[1–4]

For many years, gravitational lensing, one of the most lively fields of research in relativistic astrophysics, has been focused on lensing by (clusters of) galaxies or microlensing by individual objects where bending angles are small. Over the last 20 years, however, more and more attention has also been paid to lensing by black holes and other compact objects where bending angles can be arbitrarily large. The most important achievement in this direction has been the observation of the "shadow" of the compact object—most likely a supermassive black hole—at the center of galaxy M87 by the Event Horizon Telescope Collaboration.[2] Related theoretical investigations are primarily aiming at distinguishing the standard Kerr black holes of general relativity from other compact objects by way of their lensing features, in particular, their shadows. Among many other examples, this includes black holes in strong magnetic fields, see Refs. 3 and 4, or black holes with hairs and boson stars, see, e.g., Ref. 8. For recent reviews on shadow calculations, see Refs. 9 and 10.

When comparing Kerr black holes with other compact objects, in the first approach, it is natural to still assume stationarity and axisymmetry. This, however, does not give us sufficiently many symmetries for the complete integrability of the equations of motion for vacuum light rays, i.e., it does not allow us to reduce the equation for lightlike geodesics to the first-order form. Therefore, it is not, in general, possible to calculate lensing features, such as the shadow, analytically in a stationary and axisymmetric spacetime. This is possible if there is an additional constant of motion, known as a (generalized) Carter constant, which exists if and only if the Hamilton–Jacobi equation for lightlike geodesics separates. The best known example where a Carter constant exists is the Kerr metric. However, even on a spacetime for which a Carter constant exists, it is not, in general, true that the equations of motion for light rays *in a medium* are completely integrable. In view of applications to astrophysics, the most interesting example of a medium is a plasma. A comprehensive treatment of the light propagation and formation of a shadow in a non-magnetized and pressure-free plasma in the Kerr background is described in Ref. 11. More recently, this approach was further studied in Refs. 12 and 13. In particular, the conditions on the form of the plasma frequency have been formulated, which enable one to find a generalized Carter constant for, e.g., black holes with the surrounding plasma in some generalized gravity spacetimes, implying, together with stationarity and axisymmetry, the complete integrability of the equations for the rays.

The purpose of the present work is to discuss, within the framework of geometrical optics, the possibility of an *analytical* approach to lensing in *general* axisymmetric stationary spacetimes containing a plasma medium. If, in addition to axisymmetry and stationarity, the existence of an equatorial plane is assumed and if the plasma density shares all symmetries of the spacetime, complete integrability is guaranteed for light rays in the equatorial plane. However, for almost all applications, one is also interested in light rays off the equatorial plane, and in some interesting spacetimes, an equatorial plane does not even exist. Therefore, in this paper, we want to derive the necessary and sufficient conditions for the existence of a Carter constant for light rays in a plasma on an arbitrary stationary and axisymmetric spacetime without any further restrictions.

There are powerful, systematic studies of the separability of the geodesic equation and its first integrals in general pseudo-Riemannian (Lorentzian) spacetimes of any dimension $d > 1$. These studies analyze "separability structures" classified by the number of Killing vectors and Killing tensors, which spacetimes possess. In general, relativity, Carter's discovery of separable spacetimes[14] inspired a number of contributions on the separability of the Hamilton–Jacobi equation and the closely related problem of the separability of the wave equation. We refer, in particular, to the papers by Benenti and Francaviglia on spacetimes with two commuting Killing vector fields; for their review, see Ref. 15. Recent studies of the separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations, including black holes and other related objects in higher dimensions, are reviewed in Ref. 16. One of the most recent studies of the similar topic was performed in Ref. 17, where relativistic simulations of a massive Proca field evolving on a Kerr background were analyzed, allowing one to effectively study the effect of plasmas, leading to super-radiance. However, we emphasize that there are two important differences between these earlier works, most notably the ones by Benenti and Francaviglia, and our's: First, they do not consider a plasma, and second, they require separability of the Hamilton–Jacobi equation for *all* geodesics—timelike, lightlike, and spacelike.

This paper is organized as follows. In Secs. II and III (and in the Appendix), we derive the conditions for the existence of the Carter constant for rays in axisymmetric stationary spacetimes containing plasmas. As a preparation for calculating the shadow, in Sec. IV, the photon regions in axisymmetric and stationary spacetimes are analyzed by employing the Hamilton equations and requiring the first two derivatives of the photons' radial coordinate with respect to a curve parameter to vanish. The form of the shadows caused by black holes surrounded by plasmas is studied in Sec. V. The remaining parts of this paper present the analysis of specific examples: spacetimes of the hairy Kerr black holes are shown to enable separation in a way very similar to Kerr black holes without hair in Sec. VI. However, the Hartle–Thorne metrics describing slowly rotating relativistic stars with a quadrupole moment (with or without plasmas) do not permit the separation (Sec. VII). As the third example, the Melvin cylindrical universe with plasmas is analyzed. Besides a general interest in this case, we wish to emphasize that it serves as an instructive example of the fact that the separation of the Hamilton–Jacobi equation crucially depends on the choice of the coordinates. Whereas in the spherical-type coordinates (used in this paper so far), the Carter constant (and thus the separability) cannot be obtained, in cylindrical-type coordinates, it does work, and the Carter constant and the photon region can be found (Sec. VIII). A detailed study is performed in Sec. IX: here, the Teo wormhole metric with a plasma is investigated, the separability of the Hamilton–Jacobi equation is shown, the condition for the existence of a spherical light ray around the wormhole is derived, and the shadow caused by the Teo wormhole is found. The results both for the vacuum case and for the case when a plasma is present are analyzed thoroughly; additionally, they are illustrated and characterized graphically. Short conclusions follow.

II. LIGHT PROPAGATION IN A NON-MAGNETIZED PRESSURELESS PLASMA

Considering a description of the ray propagation through a refractive medium used by Synge,[18] a respective Hamiltonian takes the form

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\Big[g^{\beta\delta}p_\beta p_\delta - (n^2-1)(p_\gamma V^\gamma)^2\Big], \tag{1}$$

where p_α is the canonical 4-momentum (called the frequency 4-vector in Ref. 18) and V^α is the 4-velocity of the medium, i.e.,

$$\omega(x^\alpha) = -p_\alpha V^\alpha \tag{2}$$

is the frequency measured in the rest system of the medium. The rays are, then, the solutions to Hamilton's equations,

$$\dot{x}^\alpha = \frac{\partial\mathcal{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial\mathcal{H}}{\partial x^\alpha}, \quad \mathcal{H}(x,p) = 0. \tag{3}$$

For a non-magnetized pressureless plasma, the refractive index n depends on the photon frequency $\omega(x^\alpha)$ in the form

$$n^2 = 1 - \frac{\omega_{\rm pl}^2(x^\alpha)}{\omega^2(x^\alpha)}. \tag{4}$$

Here, $\omega_{\rm pl}$ is the plasma frequency whose square equals the electron density up to a constant factor. Then, Hamiltonian (1) can be rewritten as

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\Big[g^{\beta\delta}(x^\alpha)p_\beta p_\delta + \omega_{pl}^2(x^\alpha)\Big], \tag{5}$$

i.e., it is independent of the medium velocity. The basic results of Synge are summarized in a physical way in Ref. 19. Therein, the concept of the photons (light particles) associated with wave packets in media and the Minkowski energy–momentum tensor for an electromagnetic field in a medium are also discussed. Hamiltonian (5) can be derived from Maxwell's equations on a curved background with a two-fluid plasma model; see the work of Breuer and Ehlers.[20]

Hamiltonian (5) demonstrates that the light rays in a plasma are determined by the same equations as the trajectories of massive particles with a "spacetime-dependent mass function." Moreover, if the plasma density is nowhere zero, we can introduce the conformally rescaled metric,

$$\tilde{g}_{\mu\nu}(x^\alpha) = \omega_{\rm pl}^2(x^\beta)g_{\mu\nu}(x^\gamma), \quad \tilde{g}^{\mu\nu}(x^\alpha) = \omega_{\rm pl}^{-2}(x^\beta)g^{\mu\nu}(x^\gamma), \tag{6}$$

and the modified Hamiltonian,

$$\tilde{\mathcal{H}}(x^\alpha, p_\beta) = \omega_{\rm pl}^{-2}(x^\gamma)\mathcal{H}(x^\rho, p_\sigma) = \frac{1}{2}\big(\tilde{g}^{\mu\nu}(x^\alpha)p_\mu p_\nu + 1\big). \tag{7}$$

As multiplying the Hamiltonian with a nowhere vanishing function leaves the solutions to (3) invariant up to parameterization, the solutions to (3) actually coincide with the (non-affinely parameterized) timelike geodesics of metric (6), i.e., the light rays in the plasma coincide with the trajectories of freely falling massive particles for which the separability of the Hamilton–Jacobi equation was originally discussed by Carter in 1968.[21] However, as metric (6) cannot be globally introduced if the plasma density is zero on some part of the spacetime, we will not use it in the following.

III. DERIVATION OF THE CARTER CONSTANT IN A GENERAL AXIALLY SYMMETRIC STATIONARY SPACETIME IN PLASMA

Let us assume an axially symmetric stationary metric given in coordinates $(t, \varphi, r, \vartheta)$, with the metric coefficients independent of t and φ. It is our goal to find out under what conditions the Hamilton–Jacobi equation for light rays in a plasma separates, i.e., under what conditions a (generalized) Carter constant exists. We prove in the Appendix that the following two statements are true: Separation is possible only if the plasma density is independent of t and φ, and if the Hamilton–Jacobi equation separates at all, then it separates in coordinates in which the metric coefficients g_{tr}, $g_{t\vartheta}$, $g_{\varphi r}$, $g_{\varphi\vartheta}$, and $g_{r\vartheta}$ vanish, so we may write the metric in the following form:

$$ds^2 = -A(r,\vartheta)dt^2 + B(r,\vartheta)dr^2 + 2P(r,\vartheta)dt d\varphi + D(r,\vartheta)d\vartheta^2 + C(r,\vartheta)d\varphi^2. \tag{8}$$

We assume that the Killing vector fields $\partial/\partial t$ and $\partial/\partial\varphi$ span timelike surfaces, which requires $AC + P^2 > 0$, $B > 0$, and $D > 0$. The metric coefficients may depend on arbitrarily many parameters; for example, for a Kerr black hole, they depend on a mass parameter and on a spin parameter. We emphasize that the question of whether or not the Hamilton–Jacobi equation for light rays separates is a purely local question, i.e., the range of the coordinates is quite irrelevant. In particular, it is irrelevant that t runs over all of $\mathbb{R}$ and φ runs over a circle. In this sense, the following consideration is not restricted to spherical coordinates; what matters is that we have two commuting Killing vector fields that span timelike surfaces. In the following, we stick with the $(t, \varphi, r, \vartheta)$ notation, but we emphasize that the coordinates could have any meaning. It turns out that standard spherical coordinates can be inappropriate for finding the Carter constant, while in another framework respecting symmetries of the given spacetime, it is quite straightforward. This aspect is further discussed in detail in Example 3.

Non-vanishing terms of the inverse metric to (8) are

$$\begin{aligned} g^{rr} &= \frac{1}{B(r,\vartheta)}, \quad g^{\vartheta\vartheta} = \frac{1}{D(r,\vartheta)}, \quad g^{\varphi\varphi} = \frac{A(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}, \\ g^{tt} &= \frac{-C(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}, \quad g^{t\varphi} = \frac{P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}. \end{aligned} \tag{9}$$

As proven in the Appendix, separation can hold only if the plasma frequency $\omega_{pl}^2(x^\alpha)$ is a function solely of the coordinates r and ϑ, i.e., $\omega_{pl}^2(r,\vartheta)$. In a spacetime described by metric (8), Hamiltonian (5), then, takes the form

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[\frac{p_r^2}{B(r,\vartheta)} + \frac{p_\vartheta^2}{D(r,\vartheta)} + \frac{p_\varphi^2 A(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} - \frac{p_t^2 C(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} + \frac{2p_t p_\varphi P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} + \omega_{pl}^2(r,\vartheta)\right]. \tag{10}$$

Because it obviously holds that $\frac{\partial\mathcal{H}}{\partial t} = 0$ and $\frac{\partial\mathcal{H}}{\partial\varphi} = 0$, we know that p_t and p_φ are constants of motion. If the spacetime is asymptotically flat, and for a light ray that reaches infinity, the component $-p_t$ is the frequency measured by a stationary observer at infinity (see, e.g., Ref. 11).

To stress its physical meaning, let us denote it as ω_0. The third constant of motion in this system is $\mathcal{H}(x^\alpha, p_\alpha) = 0$. However, when applying this formula, it is useful to rewrite the Hamiltonian $\mathcal{H}(x^\alpha, p_\alpha)$ as a function of x^α and $\frac{\partial S}{\partial x^\alpha}$ to get the Hamilton–Jacobi equation. We now require the action S to be separated as follows:

$$S(t,\varphi,r,\vartheta) = -\omega_0 t + p_\varphi \varphi + S_r(r) + S_\vartheta(\vartheta). \tag{11}$$

Then, it is possible to write the Hamilton–Jacobi equation

$$0 = \mathcal{H}\left(x^\alpha, \frac{\partial S}{\partial x^\beta}\right) \tag{12}$$

in the form

$$\begin{aligned} 0 = &\frac{1}{B(r,\vartheta)}\left(\frac{dS_r(r)}{dr}\right)^2 + \frac{1}{D(r,\vartheta)}\left(\frac{dS_\vartheta(\vartheta)}{d\vartheta}\right)^2 + \frac{p_\varphi^2 A(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)} \\ &- \frac{\omega_0^2 C(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)} - \frac{2\omega_0 p_\varphi P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)} + \omega_{pl}^2(r,\vartheta). \end{aligned} \tag{13}$$

Here, it is crucial that this condition has to hold for all p_φ and for all ω_0. The only freedom we have is to multiply this equation with a function $F(r,\vartheta)$,

$$0 = \frac{F(r,\vartheta)}{B(r,\vartheta)}\left(\frac{dS_r(r)}{dr}\right)^2 + \frac{F(r,\vartheta)}{D(r,\vartheta)}\left(\frac{dS_\vartheta(\vartheta)}{d\vartheta}\right)^2 + F(r,\vartheta)\omega_{pl}^2(r,\vartheta) + \frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)}\left[p_\varphi^2 A(r,\vartheta) - \omega_0^2 C(r,\vartheta) - 2\omega_0 p_\varphi P(r,\vartheta)\right], \tag{14}$$

where $F(r,\vartheta)$ is arbitrary except for the condition that it must not have any zeros. Separability holds if and only if the right-hand side of (14) is for all p_φ and all ω_0, the sum of a function of r alone and a function of ϑ alone. We first consider the terms that are independent of p_φ and ω_0. As for generic light rays, $dS_r(r)/dr$ and $dS_\vartheta(\vartheta)/d\vartheta$ are non-zero, we find that separability can hold only if

$$\frac{F(r,\vartheta)}{B(r,\vartheta)} \equiv \mathcal{F}(r) \quad \text{and} \quad \frac{F(r,\vartheta)}{D(r,\vartheta)} \equiv \mathcal{G}(\vartheta), \tag{15}$$

which implies

$$\frac{B(r,\vartheta)}{D(r,\vartheta)} = \frac{\mathcal{G}(\vartheta)}{\mathcal{F}(r)}. \tag{16}$$

This is the first important condition for separability we have found: If the quotient of $B(r,\vartheta)$ and $D(r,\vartheta)$ is not of this form, we know that separability cannot hold, neither for light rays in vacuum nor in any plasma density. If (16) does hold, we get our function $F(r,\vartheta)$ from the first or, equivalently, from the second equation in (15). As $\mathcal{G}(\vartheta)$ and $\mathcal{F}(r)$ are unique up to a common non-zero constant factor, $F(r,\vartheta)$ is fixed up to a non-zero constant factor. Here, it is crucial to note that $F(r,\vartheta)$ is determined by the metric coefficients, i.e., that it is independent of the plasma. As we assume that $B(r,\vartheta)$ and $D(r,\vartheta)$ are positive, it is also clear that $\mathcal{F}(r)$ and $\mathcal{G}(\vartheta)$ can be chosen both positive; hence, $F(r,\vartheta)$ is positive and it is guaranteed that it has, indeed, no zeros.

Plugging conditions (15) into (13) yields

$$0 = \mathcal{F}(r)\left(\frac{dS_r(r)}{dr}\right)^2 + \mathcal{G}(\vartheta)\left(\frac{dS_\vartheta(\vartheta)}{d\vartheta}\right)^2 + F(r,\vartheta)\omega_{pl}^2(r,\vartheta) + \frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)}\left(p_\varphi^2 A(r,\vartheta) - \omega_0^2 C(r,\vartheta) - 2\omega_0 p_\varphi P(r,\vartheta)\right). \tag{17}$$

Looking still at the terms that are independent of p_φ and ω_0, we now see that the separability condition can hold only if the plasma frequency $\omega_{pl}^2(r,\vartheta)$ is of the form

$$\omega_{pl}^2(r,\vartheta) = \frac{f_r(r) + f_\vartheta(\vartheta)}{F(r,\vartheta)}, \tag{18}$$

where $f_r(r)$ is an arbitrary function of r and $f_\vartheta(\vartheta)$ is an arbitrary function of ϑ. Here, we have to use the positive function $F(r,\vartheta)$ that was determined in the previous step, uniquely up to a constant factor, by the metric alone. We now consider the terms in (17) that are proportional to p_φ^2, ω_0^2, and $p_\varphi\omega_0$, respectively. We find that the separability condition requires that $A(r,\vartheta)$, $C(r,\vartheta)$, and $P(r,\vartheta)$ must meet the conditions

$$\frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)} X(r,\vartheta) = X_r(r) + X_\vartheta(\vartheta), \tag{19}$$

where X stands for A, C, or P.

If the separability conditions are satisfied, we can rewrite (14) as

$$\mathcal{F}(r)\left(\frac{dS_r(r)}{dr}\right)^2 + f_r(r) + p_\varphi^2 A_r(r) - \omega_0^2 C_r(r) - 2\omega_0 p_\varphi P_r(r)$$
$$= -\mathcal{G}(\vartheta)\left(\frac{dS_\vartheta(\vartheta)}{d\vartheta}\right)^2 - f_\vartheta(\vartheta) - p_\varphi^2 A_\vartheta(\vartheta) + \omega_0^2 C_\vartheta(\vartheta) + 2\omega_0 p_\varphi P_\vartheta(\vartheta) \equiv -\mathcal{K}. \quad (20)$$

From this equation, we read that $\mathcal{K}$ is independent of both r and ϑ, i.e., that it is a constant of motion. We refer to $\mathcal{K}$ as the (generalized) Carter constant.

Two observations are crucial: If on the given spacetime the Hamilton–Jacobi equation for vacuum light rays does not separate, then it does not separate for light rays in a plasma either, whatever the plasma density may be. In addition, if the Hamilton–Jacobi equation for vacuum light rays *does* separate, then there is an entire family of plasma densities, given by (18) with $F(r,\vartheta)$ determined by the metric but $f_r(r)$ and $f_\vartheta(\vartheta)$ arbitrary such that it separates for light rays in this plasma as well.

We end this section by showing that the Carter constant is associated with a conformal Killing tensor field not only in vacuum but also in a plasma. To that end, it is important to realize that (20) gives us the Carter constant only on the hypersurface $\mathcal{H} = 0$. To extend it to the entire cotangent bundle, we define

$$\mathcal{K}(x^\alpha, p_\beta) = K^{\mu\nu}(x^\alpha)p_\mu p_\nu - \frac{1}{2}f_r(r) + \frac{1}{2}f_\vartheta(\vartheta), \quad (21)$$

where

$$K^{\mu\nu}(x^\alpha)p_\mu p_\nu = \frac{1}{2}\mathcal{G}(\vartheta)p_\vartheta^2 - \frac{1}{2}\mathcal{F}(r)p_r^2 + \frac{1}{2}(A_\vartheta(\vartheta) - A_r(r))p_\varphi^2$$
$$- \frac{1}{2}(C_\vartheta(\vartheta) - C_r(r))p_t^2 + (P_\vartheta(\vartheta) - P_r(r))p_\varphi p_t. \quad (22)$$

This defines a symmetric second-rank tensor field $K^{\mu\nu}(x^\alpha)$ that depends on the metric coefficients but not on the plasma density. By a straightforward calculation, this function $\mathcal{K}(x^\alpha, p_\beta)$ can be equivalently rewritten in the following two ways:

$$\mathcal{K}(x^\alpha, p_\beta) = -\mathcal{F}(r)p_r^2 - A_r(r)p_\varphi^2 + C_r(r)p_t^2 - 2P_r(r)p_\varphi p_t - f_r(r) + F(r,\vartheta)\mathcal{H}(x^\alpha, p_\beta), \quad (23)$$

$$\mathcal{K}(x^\alpha, p_\beta) = \mathcal{G}(\vartheta)p_\vartheta^2 + A_\vartheta(\vartheta)p_\varphi^2 - C_\vartheta(\vartheta)p_t^2 + 2P_\vartheta(\vartheta)p_\varphi p_t + f_\vartheta(\vartheta) - F(r,\vartheta)\mathcal{H}(x^\alpha, p_\beta). \quad (24)$$

Restricting these two expressions to the hypersurface $\mathcal{H} = 0$ shows that the function $\mathcal{K}$ defined in (21) gives us, indeed, the Carter constant, as it was introduced in (20). The fact that $\mathcal{K}$ is a constant of motion means that the Poisson bracket $\{\mathcal{K}, \mathcal{H}\}$ vanishes on the hypersurface $\mathcal{H} = 0$ for every choice of $f_r(r)$ and $f_\vartheta(\vartheta)$. If we choose $f_r(r) = 0$ and $f_\vartheta(\vartheta) = 0$, we find that the Poisson bracket $\{K^{\mu\nu}(x^\alpha)p_\mu p_\nu, g^{\rho\sigma}(x^\beta)p_\rho p_\sigma\}$ vanishes on the hypersurface $g^{\rho\sigma}(x^\beta)p_\rho p_\sigma = 0$. This demonstrates that $K^{\mu\nu}(x^\alpha)$ is a conformal Killing tensor field of the spacetime metric.

IV. PHOTON REGION IN A GENERAL AXIALLY SYMMETRIC SPACETIME WITH PLASMA

For simplicity, let us further generally write A_r instead of $A_r(r)$, etc., keeping in mind that these functions depend on the argument that they carry as an index. We can now apply the relations $\frac{dS_r}{dr} = p_r$ and $\frac{dS_\vartheta}{d\vartheta} = p_\vartheta$. Hence, one gets

$$\mathcal{F}(r)p_r^2 = -\mathcal{K} - f_r - p_\varphi^2 A_r + \omega_0^2 C_r + 2\omega_0 p_\varphi P_r, \quad (25)$$

$$\mathcal{G}(\vartheta)p_\vartheta^2 = \mathcal{K} - f_\vartheta - p_\varphi^2 A_\vartheta + \omega_0^2 C_\vartheta + 2\omega_0 p_\varphi P_\vartheta. \quad (26)$$

The photon region is the set of all events through which there is a light ray that is completely contained in a hypersurface r = constant. For the sake of brevity, we will call such light rays "spherical" in the following, even though r is not necessarily a radius coordinate. Along each spherical light ray, the equations $\dot{r} = \ddot{r} = 0$ have to hold, where the overdot denotes the derivative with respect to the same parameter as in Hamilton's equations. The equations of motion

$$\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r} = \frac{p_r}{B(r,\vartheta)}, \quad \dot{\vartheta} = \frac{\partial\mathcal{H}}{\partial p_\vartheta} = \frac{p_\vartheta}{D(r,\vartheta)} \quad (27)$$

give

$$B^2(r,\vartheta)\,\mathcal{F}(r)\dot{r}^2 = -\mathcal{K} - f_r - p_\varphi^2 A_r + \omega_0^2 C_r + 2\omega_0 p_\varphi P_r \quad (28)$$

and

$$D^2(r,\vartheta)\mathcal{G}(\vartheta)\dot{\vartheta}^2 = \mathcal{K} - f_\vartheta - p_\varphi^2 A_\vartheta + \omega_0^2 C_\vartheta + 2\omega_0 p_\varphi P_\vartheta. \tag{29}$$

Hence, along every spherical light ray, the following two equations have to hold:

$$0 = -\mathcal{K} - f_r - p_\varphi^2 A_r + \omega_0^2 C_r + 2\omega_0 p_\varphi P_r \equiv R(r), \tag{30}$$

$$0 = -f_r' - p_\varphi^2 A_r' + \omega_0^2 C_r' + 2\omega_0 p_\varphi P_r' \equiv R'(r). \tag{31}$$

Here, the derivative with respect to r is denoted as $'$. This gives us the constants of motion $\mathcal{K}$ and p_φ for every spherical light ray, i.e.,

$$p_\varphi = \frac{\omega_0 P_r'}{A_r'}\left(1 \pm \sqrt{1 - \frac{A_r'}{\omega_0^2 P_r'^2}\left(f_r' - \omega_0^2 C_r'\right)}\right), \tag{32}$$

$$\mathcal{K} = \frac{A_r}{A_r'}\left(f_r' - \omega_0^2 C_r'\right) + \omega_0^2 C_r + 2\frac{\omega_0^2 P_r'}{A_r'}\left(P_r - \frac{A_r P_r'}{A_r'}\right)\left(1 \pm \sqrt{1 - \frac{A_r'}{\omega_0^2 P_r'^2}\left(f_r' - \omega_0^2 C_r'\right)}\right) - f_r. \tag{33}$$

The other set of the equations of motion takes the form

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial p_\varphi} = \frac{p_\varphi A(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} - \frac{\omega_0 P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}, \tag{34}$$

$$\dot{t} = \frac{\partial \mathcal{H}}{\partial p_t} = \frac{\omega_0 C(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} + \frac{p_\varphi P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}. \tag{35}$$

As the left-hand side of (29) cannot be negative, the inequality

$$\mathcal{K} - f_\vartheta \geq p_\varphi^2 A_\vartheta - \omega_0^2 C_\vartheta - 2\omega_0 p_\varphi P_\vartheta \tag{36}$$

has to hold. Inserting (32) and (33) leads to

$$\begin{aligned}
&\frac{A_r}{A_r'}\left(\frac{f_r'}{\omega_0^2} - C_r'\right) + C_r + 2\frac{P_r'}{A_r'}\left(P_r - \frac{A_r P_r'}{A_r'}\right)\left(1 \pm \sqrt{1 - \frac{A_r'}{P_r'^2}\left(\frac{f_r'}{\omega_0^2} - C_r'\right)}\right) - \frac{f_r + f_\vartheta}{\omega_0^2} \\
&\geq -\frac{A_\vartheta}{A_r'}\left(\frac{f_r'}{\omega_0^2} - C_r'\right) - C_\vartheta + 2\frac{P_r'}{A_r'}\left(\frac{A_\vartheta P_r'}{A_r'} - P_\vartheta\right)\left(1 \pm \sqrt{1 - \frac{A_r'}{P_r'^2}\left(\frac{f_r'}{\omega_0^2} - C_r'\right)}\right).
\end{aligned} \tag{37}$$

At every point (r,ϑ) where this condition holds (either for the plus or for the minus sign before the square root), there is a spherical light ray, i.e., the inequality (37) determines the photon region.

Spherical light rays may be stable or unstable with respect to perturbations in the r-direction. The unstable ones are particularly important because they can serve as limit curves for light rays that approach the photon region from far away. A spherical light ray is unstable if

$$0 < R''(r) = -f_r'' - p_\varphi^2 A_r'' + \omega_0^2 C_r'' + 2\omega_0 p_\varphi P_r''. \tag{38}$$

V. BLACK HOLE SHADOW IN AN AXIALLY SYMMETRIC AND STATIONARY SPACETIME WITH PLASMA

We will now demonstrate that the separability of the Hamilton–Jacobi equation for light rays allows us to derive an analytical formula for the boundary curve of the shadow. We will do this for the case that our spacetime describes a black hole, but we mention that the same methodology also works for some other compact objects, e.g., for wormholes, see Example 4.

We want to calculate the shadow for an observer located at coordinates (r_O, ϑ_O) outside of the black hole horizon. To that end, we introduce an orthonormal tetrad

$$e_0 = Y_1\partial_t + Y_2\partial_\varphi|_{(r_O,\vartheta_O)}, \tag{39}$$

$$e_1 = \frac{1}{\sqrt{D(r,\vartheta)}}\partial_\vartheta\bigg|_{(r_O,\vartheta_O)}, \tag{40}$$

$$e_2 = Y_3\partial_t + Y_4\partial_\varphi|_{(r_O,\vartheta_O)}, \tag{41}$$

$$e_3 = -\frac{1}{\sqrt{B(r,\vartheta)}}\partial_r\bigg|_{(r_O,\vartheta_O)}. \tag{42}$$

The coefficients Y_1, Y_2, Y_3, and Y_4 are chosen so that the orthonormality conditions $g(e_0,e_0) = -1$, $g(e_2,e_2) = 1$, and $g(e_0,e_2) = 0$ hold. Their concrete form can be derived for any given metric. We assume that e_0 is the four-velocity of the observer. The orthonormality conditions for our general form of metric (8) read

$$-A(r,\vartheta)Y_1^2 + 2P(r,\vartheta)Y_1Y_2 + C(r,\vartheta)Y_2^2 = -1, \tag{43}$$

$$-A(r,\vartheta)Y_3^2 + 2P(r,\vartheta)Y_3Y_4 + C(r,\vartheta)Y_4^2 = 1, \tag{44}$$

$$-A(r,\vartheta)Y_1Y_3 + P(r,\vartheta)(Y_1Y_4 + Y_2Y_3) + C(r,\vartheta)Y_2Y_4 = 0. \tag{45}$$

One can see that there are actually only three equations (43)–(45) for four unknowns, which means that one of the components can be chosen arbitrarily. This reflects the fact that we can choose for the four-velocity any normalized timelike vector in the two-space spanned by ∂_t and ∂_φ.

A tangent vector to a light ray $\lambda(s) = (r(s), \vartheta(s), \varphi(s), t(s))$ can be written as

$$\dot\lambda = \dot r\partial_r + \dot\vartheta\partial_\vartheta + \dot\varphi\partial_\varphi + \dot t\partial_t. \tag{46}$$

Here, the overdot denotes the derivative with respect to s, which is the parameter that is used in Hamilton's equations. At the observation event, the same tangent vector can be written as

$$\dot\lambda = -\alpha e_0 + \beta(\sin\theta\cos\psi e_1 + \sin\theta\sin\psi e_2 + \cos\theta e_3)|_{(r_O,\vartheta_O)}. \tag{47}$$

Factors α, β are positive. Coordinates θ and ψ denote the celestial coordinates of the observer—the colatitude and the azimuthal angle, respectively. Due to the form of Hamiltonian (1), the light rays are parameterized as $g(\dot\lambda,\dot\lambda) = -\omega_{pl}^2$, and, approve thus,

$$\alpha^2 - \beta^2 = \omega_{pl}^2|_{(r_O,\vartheta_O)}. \tag{48}$$

Furthermore, α can be derived as

$$\alpha = g(\dot\lambda, e_0) = g(\dot\lambda, Y_1\partial_t + Y_2\partial_\varphi) = Y_1(\dot t g_{tt} + \dot\varphi g_{t\varphi}) + Y_2(\dot t g_{t\varphi} + \dot\varphi g_{\varphi\varphi}) = Y_1(-\omega_0) + Y_2 p_\varphi, \tag{49}$$

and then,

$$\beta = \sqrt{(-Y_1\omega_0 + Y_2 p_\varphi)^2 - \omega_{pl}^2}\,. \tag{50}$$

Here, all expressions have to be evaluated at (r_O, ϑ_O). Note that our assumption $\alpha > 0$ means that the light ray goes from the observer position into the past; hence, $\omega_0 = -p_t$ is negative.

A general relation between celestial coordinates θ, ψ and constants of motion p_φ, $\mathcal{K}$ can be found, comparing factors of ∂_r and ∂_φ in (46) and (47). This yields

$$\dot r = -\beta\cos\theta\frac{1}{\sqrt{B(r,\vartheta)}}, \tag{51}$$

$$\dot\varphi = -\alpha Y_2 + \beta\sin\theta\sin\psi Y_4. \tag{52}$$

It is now desirable to plug into these general formulas the expressions for dotted variables (28), (29), (34) and (35) and factors α, β (49), and (50) derived above. One, then, gets

$$\sin\theta = \left(1 + \frac{\mathcal{K} + f_r + p_\varphi^2 A_r - \omega_0^2 C_r - 2\omega_0 p_\varphi P_r}{F(r,\vartheta)\big((-Y_1\omega_0 + Y_2 p_\varphi)^2 - \omega_{pl}^2\big)}\right)^{1/2}\Bigg|_{(r_O,\vartheta_O)}, \tag{53}$$

$$\sin\psi = \frac{(A_r + A_\vartheta + F(r,\vartheta)Y_2^2)p_\varphi - (P_r + P_\vartheta + F(r,\vartheta)Y_1Y_2)\omega_0}{F^{1/2}(r,\vartheta)Y_4\big[F(r,\vartheta)(-Y_1\omega_0 + Y_2 p_\varphi)^2 + \mathcal{K} - f_\vartheta + p_\varphi^2 A_r - \omega_0^2 C_r - 2\omega_0 p_\varphi P_r\big]^{1/2}}\Bigg|_{(r_O,\vartheta_O)}. \tag{54}$$

For discussing the shadow, we have to consider all light rays that issue from the observer position into the past. If there is only one photon region outside of the horizon and if it consists of *unstable* spherical light rays, the boundary of the shadow is determined by those light rays that asymptotically approach one of these spherical light rays. As the former must have the same constants of motion p_φ and $\mathcal{K}$ as the latter, we can insert (32) and (33) into (54) to get θ and ψ as functions of the radius coordinate $r = r_p$ on which p_φ and $\mathcal{K}$ depend. This gives us the boundary of the shadow on the observer's sky as a curve parameterized by r_p.

Minimum and maximum values of r_p can be obtained from the condition $\sin\psi = \pm 1$. This is achieved when

$$\begin{aligned}&(A_r + A_\vartheta + F(r,\vartheta)Y_2^2)p_\varphi - (P_r + P_\vartheta + F(r,\vartheta)Y_1Y_2)\omega_0\big|_{(r_O,\vartheta_O)}\\ &\quad= \pm F(r,\vartheta)Y_4\big[F(r,\vartheta)\big((-Y_1\omega_0 + Y_2p_\varphi)^2 - \omega_{pl}^2\big) + \mathcal{K} + f_r + p_\varphi^2 A_r - \omega_0^2 C_r - 2\omega_0 p_\varphi P_r\big]^{1/2}\Big|_{(r_O,\vartheta_O)}.\end{aligned} \tag{55}$$

In a plasma, the shadow depends on ω_0. We have already mentioned that in the case of asymptotic flatness and for a light ray that reaches infinity, ω_0 is the frequency measured by a stationary observer at infinity. As we parameterize our light rays in the past-oriented direction, ω_0 is negative and the positive frequency ω_{obs} measured by our observer at (r_O, ϑ_O) whose four-velocity is determined by the tetrad coefficients Y_1 and Y_2 is

$$\omega_{\text{obs}} = Y_1(-\omega_0) + Y_2 p_\varphi. \tag{56}$$

If $Y_2 = 0$, all light rays with the same ω_0 give the same ω_{obs}; this is not the case if $Y_2 \neq 0$.

VI. EXAMPLE 1: THE HAIRY KERR METRIC

To demonstrate how our general formula works, let us now apply it to the hairy Kerr metric. This is a generalized case to the Kerr metric, and the obtained expressions can be, thus, easily compared with the results derived in Ref. 11. The metric describing a generalized Kerr black hole in the Boyer–Lindquist coordinates reads (e.g., Ref. 22)

$$ds^2 = -\left(1 - \frac{2rM(r)}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 - \frac{4arM(r)}{\rho^2}\sin^2\vartheta\, dt\, d\varphi + \left(r^2 + a^2 + \frac{2a^2 rM(r)}{\rho^2}\sin^2\vartheta\right)\sin^2\vartheta\, d\varphi^2, \tag{57}$$

where $\Delta = r^2 + a^2 - 2M(r)r$, $\rho^2 = r^2 + a^2\cos^2\vartheta$. The Kerr metric can be obtained as a special case when $M(r) = m = \text{const}$.

In this case, the relevant terms become

$$B(r,\vartheta) = \frac{\rho^2}{\Delta}, \quad D(r,\vartheta) = \rho^2, \tag{58}$$

$$F(r,\vartheta) = \rho^2, \quad \mathcal{F}(r) = \Delta, \quad \mathcal{G}(\vartheta) = 1, \tag{59}$$

$$A_r = -\frac{a^2}{\Delta}, \quad A_\vartheta = \sin^{-2}\vartheta, \tag{60}$$

$$C_r = \frac{(r^2 + a^2)^2}{\Delta}, \quad C_\vartheta = -a^2\sin^2\vartheta, \tag{61}$$

$$P_r = -\frac{a(r^2 + a^2)}{\Delta}, \quad P_\vartheta = a. \tag{62}$$

These are formally the same expressions as those obtained for the Kerr metric, but Δ differs, containing a general function $M(r)$.

Assuming that $\omega_{pl}^2(r,\vartheta) = (f_r + f_\vartheta)/F(r,\vartheta)$ and applying the formulas introduced above lead to

$$\mathcal{F}(r)\left(\frac{dS_r}{dr}\right)^2 + f_r + p_\varphi^2 A_r - \omega_0^2 C_r - 2\omega_0 p_\varphi P_r = \Delta\left(\frac{dS_r}{dr}\right)^2 + f_r - \frac{1}{\Delta}(ap_\varphi + (r^2+a^2)\omega_0)^2$$

and

$$-\mathcal{G}(\vartheta)\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 - f_\vartheta - p_\varphi^2 A_\vartheta + \omega_0^2 C_\vartheta + 2\omega_0 p_\varphi P_\vartheta = -\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 - f_\vartheta - \left(\frac{p_\varphi}{\sin\vartheta} + a\sin\vartheta\omega_0\right)^2.$$

The obtained expressions formally agree with relation (27) introduced in Ref. 11.

A difference from the Kerr metric occurs in the formula for the photon region. From general expression (37), one gets

$$\left[\frac{r^2\Delta}{(r-M-rM')^2}\left(1 \pm \sqrt{1 - \frac{f_r'(r-M-rM')}{2r^2\omega_0^2}}\right)^2 - \frac{f_r + f_\vartheta}{\omega_0^2}\right]a^2\sin^2\vartheta \tag{63}$$

$$\geq \left[\frac{1}{r-M-rM'}\left(M(a^2-r^2) + rM'(r^2+a^2) \pm r\Delta\sqrt{1 - \frac{f_r'(r-M-rM')}{2r^2\omega_0^2}}\right) + a^2\sin^2\vartheta\right]^2. \tag{64}$$

The Kerr case derived in Ref. 11 is, indeed, obtained when $M' = 0$.

Let us note that the expression for the hairy Kerr black hole shadow would formally be the same as introduced in Ref. 11 for the Kerr black hole though Δ can be more general as introduced above. For this reason, the formula for the hairy Kerr black hole shadow is not explicitly given here. Although there is a formal correspondence of the obtained expressions, the physical situation described by these two metrics can be significantly different. A natural assumption is that in a physically relevant case, the function $M(r)$ suitably decays with increasing r, and the matter stress tensor arises due to a non-constant $M(r)$ satisfying energy conditions.

VII. EXAMPLE 2: THE HARTLE–THORNE METRIC

In the Appendix of Ref. 23, a form of the Hartle–Thorne metric for the external gravitational field of a rotating star, accurate to the second order in the angular velocity, can be found, namely,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)\left\{1 + 2P_2(\cos\vartheta)\left[\frac{J^2}{Mr^3}\left(1 + \frac{M}{r}\right) + \frac{5}{8}\frac{Q - J^2/M}{M^3}Q_2^2\left(\frac{r}{M} - 1\right)\right]\right\}dt^2 \\ &+ \left(1 - \frac{2M}{r} + \frac{2J^2}{r^4}\right)^{-1}\left\{1 - 2P_2(\cos\vartheta)\left[\frac{J^2}{Mr^3}\left(1 - \frac{5M}{r}\right) + \frac{5}{8}\frac{Q - J^2/M}{M^3}Q_2^2\left(\frac{r}{M} - 1\right)\right]\right\}dr^2 \\ &+ r^2\left\{1 + 2P_2(\cos\vartheta)\left[-\frac{J^2}{Mr^3}\left(1 + \frac{2M}{r}\right) + \frac{5}{8}\frac{Q - J^2/M}{M^3}\left\langle\frac{2M}{\sqrt{r(r-2M)}}Q_2^1\left(\frac{r}{M} - 1\right)\right.\right.\right. \\ &\left.\left.\left. - Q_2^2\left(\frac{r}{M} - 1\right)\right\rangle\right]\right\} \times \left\{d\vartheta^2 + \sin^2\vartheta\left(d\varphi - \frac{2J}{r^3}dt\right)^2\right\}. \end{aligned} \tag{65}$$

Here, M, J, and Q are constants. M determines the mass, J stands for the total angular momentum, and Q is the quadrupole moment of the star. Function $P_2(\cos\vartheta)$ denotes the Legendre polynomial of order 2 of the argument $\cos\vartheta$, and $Q_n^m\left(\frac{r}{M} - 1\right)$ denotes the associated Legendre functions of the second kind of the argument $\frac{r}{M} - 1$.

It was observed already by Glampedakis and Babak[24] that for the Hartle–Thorne metric in the chosen coordinates, the Hamilton–Jacobi equation for geodesics separates only in the Schwarzschild case where $J = 0$ and $Q = 0$; in all other cases, including the Kerr case $Q = J^2/M \neq 0$, separability fails. As this is true, in particular, for *lightlike* geodesics, it is clear that separability cannot hold in the *chosen coordinates* for light rays in a plasma, whatever the plasma density may be. In this section, we will rederive this result with the help of our general equations.

For a better transparency, let us introduce

$$A_1 = 1 - \frac{2M}{r} + \frac{2J^2}{r^4}, \qquad j = \frac{J^2}{Mr^3},$$
$$K = \frac{5}{8}\frac{Q - J^2/M}{M^3}, \qquad j_1 = \frac{2J}{r^3}, \tag{66}$$

and let us write Q_n^m instead of $Q_n^m\left(\frac{r}{M} - 1\right)$. Let us further denote

$$\mathcal{M}_A = 1 + 2P_2(\cos\vartheta)\left[j\left(1 + \frac{M}{r}\right) + KQ_2^2\right], \tag{67}$$

$$\mathcal{M}_B = 1 - 2P_2(\cos\vartheta)\left[j\left(1 - \frac{5M}{r}\right) + KQ_2^2\right], \tag{68}$$

$$\mathcal{M}_\varphi = 1 + 2P_2(\cos\vartheta)\left[-j\left(1 + \frac{2M}{r}\right) + K\left(\frac{2M}{\sqrt{r(r-2M)}}Q_2^1 - Q_2^2\right)\right]. \tag{69}$$

Then, the metric terms introduced in a general form in (8) are

$$A(r,\vartheta) = A_1\mathcal{M}_A - j_1^2 r^2 \sin^2\vartheta\mathcal{M}_\varphi, \qquad B(r,\vartheta) = A_1^{-1}\mathcal{M}_B,$$
$$C(r,\vartheta) = r^2\sin^2\vartheta\mathcal{M}_\varphi, \qquad D(r,\vartheta) = r^2\mathcal{M}_\varphi,$$
$$P(r,\vartheta) = -j_1 r^2\sin^2\vartheta\mathcal{M}_\varphi. \tag{70}$$

It can be seen that the ratio of terms $B(r,\vartheta)$ and $D(r,\vartheta)$ gives

$$\frac{B(r,\vartheta)}{D(r,\vartheta)} = \frac{\mathcal{M}_B}{A_1 r^2\mathcal{M}_\varphi}. \tag{71}$$

Separability requires the right-hand side to be a function of ϑ alone divided by a function of r alone. From the way in which $\mathcal{M}_B$ and $\mathcal{M}_\varphi$ depend on r and ϑ, we read that this is true only if $J = 0$ and $Q = 0$, i.e., in the Schwarzschild case. In the Kerr case $Q = J^2/M(\neq 0)$, it is possible to change to Boyer–Lindquist coordinates in which separability is well-known to hold. The explicit form of this coordinate transformation can be found in the above-mentioned paper by Glampedakis and Babak.[24]

VIII. EXAMPLE 3: THE MELVIN UNIVERSE

In this section, we discuss the specific example of the Melvin universe, which is a solution to the Einstein–Maxwell equations with a uniform magnetic field. It was found by Bonnor[25] and, then, independently rediscovered by Melvin.[26] For a detailed discussion of the geodesics in this spacetime, we refer to Ref. 27.

The metric of the Melvin universe can be written as

$$ds^2 = \tilde{a}^2\left[(1+\rho^2)^2(-dt^2 + d\rho^2 + dz^2) + \rho^2(1+\rho^2)^{-2}d\varphi^2\right], \tag{72}$$

where $\tilde{a}$ is a positive constant that plays the role of an overall scaling factor, t is a time coordinate, and ρ, z, and φ are the usual cylindrical polar coordinates. We denote $\Lambda(\rho) \equiv 1 + \rho^2$.

A. Separation in the spherical coordinates

To obtain a Carter constant, the separated terms have to be found. Because our general formulas defined above are written in the spherical coordinates $(t, r, \vartheta, \varphi)$, we transform the cylindrical coordinates (t, ρ, z, φ) used in (72) by putting

$$t = t, \tag{73}$$
$$\rho = r\sin\vartheta, \tag{74}$$
$$z = r\cos\vartheta, \tag{75}$$
$$\varphi = \varphi. \tag{76}$$

In the spherical coordinates, the Melvin metric can be, hence, rewritten in the form

$$ds^2 = \tilde{a}^2\left[\Lambda^2(r,\vartheta)(-dt^2 + dr^2 + r^2 d\vartheta^2) + r^2\sin^2\vartheta\Lambda^{-2}(r,\vartheta)d\varphi^2\right], \tag{77}$$

where $\Lambda(r,\vartheta) = 1 + r^2\sin^2\vartheta$. If we had $\Lambda(r,\vartheta) = 1$, this would be Minkowski space.

The individual metric terms, when using the notation of Sec. III, read

$$A(r,\vartheta) = \tilde{a}^2\Lambda^2(r,\vartheta), \quad B(r,\vartheta) = \tilde{a}^2\Lambda^2(r,\vartheta), \tag{78}$$

$$C(r,\vartheta) = \tilde{a}^2 r^2\sin^2\vartheta\,\Lambda^{-2}(r,\vartheta), \quad D(r,\vartheta) = \tilde{a}^2 r^2\Lambda^2(r,\vartheta), \quad P(r,\vartheta) = 0. \tag{79}$$

This implies

$$\frac{B(r,\vartheta)}{D(r,\vartheta)} = r^{-2} \quad \Rightarrow \quad F(r,\vartheta) = \tilde{a}^2 r^2\Lambda^2(r,\vartheta), \quad \mathcal{F}(r) = r^2, \quad \mathcal{G}(\vartheta) = 1 \tag{80}$$

and

$$A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta) = \tilde{a}^4 r^2\sin^2\vartheta, \tag{81}$$

$$\frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}A(r,\vartheta) = \Lambda^4(r,\vartheta)\sin^{-2}\vartheta = \sin^6\vartheta(r^2 + \sin^{-2}\vartheta)^4 \neq A_r + A_\vartheta, \tag{82}$$

$$\frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}C(r,\vartheta) = r^2 = C_r, \tag{83}$$

$$\frac{F(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)}P(r,\vartheta) = 0. \tag{84}$$

We see that the term $A(r,\vartheta)$ is not in a separated form, and the Carter constant cannot be obtained.

B. Separation in the cylindrical coordinates

If, however, the cylindrical-type coordinates ρ, z in which the Melvin metric was originally given [see (72)] are used instead, the separation can be performed. Following our original notation of Sec. III, let us introduce

$$A(\rho,z) = \tilde{a}^2\Lambda^2(\rho), \quad B(\rho,z) = \tilde{a}^2\Lambda^2(\rho), \tag{85}$$

$$C(\rho,z) = \tilde{a}^2\rho^2\Lambda^{-2}(\rho), \quad D(\rho,z) = \tilde{a}^2\Lambda^2(\rho), \quad P(\rho,z) = 0. \tag{86}$$

All these terms are solely functions of ρ. Proceeding like before, we find

$$\frac{B(\rho,z)}{D(\rho,z)} = 1 \quad \Rightarrow \quad F(\rho,z) = \tilde{a}^2\Lambda^2(\rho), \quad \mathcal{F}(\rho) = 1, \quad \mathcal{G}(z) = 1 \tag{87}$$

and

$$A(\rho,z)C(\rho,z) + P^2(\rho,z) = \tilde{a}^4\rho^2, \tag{88}$$

$$A_\rho = \rho^{-2}\Lambda^4(\rho), \quad A_z = 0, \tag{89}$$

$$C_\rho = 1, \quad C_z = 0, \tag{90}$$

$$P_\rho = 0, \quad P_z = 0. \tag{91}$$

Note that these results are not unique: We can always add a constant to X_ρ and subtract the same constant from X_z, where X stands for A, C, or P.

Applying (20) leads to

$$\left(\frac{dS_\rho}{d\rho}\right)^2 + f_\rho(\rho) + p_\varphi^2\rho^{-2}\Lambda^4(\rho) - \omega_0^2 = -\left(\frac{dS_z}{dz}\right)^2 - f_z(z) \equiv -\mathcal{K}, \tag{92}$$

which corresponds to the results introduced in Ref. 27.

Thus, the photon region is given by the relation

$$\frac{\rho\Lambda(\rho)f_\rho'(\rho)}{2(3\rho^2-1)}+\omega_0^2-f_\rho(\rho)-f_z(z)\geq 0. \tag{93}$$

If there is no plasma, the entire spacetime is the photon region. Note that in the Melvin universe, the photon region is filled with "cylindrical light rays," rather than with spherical light rays. As these cylindrical light rays are not trapped within a spatially compact region, there is no meaningful notion of a "shadow" in the Melvin spacetime.

We mention that there are exact solutions to the Einstein–Maxwell equations that describe a Schwarzschild or a Kerr black hole immersed in a Melvin universe; see the work of Ernst[28] for the Schwarzschild case and the work of Ernst and Wild[29] for the Kerr case. The shadow of such a black hole was recently discussed in Refs. 3 and 4, respectively. However, in these cases, the equations for the light rays are not separable.

IX. EXAMPLE 4: THE TEO WORMHOLE METRIC

To show another example where our general formulas can be used, let us turn to a stationary and axisymmetric metric describing a rotating traversable wormhole obtained by Teo.[30] Our general results will give us, for light rays in a plasma on a such spacetime, the necessary and sufficient conditions for the separability of the Hamilton–Jacobi equation. This will allow us to analytically determine the photon region and the shadow. We note that, without a plasma and for a subclass of Teo metrics, the shadow was already calculated by Nedkova *et al.*; see Ref. 31 and cf. Refs. 32 and 33.

This system is described by the metric

$$ds^2=-N^2dt^2+\left(1-\frac{b}{r}\right)^{-1}dr^2+r^2K^2d\vartheta^2+r^2K^2\sin^2\vartheta(d\varphi-\omega dt)^2, \tag{94}$$

where N, K, b, and ω are functions of r and ϑ. The given metric is supposed to be asymptotically flat. To meet this assumption, the introduced functions must at $r\to\infty$ obey

$$\begin{aligned} N=1-\frac{M}{r}+\mathcal{O}\left(\frac{1}{r^2}\right),\quad K=1+\mathcal{O}\left(\frac{1}{r}\right),\\ \frac{b}{r}=\mathcal{O}\left(\frac{1}{r}\right),\quad \omega=\frac{2J}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right). \end{aligned} \tag{95}$$

The chosen coordinates are supposed to cover the spacetime region between the "neck," which is defined by the equation $b(r,\vartheta)=r$, and infinity. On this domain, the functions N, b, and K have to be strictly positive. Moreover, it is assumed that $\partial b(r,\vartheta)/\partial\vartheta\to 0$ and $b(r,\vartheta) > r\partial b(r,\vartheta)/\partial r$ if the neck is approached. Then, one can glue two copies of the spacetime together at the neck to get a wormhole that connects two asymptotically flat ends.

According to our notation introduced in (8), it can be easily seen that

$$\begin{aligned} A(r,\vartheta)=N^2-r^2K^2\omega^2\sin^2\vartheta,\quad B(r,\vartheta)=\left(1-\frac{b}{r}\right)^{-1},\\ C(r,\vartheta)=r^2K^2\sin^2\vartheta,\quad D(r,\vartheta)=r^2K^2,\quad P(r,\vartheta)=-\omega r^2K^2\sin^2\vartheta. \end{aligned} \tag{96}$$

This leads to [by using (15) and (19)]

$$F(r,\vartheta)=r^2K^2,\quad \mathcal{F}(r)=r^2K^2\left(1-\frac{b}{r}\right),\quad \mathcal{G}(\vartheta)=1, \tag{97}$$

$$A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)=N^2r^2K^2\sin^2\vartheta, \tag{98}$$

$$A_r=-\frac{r^2K^2}{N^2}\omega^2,\quad A_\vartheta=\sin^{-2}\vartheta, \tag{99}$$

$$C_r=\frac{r^2K^2}{N^2},\quad C_\vartheta=0, \tag{100}$$

$$P_r=-\frac{r^2K^2}{N^2}\omega,\quad P_\vartheta=0. \tag{101}$$

For this general form, we find

$$A_r' = -2\frac{rK^2}{N^2}\omega^2 - r^2\omega^2\left(\frac{K^2}{N^2}\right)' - \frac{r^2K^2}{N^2}(\omega^2)', \tag{102}$$

$$C_r' = 2\frac{rK^2}{N^2} + r^2\left(\frac{K^2}{N^2}\right)', \tag{103}$$

$$P_r' = -2\frac{rK^2}{N^2}\omega - r^2\omega\left(\frac{K^2}{N^2}\right)' - \frac{r^2K^2}{N^2}\omega'. \tag{104}$$

These relations show that to be able to perform the separability of variables, each of the terms

$$\frac{K}{N}, \quad K^2\left(1-\frac{b}{r}\right), \quad \omega$$

must be a function of r only.

When the separability condition holds, a new radial coordinate ℓ can be introduced, obeying

$$d\ell = \pm\left(K\sqrt{1-\frac{b}{r}}\right)^{-1} dr. \tag{105}$$

This coordinate describes the radial length in a new metric obtained by a conformal transformation $g_{\mu\nu} \mapsto K^{-2}g_{\mu\nu}$. While the original Teo coordinates (r, θ, ϕ, t) describe only one-half of the spacetime (from an asymptote up to the neck), the new coordinates (ℓ, θ, ϕ, t) cover the whole spacetime because ℓ runs from $-\infty$ to ∞ and, thus, from one asymptotic end to the other. In the following, however, we will use the original Teo coordinates.

Plugging the separated terms into (20) gives the equation for the Carter constant in the form

$$r^2K^2\left(1-\frac{b}{r}\right)\left(\frac{dS_r}{dr}\right)^2 + f_r(r) - p_\varphi^2\frac{r^2K^2}{N^2}\omega^2 - \omega_0^2\frac{r^2K^2}{N^2} + 2\omega_0 p_\varphi\frac{r^2K^2}{N^2}\omega$$
$$= -\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 - f_\vartheta(\vartheta) - p_\varphi^2\sin^{-2}\vartheta \equiv -\mathcal{K}. \tag{106}$$

The equations for the derivatives of S_r and S_ϑ can, then, be rewritten as follows:

$$N^2\left(1-\frac{b}{r}\right)\left(\frac{dS_r}{dr}\right)^2 = (\omega_0 - \omega p_\varphi)^2 - \frac{N^2}{r^2K^2}(\mathcal{K} + f_r(r)), \tag{107}$$

$$\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 = \mathcal{K} - f_\vartheta(\vartheta) - p_\varphi^2\sin^{-2}\vartheta. \tag{108}$$

For the special case that the plasma density is zero and that each of the metric coefficients N, b, K, and ω separately depends on r only, these equations have already been given by Nedkova *et al.*[31]

The general expressions (32) and (33) for $\mathcal{K}$ and p_φ in this case give

$$\omega p_\varphi = \frac{\omega_0\left(\mathcal{Q}' + \mathcal{Q}\frac{\omega'}{\omega}\right) \pm \omega_0\sqrt{\mathcal{Q}^2\left(\frac{\omega'}{\omega}\right)^2 + \frac{f_r'}{\omega_0^2}\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)}}{\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}}, \tag{109}$$

$$\mathcal{K} = \frac{\omega_0^2\mathcal{Q}\left(\mathcal{Q}\frac{\omega'}{\omega} \mp \sqrt{\mathcal{Q}^2\left(\frac{\omega'}{\omega}\right)^2 + \frac{f_r'}{\omega_0^2}\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)}\right)^2}{\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)^2} - f_r, \tag{110}$$

where

$$\mathcal{Q} \equiv \frac{r^2K^2}{N^2}$$

is a function of r only and $'$ denotes the derivative with respect to r.

Inserting these expressions for p_φ and $\mathcal{K}$ into (108) leads to

$$\mathcal{Q}\left(\mathcal{Q}\frac{\omega'}{\omega} \mp \sqrt{\mathcal{Q}^2\left(\frac{\omega'}{\omega}\right)^2 + \frac{f_r'}{\omega_0^2}\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)}\right)^2 - \frac{f_r + f_\vartheta}{\omega_0^2}\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)^2$$
$$\geq \sin^{-2}\vartheta\,\omega^{-2}\left(\mathcal{Q}' + \mathcal{Q}\frac{\omega'}{\omega} \pm \sqrt{\mathcal{Q}^2\left(\frac{\omega'}{\omega}\right)^2 + \frac{f_r'}{\omega_0^2}\left(\mathcal{Q}' + 2\mathcal{Q}\frac{\omega'}{\omega}\right)}\right)^2, \tag{111}$$

which is the condition for the existence of a spherical light ray around the Teo wormhole. Alternatively, this expression can be obtained by applying the general formula (37).

With p_φ and K expressed as functions of the radius coordinate $r = r_p$ by (109) and (110), respectively, Eqs. (53) and (54) give us the boundary curve of the shadow parameterized by r_p,

$$\sin\theta = \left(\frac{\mathcal{K} - f_\vartheta}{\mathcal{Q}\,(\omega p_\varphi - \omega_0)^2 - (f_r + f_\vartheta)}\right)^{1/2}\Bigg|_{(r_O,\vartheta_O)}, \tag{112}$$

$$\sin\psi = \frac{p_\varphi}{\sin\vartheta\sqrt{\mathcal{K} - f_\vartheta}}\Bigg|_{(r_O,\vartheta_O)}. \tag{113}$$

Note that the orthonormal tetrad

$$e_0 = \frac{1}{N}(\partial_t + \omega\partial_\varphi)\Big|_{(r_O,\vartheta_O)}, \tag{114}$$

$$e_1 = \frac{1}{rK}\partial_\vartheta\Big|_{(r_O,\vartheta_O)}, \tag{115}$$

$$e_2 = \frac{1}{rK\sin\vartheta}\partial_\varphi\Big|_{(r_O,\vartheta_O)}, \tag{116}$$

$$e_3 = -\left(1 - \frac{b}{r}\right)^{1/2}\partial_r\Big|_{(r_O,\vartheta_O)} \tag{117}$$

was applied in order to obtain the relations for the wormhole shadow.

As a specific example, we consider the Teo wormhole of the form

$$ds^2 = \Omega(r,\vartheta)\left(-dt^2 + \frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2 d\vartheta^2 + r^2\sin^2\vartheta\left(d\varphi - \frac{2\,a\,dt}{r^3}\right)^2\right) \tag{118}$$

with

$$\Omega(r,\vartheta) = 1 + \frac{(4\,a\,\cos\vartheta)^2}{r_0^3\,r}, \tag{119}$$

where r_0 is a positive constant with the dimension of length and a is a constant with the dimension of length squared. The radius coordinate ranges from r_0 to infinity. The neck is situated at r_0 where, indeed, the condition $\left(r/\Omega(r,\vartheta)\right)\left(1 - r_0^2/r^2\right) = 0$ is satisfied.

Individual terms relevant for the photon region and the shadow of this object, thus, are

$$A(r,\vartheta) = \Omega(r,\vartheta)\left(1 - \frac{4a^2}{r^4}\sin^2\vartheta\right), \quad B(r,\vartheta) = \Omega(r,\vartheta)\left(1 - \frac{r_0^2}{r^2}\right)^{-1}, \tag{120}$$

$$C(r,\vartheta) = \Omega(r,\vartheta)r^2\sin^2\vartheta, \quad D(r,\vartheta) = \Omega(r,\vartheta)r^2, \quad P(r,\vartheta) = -\frac{2a}{r}\Omega(r,\vartheta)\sin^2\vartheta. \tag{121}$$

The separated terms are

$$F(r,\vartheta)=\Omega(r,\vartheta)r^2,\quad \mathcal{F}(r)=r^2\left(1-\frac{r_0^2}{r^2}\right),\quad \mathcal{G}(\vartheta)=1, \tag{122}$$

$$A(r,\vartheta)C(r,\vartheta)+P^2(r,\vartheta)=\Omega^2(r,\vartheta)r^2\sin^2\vartheta, \tag{123}$$

$$A_r=-\frac{4a^2}{r^4},\quad A_\vartheta=\sin^{-2}\vartheta, \tag{124}$$

$$C_r=r^2,\quad C_\vartheta=0, \tag{125}$$

$$P_r=-\frac{2a}{r},\quad P_\vartheta=0. \tag{126}$$

The Carter constant exists if the plasma density is of the form

$$\omega_{\rm pl}(r,\vartheta)^2=\frac{f_r(r)+f_\vartheta(\vartheta)}{\Omega(r,\vartheta)r^2}. \tag{127}$$

The equations of motion (28), (29), (34), and (35) read

$$\Omega^2(r,\vartheta)r^4\dot r^2=\left(r^2-r_0^2\right)\left(-\mathcal{K}+\left(\frac{2a}{r^2}p_\varphi-\omega_0 r\right)^2-f_r(r)\right), \tag{128}$$

$$\Omega^2(r,\vartheta)r^4\dot\vartheta^2=\mathcal{K}-\frac{p_\varphi^2}{\sin^2\vartheta}-f_\vartheta, \tag{129}$$

$$\dot\varphi=\frac{p_\varphi\left(r^4-4a^2\sin^2\vartheta\right)+2\,a\,\omega_0\,r^5\sin^2\vartheta}{\Omega(r,\vartheta)\,r^6\sin^2\vartheta}, \tag{130}$$

$$\dot t=\frac{\omega_0\,r^3-2\,a\,p_\varphi}{\Omega(r,\vartheta)\,r^3}. \tag{131}$$

For spherical light rays, the right-hand side of (128) and its derivative must be equal to zero. Evaluating these two equations, we see that they are always satisfied at the neck, at $r=r_0$, and the Carter constant of the corresponding light rays is a function of p_φ, i.e.,

$$\mathcal{K}(p_\varphi)=\left(\frac{2a}{r^2}p_\varphi-\omega_0 r\right)^2-f_r(r). \tag{132}$$

In vacuum, these spherical light rays at the neck are unstable, but in the plasma, some of them may become stable depending on the special form of the function $f_r(r)$. For $r\neq r_0$, setting the right-hand side of (128) and its derivative equal to zero and solving these two equations for p_φ and $\mathcal{K}$ show that for a light ray on a sphere of radius r_p, these constants of motion are

$$p_\varphi(r_p)=\frac{\omega_0 r_p^2}{8a}\left(r_p\mp\sqrt{9r_p^2-4r_p\frac{f_r'(r_p)}{\omega_0^2}}\right), \tag{133}$$

$$\mathcal{K}(r_p)=\frac{\omega_0^2}{16}\left(3r_p\pm\sqrt{9r_p^2-4r_p\frac{f_r'(r_p)}{\omega_0^2}}\right)^2-f_r(r_p). \tag{134}$$

Hence, in addition to the photon sphere at the neck, we also have, in general, a photon region, given by (37) specified to the case at hand,

$$\left(3r\pm\sqrt{9r^2-4r\frac{f_r'(r)}{\omega_0^2}}\right)^2-16\frac{f_r(r)+f_\vartheta(\vartheta)}{\omega_0^2}\geq\frac{r^4}{4a^2\sin^2\vartheta}\left(r\mp\sqrt{9r^2-4r\frac{f_r'(r)}{\omega_0^2}}\right)^2. \tag{135}$$

From (128), one gets the condition for the spherical light orbits in the photon region to be unstable,

$$0<R''(r)=-f_r''(r)+2\left(\frac{40a^2}{r^6}p_\varphi^2+\omega_0^2-\frac{4a}{r^3}\omega_0 p_\varphi\right). \tag{136}$$

In vacuum, only the upper sign in (135) is possible, and for $a < r_0^2/6$, the photon region does not exist, i.e., only the unstable photon orbits in the neck can serve as limit curves for light rays that determine the boundary of the shadow. For $a > r_0^2/6$, the photon region exists. It is divided by the photon sphere at the neck into two symmetric parts. The boundary curve of the shadow is partly formed by light rays that spiral toward the photon sphere and partly by light rays that spiral toward unstable spherical orbits in the photon region at radii $r_p \neq r_0$. In a plasma, the new feature is that the photon region may become detached from the neck with the spherical orbits at the neck being stable. The boundary curve of the shadow is, then, entirely determined by light rays that spiral toward spherical orbits in the component of the photon region that is on the same side of the neck as the observer. The photon orbits for a wormhole with the choice of $a = r_0^2/3$ and $a = 0.8r_0^2$, respectively, are shown in Fig. 1. Note that if the plasma profile is chosen as $f_r = 4\omega_0^2 r_0^2 (r_0/r)^{1/2}$ and $f_\vartheta = 0$, the photon region of the Teo wormhole with $a = r_0^2/3$ does not exist.

For calculating the shadow, we choose the same tetrad as in (114)–(117), which now takes the following form:

$$e_0 = \frac{1}{\sqrt{\Omega(r,\vartheta)}}\left(\partial_t + \frac{2a}{r^3}\partial_\varphi\right)\Big|_{(r_O,\vartheta_O)}, \tag{137}$$

$$e_1 = \frac{1}{\sqrt{\Omega(r,\vartheta)}r}\partial_\vartheta\Big|_{(r_O,\vartheta_O)}, \tag{138}$$

$$e_2 = \frac{1}{\sqrt{\Omega(r,\vartheta)}r\,\sin\,\vartheta}\partial_\varphi\Big|_{(r_O,\vartheta_O)}, \tag{139}$$

$$e_3 = -\frac{1}{\sqrt{\Omega(r,\vartheta)}}\left(1 - \frac{r_0^2}{r^2}\right)^{1/2}\partial_r\Big|_{(r_O,\vartheta_O)}. \tag{140}$$

Here, the observer position r_O should not be confused with the radius r_0 of the neck. Comparing the coefficients of ∂_t and ∂_φ in (46) with those in (47), then, yields

$$\alpha = \frac{2ap_\varphi - \omega_0 r_O^3}{\sqrt{\Omega(r_O,\vartheta_O)}\,r_O^3}, \tag{141}$$

$$\beta = \frac{p_\varphi}{\sqrt{\Omega(r_O,\vartheta_O)}\,r_O\,\sin\,\vartheta_O\,\sin\,\psi\,\sin\,\theta}. \tag{142}$$

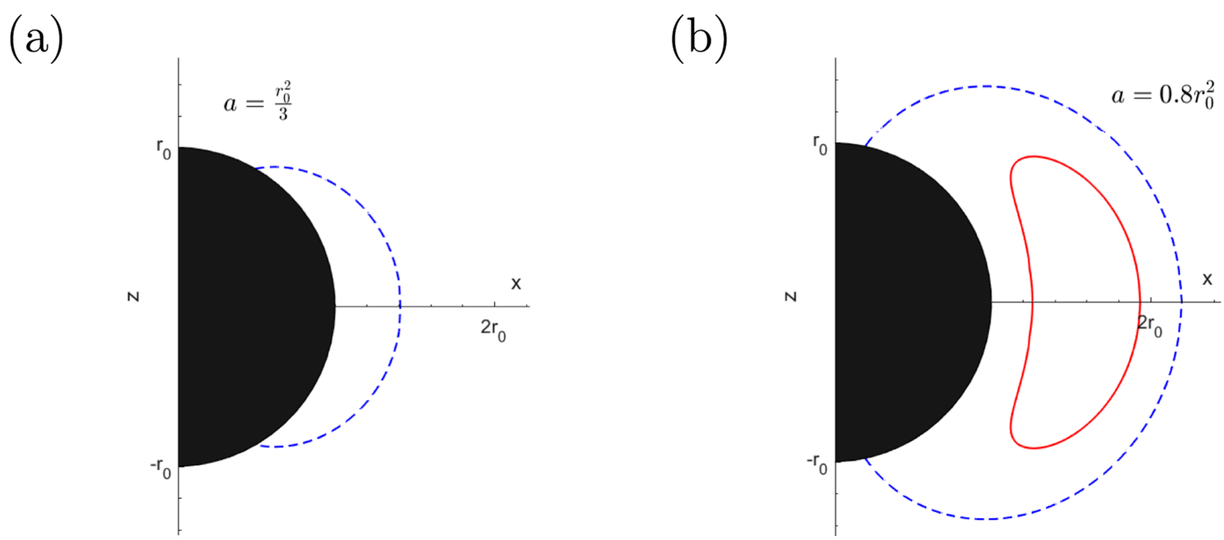

FIG. 1. Photon regions for the Teo wormhole in the case when (a) $a = r_0^2/3$ and (b) $a = 0.8r_0^2$. The blue dashed lines show the boundary of the photon region in the vacuum case, while the red solid line corresponds to the case that $f_r = 4\omega_0^2 r_0^2 (r_0/r)^{1/2}$ and $f_\vartheta = 0$.

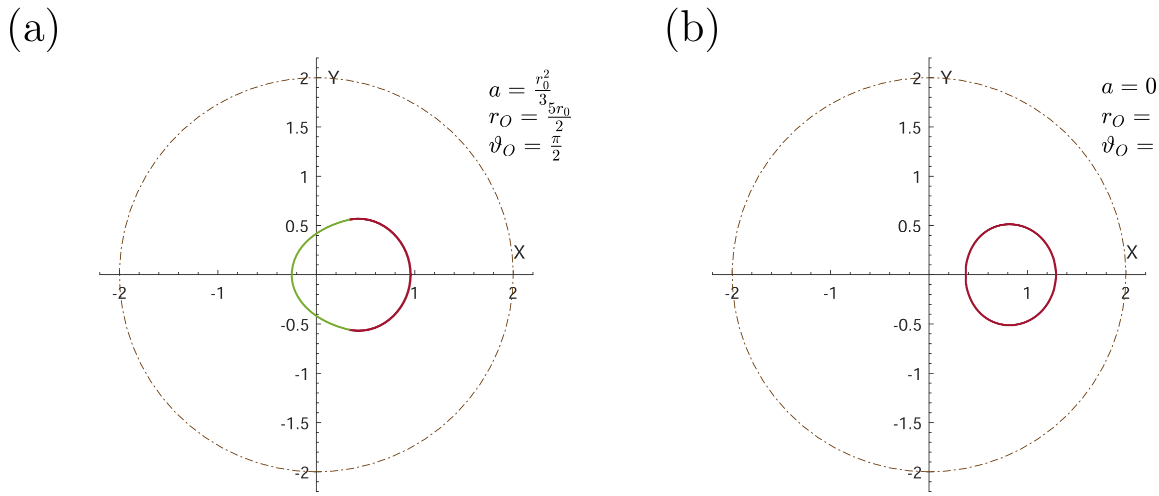

FIG. 2. Shadow of the Teo wormhole for an observer at $r_O = 5r_0/2$ and $\vartheta_O = \pi/2$ in the case when (a) $a = r_0^2/3$, $f_r = 0$, and $f_\vartheta = 0$ and (b) $a = 0.8r_0^2$, $f_r = 4\omega_0^2 r_0^2 (r_0/r)^{1/2}$, and $f_\vartheta = 0$. The thick purple curves show the boundary of the shadow that is given by light rays spiraling toward spherical light rays in the photon region, and the thin green curve corresponds to the shadow boundary formed by light rays that spiral toward the photon sphere at the neck, i.e., when $r = r_0$. The dashed-dotted circle shows the position of the celestial equator.

Here, we have used (130) and (131). Similarly, we compare the coefficients of ∂_r and ∂_ϑ. Inserting the resulting expressions for $\dot{r}$ and $\dot{\vartheta}$ into (128) and (129), substituting for β from (142), and solving for the celestial coordinates ψ and θ give

$$\tan^2\theta = \frac{\mathcal{K} - f_\vartheta(\vartheta_O)}{\left(\frac{2a}{r_O^2}p_\varphi - \omega_0 r_O\right)^2 - \mathcal{K} - f_r(r_O)}, \tag{143}$$

$$\sin^2\psi = \frac{p_\varphi^2}{(\mathcal{K} - f_\vartheta(\vartheta_O))\sin^2\vartheta_O}. \tag{144}$$

Inserting $p_\varphi = p_\varphi(r_p)$ and $\mathcal{K} = \mathcal{K}(r_p)$ from (133) and (134) gives the part of the boundary curve of the shadow that is formed by light rays that spiral toward spherical light rays in the photon region; this curve is parameterized by r_p. Inserting $K = K(p_\varphi)$ from (132) gives the part of the boundary curve of the shadow that corresponds to light rays that spiral toward the photon sphere at the neck; this curve is parameterized by p_φ.

The shadow as seen by an observer located at $r_O = 5r_0/2$ and $\vartheta_O = \pi/2$ in vacuum and in a plasma, respectively, is shown in Fig. 2. The dimensionless Cartesian coordinates used in Ref. 11 were applied to depict the shadow curve. They read

$$X(r) = -2\tan\left(\frac{\theta(r)}{2}\right)\sin(\psi(r)), \tag{145}$$

$$Y(r) = -2\tan\left(\frac{\theta(r)}{2}\right)\cos(\psi(r)). \tag{146}$$

The coordinates are defined in the plane that is tangent to the celestial sphere at the pole $\theta = 0$, and the stereographic projection onto it is performed.

X. CONCLUSIONS

Axisymmetric stationary spacetimes represent a natural arena for objects that are of astrophysical interest. If such objects are surrounded by plasmas, the light rays are affected by both the curved geometry and properties of the medium. We studied such situations starting from the elegant Hamiltonian formalism of the light rays propagating in general stationary axisymmetric spacetimes with a non-magnetized (hence locally isotropic) and pressure-free plasma, which is refractive and dispersive (Sec. II). If the plasma density shares the symmetry of the spacetime, the Killing vector fields associated with stationarity and axisymmetry give us two constants of motion in addition to the Hamiltonian. If the spacetime admits an equatorial plane and if the plasma density is symmetric with respect to the equatorial plane, this gives us enough constants of motion for integrating the light rays in the equatorial plane and also along the axis of symmetry. For general light rays, however, the equations of motion are not completely integrable. We investigated what conditions arise on the axisymmetric, stationary metric in the coordinates adapted to the symmetries [see Eq. (8)] and on the plasma density if we require the Hamilton–Jacobi equation for the rays to be separable. Similarly to the most relevant case of this type, the Kerr metric, another constant of motion, the "generalized Carter constant," must exist. In Sec. III, we found these conditions [see Eqs. (15), (18), and (19)]. Assuming such a constant to exist and employing the Hamilton equations of motion, we determined the photon region by calculating the position of the spherical light rays (Sec. IV) and the black hole shadow in a stationary and axisymmetric spacetime with plasmas (Sec. V). As the separability condition is local in nature, our results are also applicable to all other cases where we have two commuting Killing vector fields that span timelike two-surfaces. A noteworthy observation is that, on a spacetime with such symmetries, the separability condition cannot hold for light rays in any plasma density if it does not hold for light rays in vacuum.

After these considerations, we analyzed several examples in Secs. VI–IX. First, we showed how our general formulas work for the so-called "hairy Kerr metrics" arising by assuming the mass in the Kerr metric to be a function of the radial (Boyer–Lindquist-type) coordinate rather than a constant. Next, we turned to the Hartle–Thorne approximate metric representing exterior regions of slowly rotating objects with a quadrupole moment and verified with the help of our general results that in the chosen coordinates, the full separability of variables in the Hamilton–Jacobi equation cannot be achieved unless in the limiting Schwarzschild case. An instructive example demonstrating the importance of the choice of a suitable coordinate system was presented in our discussion of the Melvin universe arising due to the strong "uniform" magnetic field; again, we also assumed the plasma to be present. The equations for rays are separable in the "cylindrical-type" coordinates, but not in the "spherical-type" coordinates.

The most detailed discussion was devoted to Teo's rotating traversable wormhole spacetime with plasmas. The separability of variables (the existence of the Carter constant) was shown to be possible if certain metric terms are only functions of the radial-type coordinate and if the plasma density is of a certain separated form. The photon region and the shadow of the wormhole with plasmas were determined, and the results were compared with those of Nedkova *et al.*[31] In Fig. 1, the shapes of the photon regions around a specific Teo wormhole in vacuum and with some specific plasma distribution are constructed; in Fig. 2, the shadows are compared.

Of course, there exist more cases of stationary and axisymmetric spacetimes with plasmas in which the Carter-type constant will exist, and the separability of the Hamilton–Jacobi equation for rays will be feasible. For example, the "specific variant" of the old Lense–Thirring metric (in their original asymptotic form) became recently of interest, see, e.g., Ref. 34, because they can be applied as a perfectly good approximation for the gravitational field generated by rotating sources with angular momentum.

DEDICATION

Dedicated to the 60th birthday of our colleague and friend Oldřich Semerák, who did much important work on the motion of particles in strong gravitational fields, in particular, around black holes.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Barbora Bezděková: Investigation (equal); Project administration (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Volker Perlick**: Investigation (equal); Methodology (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Jiří Bičák**: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX: THE DERIVATION OF THE BASIC METRIC (8)

In this appendix, we assume an axially symmetric and stationary metric, i.e., a metric given in coordinates $(t, \varphi, r, \vartheta)$ with the metric coefficients being independent of t and φ, and we prove the following two statements: The Hamilton–Jacobi equation for light rays in a plasma can separate only if the plasma density is independent of t and φ, and if separation holds in the chosen coordinates, then it holds in coordinates with $g_{tr} = g_{t\vartheta} = g_{\varphi r} = g_{\varphi\vartheta} = g_{r\vartheta} = 0$, i.e., in coordinates such that the metric takes the form of Eq. (8). To that end, we assume that the two Killing vector fields, $\partial/\partial t$ and $\partial/\partial\varphi$, span two-dimensional hypersurfaces of signature $(+,-)$. This assumption implies that g^{rr} and $g^{\vartheta\vartheta}$ are non-zero because a timelike vector cannot be tangential to a lightlike hypersurface.

We want to solve the Hamilton–Jacobi equation,

$$\mathcal{H}\left(x^\alpha, \frac{\partial S}{\partial x^\beta}\right) = 0, \tag{A1}$$

under the assumption that S satisfies the separation form,

$$S(x^\alpha) = S_t(t) + S_\varphi(\varphi) + S_r(r) + S_\vartheta(\vartheta). \tag{A2}$$

Then, the Hamilton–Jacobi equation reads

$$g^{tt}\left(\frac{dS_t}{dt}\right)^2 + 2g^{t\varphi}\frac{dS_t}{dt}\frac{dS_\varphi}{d\varphi} + g^{\varphi\varphi}\left(\frac{dS_\varphi}{d\varphi}\right)^2 + 2g^{tr}\frac{dS_t}{dt}\frac{dS_r}{dr} + 2g^{t\vartheta}\frac{dS_t}{dt}\frac{dS_\vartheta}{d\vartheta} + 2g^{\varphi r}\frac{dS_\varphi}{d\varphi}\frac{dS_r}{dr}$$

$$+ g^{\varphi\vartheta}\frac{dS_\varphi}{d\varphi}\frac{dS_\vartheta}{d\vartheta} + g^{rr}\left(\frac{dS_r}{dr}\right)^2 + 2g^{r\vartheta}\frac{dS_r}{dr}\frac{dS_\vartheta}{d\vartheta} + g^{\vartheta\vartheta}\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 + \omega_{\mathrm{pl}}^2 = 0, \tag{A3}$$

where ω_{pl} is the plasma frequency. Differentiation with respect to t yields

$$\left(g^{tt}\frac{dS_t}{dt} + g^{t\varphi}\frac{dS_\varphi}{d\varphi} + g^{tr}\frac{dS_r}{dr} + g^{t\vartheta}\frac{dS_\vartheta}{d\vartheta}\right)\frac{d^2S_t}{dt^2} + \omega_{\mathrm{pl}}\frac{\partial\omega_{\mathrm{pl}}}{\partial t} = 0. \tag{A4}$$

As the plasma density is independent of the individual solution to the Hamilton–Jacobi equation, this can be true only if $\partial\omega_{\mathrm{pl}}/\partial t$ is zero and $p_t = dS_t/dt$ is a constant. An analogous calculation shows that $\partial\omega_{\mathrm{pl}}/\partial\varphi$ must be zero and $p_\varphi = dS_\varphi/d\varphi$ must be a constant; hence,

$$S(x) = p_t\, t + p_\varphi\, \varphi + S_r(r) + S_\vartheta(\vartheta), \tag{A5}$$

with constants p_t and p_φ. Here, we are free to multiply the left-hand side of the Hamilton–Jacobi equation with a function $F(r,\vartheta)$ that is non-zero but otherwise arbitrary. Written out in full, the Hamilton–Jacobi equation reads

$$F\left(g^{tt}p_t^2 + 2g^{t\varphi}p_tp_\varphi + g^{\varphi\varphi}p_\varphi^2 + 2g^{tr}p_t\frac{dS_r}{dr} + 2g^{t\vartheta}p_t\frac{dS_\vartheta}{d\vartheta} + 2g^{\varphi r}p_\varphi\frac{dS_r}{dr} + g^{\varphi\vartheta}p_\varphi\frac{dS_\vartheta}{d\vartheta}\right.$$
$$\left. + g^{rr}\left(\frac{dS_r}{dr}\right)^2 + 2g^{r\vartheta}\frac{dS_r}{dr}\frac{dS_\vartheta}{d\vartheta} + g^{\vartheta\vartheta}\left(\frac{dS_\vartheta}{d\vartheta}\right)^2 + \omega_{\mathrm{pl}}^2\right) = 0. \tag{A6}$$

Separability requires that the left-hand side is a function of r only plus a function of ϑ only. As this has to hold for all p_t and p_φ and as the plasma density is independent of p_t, p_φ, S_r, and S_ϑ, this gives us the following set of equations:

$$F\,g^{tt} = \rho_r + \rho_\vartheta, \quad F\,g^{t\varphi} = \lambda_r + \lambda_\vartheta, \quad F\,g^{\varphi\varphi} = \sigma_r + \sigma_\vartheta, \tag{A7}$$

$$F\,g^{tr} = \delta_r, \quad F\,g^{t\vartheta} = \delta_\vartheta, \quad F\,g^{\varphi r} = \varepsilon_r, \quad F\,g^{\varphi\vartheta} = \varepsilon_\vartheta, \tag{A8}$$

$$F\,g^{rr} = \zeta_r, \quad F\,g^{\vartheta\vartheta} = \zeta_\vartheta, \quad g^{r\vartheta} = 0, \tag{A9}$$

$$F\,\omega_{\mathrm{pl}}^2 = f_r + f_\vartheta. \tag{A10}$$

Here and in the following, functions with an index r depend on r only and functions with an index ϑ depend on ϑ only. In particular, we read from (A9) that the separability can hold only if $g^{r\vartheta} = 0$.

We now perform a coordinate transformation,

$$t \mapsto t + \alpha_r + \alpha_\vartheta, \quad \varphi \mapsto \varphi + \beta_r + \beta_\vartheta, \quad r \mapsto r, \quad \vartheta \mapsto \vartheta, \tag{A11}$$

$$dt \mapsto dt + \frac{d\alpha_r}{dr}dr + \frac{d\alpha_\vartheta}{d\vartheta}d\vartheta, \quad d\varphi \mapsto d\varphi + \frac{d\beta_r}{dr}dr + \frac{d\beta_\vartheta}{d\vartheta}d\vartheta, \quad dr \mapsto dr, \quad d\vartheta \mapsto d\vartheta, \tag{A12}$$

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \varphi} \mapsto \frac{\partial}{\partial \varphi}, \quad \frac{\partial}{\partial r} \mapsto \frac{\partial}{\partial r} - \frac{d\alpha_r}{dr}\frac{\partial}{\partial t} - \frac{d\beta_r}{dr}\frac{\partial}{\partial \varphi}, \quad \frac{\partial}{\partial \vartheta} \mapsto \frac{\partial}{\partial \vartheta} - \frac{d\alpha_\vartheta}{d\vartheta}\frac{\partial}{\partial t} - \frac{d\beta_\vartheta}{d\vartheta}\frac{\partial}{\partial \varphi}. \tag{A13}$$

Hence,

$$g^{tr} \mapsto g^{tr} + \frac{d\alpha_r}{dr} g^{rr} + \frac{d\alpha_\vartheta}{d\vartheta} g^{\vartheta r} = \frac{1}{F}\left(\delta_r + \frac{d\alpha_r}{dr}\zeta_r\right), \tag{A14}$$

$$g^{t\vartheta} \mapsto g^{t\vartheta} + \frac{d\alpha_r}{dr} g^{r\vartheta} + \frac{d\alpha_\vartheta}{d\vartheta} g^{\vartheta\vartheta} = \frac{1}{F}\left(\delta_\vartheta + \frac{d\alpha_\vartheta}{d\vartheta}\zeta_\vartheta\right), \tag{A15}$$

$$g^{\varphi r} \mapsto g^{\varphi r} + \frac{d\beta_r}{dr} g^{rr} + \frac{d\beta_\vartheta}{d\vartheta} g^{\vartheta r} = \frac{1}{F}\left(\varepsilon_r + \frac{d\beta_r}{r}\zeta_r\right), \tag{A16}$$

$$g^{\varphi\vartheta} \mapsto g^{\varphi\vartheta} + \frac{d\beta_r}{dr} g^{r\vartheta} + \frac{d\beta_\vartheta}{d\vartheta} g^{\vartheta\vartheta} = \frac{1}{F}\left(\varepsilon_\vartheta + \frac{d\beta_\vartheta}{d\vartheta}\zeta_\vartheta\right). \tag{A17}$$

As $\zeta_r = Fg^{rr}$ and $\zeta_\vartheta = Fg^{\vartheta\vartheta}$ are non-zero, we can choose functions α_r, α_ϑ, β_r, and β_ϑ such that the right-hand sides of these four equations are zero, i.e., such that in the new coordinates, the metric components g^{tr}, $g^{t\vartheta}$, $g^{\varphi r}$, and $g^{\varphi\vartheta}$ vanish. Conditions (A8) are, then, still satisfied, now with the right-hand sides equal to zero. Equations (A9) and (A10) are unchanged, whereas Eq. (A7) is still true, but now with new functions ρ_r, ρ_ϑ, λ_r, λ_ϑ, σ_r, and σ_ϑ, so separability still holds in the new coordinates if it did so in the original coordinates. By inverting the matrix $(g^{\mu\nu})$, we see that in the new coordinates, the metric takes, indeed, the form of Eq. (8).

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