Adiabatic radial perturbations of relativistic stars: Analytic solutions to an old problem

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We present a new system of equations that fully characterizes adiabatic, radial perturbations of perfect fluid stars within the theory of general relativity. The properties of the system are discussed, and, provided that the equilibrium spacetime verifies some general regularity conditions, analytical solutions for the perturbation variables are found. As illustrative examples, the results are applied to study perturbations of selected classical exact spacetimes, and the first oscillation eigenfrequencies are computed. Exploiting the new formalism, we derive an upper bound for the maximum compactness of stable, perfect fluid stars, which is equation-of-state agnostic and significantly smaller than the Buchdahl bound.

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I. INTRODUCTION

Relativistic compact stellar objects are among the most complex and, at the same time, most fascinating gravitational systems. Similar to black holes, these objects represent strong gravity systems. However, they are fundamentally different in that the nature and behavior of matter play a prominent role in their structure and evolution. This fact makes their theoretical description particularly challenging. The complexities associated with these objects explain, on the one hand, the predominant use of numerical techniques and, on the other, the necessity to develop and apply perturbative approaches to understand their properties and dynamics.

The first attempt at the description of perturbations of stars in general relativity was made by Chandrasekhar [1,2]. In those works, Chandrasekhar focused on understanding the behavior of adiabatic, radial perturbations and developed an integro-differential equation, the so-called Chandrasekhar radial pulsation equation, to describe this somewhat simpler type of perturbation.

The Chandrasekhar equation has had a crucial influence on the subsequent studies on the dynamical behavior of perturbations of self-gravitating, massive compact objects, and various methods were developed to compute the oscillation eigenfrequencies directly from the pulsation equation [3]. Various reformulations of Chandrasekhar's original equation have been proposed to ease the numerical treatment of adiabatic, radial oscillations of self-gravitating fluids with realistic equations of state (see, e.g., [4–8]). However, to our knowledge, no work has so far tackled one of the main limitations of Chandrasekhar's approach, i.e., the issue of gauge dependence. Indeed, the predictions of the Chandrasekhar equation on the stability of a given matter configuration and on the very behavior of its perturbations are intrinsically associated with the specific coordinate system considered and, therefore, conditioned by the choice of the gauge.

Recently, the authors of this paper derived a completely covariant and gauge-invariant theory of perturbations for static, locally rotationally symmetric of class II (LRS II) spacetimes [9]. LRS spacetimes are characterized by local rotational symmetry about a given spatial direction, and their nonvortical subclass, known as LRS II in the case of perfect fluid sources, can be proven to contain all isotropic nonrotating spacetimes suitable to describe stellar compact objects and even some slowly rotating ones. The new perturbation framework is based on the so-called $1+1+2$ covariant formalism [10,11]. This formalism is based on the procedure of covariant spacetime decomposition, such that at each point the spacetime is foliated by surfaces

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orthogonal to two vector fields, a timelike and a spacelike vector field. Using this covariant decomposition, it is possible to characterize different aspects of the geometry of LRS II spacetimes and the thermodynamics of the matter fields that permeate them in a geometrically clear and physically meaningful way (see, e.g., Refs. [12–21]).

The aim of this paper is to apply the covariant gauge-invariant perturbation theory of Ref. [9] to study adiabatic, radial perturbations of static, compact stellar objects composed of a perfect fluid. In the body of the text, we will, however, write the perturbation equations immediately in a noncovariant way from the classical point of view of an observer locally comoving with the matter and use the circumferential radius to identify points within the star. There are essentially two reasons for this choice. First, we aim to provide a set of equations that can be applied without any specific knowledge of the covariant formalisms. Second, the above setting is usually considered in the standard metric-based description of this type of perturbation, providing a clear, familiar interpretation of the quantities and the equations. We will then present exact solutions for the perturbations of well-known models for equilibrium spacetimes in a gauge-invariant way.

The paper is organized as follows. In Sec. II, we will present the equations describing adiabatic, radial perturbations, and propose an algorithm to obtain power series exact solutions of the system. We also give a lower bound for the minimum eigenvalue of the system. In Sec. III, we apply these methods to some classical solutions, namely, we will consider the interior Schwarzschild, Tolman IV, Kuchowicz 2-III, and Heintzmann IIa (as cataloged in [22]) spacetimes, and present the behavior of the first eigenfunctions that characterize the perturbations in the frame of the comoving observer. In Sec. IV, using results for the perturbation of the interior Schwarzschild solution, we conjecture on a general upper bound for instability of static, stellar compact objects composed of a perfect fluid, which is independent of the information on the equation of state of the perturbed matter fluid. We then draw conclusions in Sec. V. The paper also contains four Appendixes. In Appendix A, we introduce the general definitions for the $1+1+2$ covariant quantities. In Appendix B, we give the general covariant, gauge-invariant perturbation equations for adiabatic, radial perturbations found from the $1+1+2$ formalism. In Appendix C, we show how the original Chandrasekhar radial pulsation equation can be recovered from the new gauge-invariant equations, demonstrating their equivalence in the considered coordinate system, and in Appendix D we present the general intermediate matrix for the power series solutions of the system.

Throughout the article, we will work in the geometrized unit system where $8\pi G = c = 1$, and consider the metric signature $(-+++)$.

II. ADIABATIC RADIAL PERTURBATIONS

The Einstein field equations (EFE) are a set of nonlinear partial differential equations that are manifestly difficult to solve for general source matter fields. To circumvent these difficulties, it is helpful to devise perturbation schemes to linearize the field equations and study the behavior of small deviations from equilibrium solutions. In Ref. [9], a general set of covariant gauge-invariant equations was derived that can characterize linear perturbations of static, spatially compact, spherically symmetric solutions of the Einstein field equations. We will call such solutions "stars" because of their most immediate physical application. The gauge-invariant equations for linear perturbations were written in the language of the $1+1+2$ covariant decomposition formalism, which relates the geometry of the spacetime and the properties of the matter fluid with the kinematical quantities that characterize two sets of congruences: a timelike congruence and a spacelike congruence. As mentioned above, however, we will relate all quantities directly with the metric tensor and the matter fluid variables. Nonetheless, since the $1+1+2$ variables have intrinsic physical meaning, they are important to interpret the results. The reader can find the basic definitions of the $1+1+2$ formalism in Appendix A.

The evolution of general perturbations of stars is a compelling yet remarkably complicated problem in gravitation theory. Here, we will focus on studying adiabatic, radial perturbations of stars with a perfect fluid source. The article is intended to be self-contained. Nonetheless, some general properties will be simply stated, and we redirect the reader to Ref. [9] for technical details.

A. The equilibrium spacetime and the perturbation variables

We will assume that the equilibrium background spacetime is static and spherically symmetric, such that it can be characterized by a line element of the form

$$ds_0^2 = -(g_0)_{tt}dt^2 + (g_0)_{rr}dr^2 + r^2 d\Omega^2, \qquad (1)$$

where $d\Omega^2$ represents the natural line element for the unit 2-sphere, (t, r) are the standard Schwarzschild coordinates measured by an observer at spatial infinity, and the metric components $(g_0)_{tt}$ and $(g_0)_{tt}$ are functions of the circumferential radius, r, only. We will consider the setup where two solutions of the EFE are smoothly matched at a common timelike hypersurface, such that the interior of the star is described by a static, spatially compact solution with a perfect fluid source, with $r \in [0, r_b]$, where $r = r_b$ defines the boundary of the star, and the exterior spacetime, with $r > r_b$, is described by a radial branch of the vacuum Schwarzschild solution with no event horizons. The metric coefficients $(g_0)_{tt}$ and $(g_0)_{rr}$ are determined by the EFE with a zero cosmological constant

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}, \qquad (2)$$

where $R_{\alpha\beta} \coloneqq R_{\alpha\mu\beta}{}^{\mu}$ represents the Ricci tensor, $R \coloneqq R_{\mu}{}^{\mu}$ the Ricci scalar, and $T_{\alpha\beta}$ the metric stress-energy tensor described by a perfect fluid source, i.e.,

$$T_{\alpha\beta} = (\mu_0 + p_0)(u_0)_\alpha (u_0)_\beta + p_0 (g_0)_{\alpha\beta}, \tag{3}$$

where u_0 is the 4-velocity of the elements of volume of the fluid, and μ_0 and p_0 are, respectively, the energy density and the isotropic pressure of the matter fluid. Here and in the following, we will use the subscript "0" to explicitly refer to quantities of the equilibrium spacetime.

It is useful to introduce the following scalar functions:

$$\begin{aligned} \phi_0 &= \frac{2}{r\sqrt{(g_0)_{tt}}}, \\ \mathcal{A}_0 &= \frac{1}{2(g_0)_{tt}\sqrt{(g_0)_{rr}}}\frac{d(g_0)_{tt}}{dr}, \\ \mathcal{E}_0 &= \frac{1}{3}\mu_0 + p_0 - \mathcal{A}_0\phi_0. \end{aligned} \tag{4}$$

The function ϕ_0 represents the spatial expansion of the normalized radial gradient vector field, $\mathcal{A}_0$ is the radial component of the 4-acceleration of the elements of volume of the fluid, and $\mathcal{E}_0$ is the pure radial component of the electric part of the Weyl tensor, Eq. (A17), which partially describes radial tidal forces.

To characterize the perturbed spacetime with respect to the equilibrium background unambiguously, we have to choose variables that are identification gauge invariant, that is, variables that are independent of the choice of diffeomorphism between the equilibrium and the perturbed spacetimes. Following the Stewart-Walker lemma [23], variables that vanish identically in the background spacetime are gauge invariant.

Since the equilibrium spacetime is assumed static, the proper time derivatives of covariantly defined quantities vanish in the background and can be used to characterize the perturbed spacetime. Indeed, to describe the perturbations, we will consider the gauge-invariant variables

$$\mathsf{m} \coloneqq \dot{\mu}, \quad \mathsf{p} \coloneqq \dot{p}, \quad \mathsf{A} \coloneqq \dot{\mathcal{A}}, \quad \mathsf{F} \coloneqq \dot{\phi}, \quad \mathsf{E} \coloneqq \dot{\mathcal{E}}, \tag{5}$$

where the "dot" represents the proper time derivative of an observer locally comoving with the fluid. In addition to the variables in Eq. (5), we will also consider two other gauge-invariant variables: the expansion scalar θ associated with the integral curves of the u vector field, and the nontrivial radial component of the shear tensor, Σ (see Appendix A for details). The expansion scalar, θ, represents the fractional rate of change of the sectional volume of the congruence associated with the vector field u per unit of proper time τ, whereas Σ, together with θ, partially characterizes the proper time evolution of the radial inhomogeneity of the matter fluid. Then, adiabatic, radial perturbations of perfect fluid stars can be completely described in a frame comoving with the fluid, by the set of gauge-invariant variables $\{\mathsf{m}, \mathsf{p}, \mathsf{A}, \mathsf{F}, \mathsf{E}, \theta, \Sigma\}$.

B. Harmonic decomposition

As is often the case in relativistic perturbation theory, the equilibrium solution is found by considering highly symmetric setups. These symmetries can be taken into account to transform the linearized system of partial differential equations into a system of ordinary differential equations. Indeed, in the case of a static, spherically symmetric equilibrium spacetime, for a suitable choice of frame and at linear perturbation order, all quantities can be can be written in terms of the spherical harmonics, $Y_{\ell m}$, and the eigenfunctions of the Laplace operator in $\mathbb{R}$, $e^{i\upsilon\tau}$, such that a perturbed first-order scalar quantity χ can be written as

$$\chi = \sum_{\upsilon}\left(\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\Psi_\chi^{(\upsilon,\ell)} Y_{\ell m}\right) e^{i\upsilon\tau}, \tag{6}$$

where $\sum_\upsilon$ represents either a discrete sum or an integral in υ, depending on the boundary conditions of the problem, τ is the proper time measured by an observer comoving with the fluid in the background spacetime, υ represent the eigenfrequencies, and $\Psi_\chi^{(\upsilon,\ell)}$ are the harmonic coefficients, which depend only on the radial coordinate r.

In the case of isotropic perturbations, all coefficients $\Psi_\chi^{(\upsilon,\ell)}$ with $\ell \geq 1$ are identically zero, that is, dipole and higher-order angular multipoles must be trivial; otherwise, the perturbations would induce preferred directions in the system. Moreover, it was rigorously shown in Ref. [9] that, under certain regularity conditions, for this type of perturbation, the eigenfrequencies are such that υ^2 are countable, are real, are simple, have a minimum, and are unbounded from above. Therefore, in the study of adiabatic, isotropic perturbations a gauge-invariant, first-order scalar quantity χ can be decomposed as

$$\chi = \sum_{\upsilon^2=\{\upsilon_0^2,\upsilon_1^2,\ldots\}} \Psi_\chi^{(\upsilon)} Y_{00} e^{i\upsilon\tau}, \tag{7}$$

where the radial harmonic coefficients depend on the radial coordinate only: $\Psi_\chi^{(\upsilon)} = \Psi_\chi^{(\upsilon)}(r)$, and we have omitted the (trivial) dependency on ℓ to lighten the notation.

Alternatively, instead of considering the proper time τ, we can consider the time coordinate t. In that case, given the equation (see Ref. [9])

$$\upsilon(r) = \frac{\lambda}{\sqrt{(g_0)_{tt}}}, \tag{8}$$

relating the eigenfrequencies, υ, measured by an observer comoving with the fluid, with the constant eigenfrequencies λ, measured by a static observer at spatial infinity, a gauge-invariant, first-order scalar quantity χ can be equivalently given by Eq. (7) or

$$\chi = \sum_{\lambda^2=\{\lambda_0^2,\lambda_1^2,\ldots\}} \Psi_\chi^{(\lambda)} Y_{00} e^{i\lambda t}. \tag{9}$$

C. Gauge-invariant equation of state and perturbation equations

The perturbation variables proposed previously in this section completely describe the dynamical evolution of the adiabatically, radially perturbed spacetime. Nonetheless, to close the system, we need to characterize the perturbed matter fluid by providing an equation of state. As a simplifying assumption, we will consider that the perturbed matter fluid verifies a barotropic equation of state, such that

$$p = f(\mu), \tag{10}$$

where f is assumed to be twice differentiable in an open neighborhood of μ_0. Then, at linear order

$$\mathsf{p} \approx f'(\mu_0)\mathsf{m}, \tag{11}$$

where prime represents the derivative with respect to the function's parameter, so that $f'(\mu_0)$ represents the square of the adiabatic speed of sound in the perturbed fluid, to be assumed nonvanishing in the interior of the perturbed star. Notice that the function f does not have to be equal to the equation of state of the equilibrium configuration. Indeed, the equilibrium fluid is not even required to verify a barotropic equation of state. This freedom is physically relevant, as it was noted, for instance, in Refs. [6,24] where distinct adiabatic indexes were considered for the equilibrium and the perturbed star. Moreover, as we will see, in the case of an interior Schwarzschild background spacetime, the choice of f is instrumental in analyzing its stability.

In Appendix B we list the covariant, nontrivial perturbation equations for the variables $\{\mathsf{m}, \mathsf{p}, \mathsf{A}, \mathsf{F}, \mathsf{E}, \theta, \Sigma\}$ that characterize adiabatic, radial perturbations. Breaking covariance, in the Schwarzschild coordinate system (t, r) and considering the harmonic decomposition described in the previous subsection, the radial coefficients associated with those variables verify the following system of differential equations [9]:

$$\frac{d\Psi_\mathsf{p}^{(v)}}{dr} + \frac{4\mathcal{A}_0}{r\phi_0}\left(1 + \frac{1}{3f'(\mu_0)}\right)\Psi_\mathsf{p}^{(v)} = -\frac{2(\mu_0 + p_0)}{r\phi_0}(\Psi_\mathsf{A}^{(v)} + \mathcal{A}_0\Psi_\Sigma^{(v)}), \tag{12}$$

$$\frac{d\Psi_\mathsf{A}^{(v)}}{dr} + \left(\frac{6\mathcal{A}_0}{r\phi_0} - \frac{1}{r}\right)\Psi_\mathsf{A}^{(v)} = \frac{2\mathcal{E}_0}{r\phi_0(\mu_0 + p_0)f'(\mu_0)}\Psi_\mathsf{p}^{(v)} - \frac{3}{r\phi_0}\left(v^2 + \mathcal{A}_0^2 + \frac{1}{3}\mu_0 - 2\mathcal{E}_0\right)\Psi_\Sigma^{(v)}, \tag{13}$$

$$\begin{aligned}\frac{d\Psi_\Sigma^{(v)}}{dr} + \left(\frac{3}{r} - \frac{4\mathcal{A}_0}{3r\phi_0 f'(\mu_0)}\right)\Psi_\Sigma^{(v)} = {}& \frac{2}{3(\mu_0 + p_0)f'(\mu_0)}\left[\left(\frac{f''(\mu_0)}{f'(\mu_0)} + \frac{1}{\mu_0 + p_0}\right)\frac{d\mu_0}{dr} + \frac{1}{r\phi_0}\left(\frac{4}{3f'(\mu_0)} + 2\right)\mathcal{A}_0\right]\Psi_\mathsf{p}^{(v)} \\ & + \frac{4}{3r\phi_0 f'(\mu_0)}\Psi_\mathsf{A}^{(v)},\end{aligned} \tag{14}$$

and the constraints

$$(v^2 + \mathcal{A}_0\phi_0 + \mathcal{A}_0^2 - p_0)\left(\frac{2}{3}\Psi_\theta^{(v)} - \Psi_\Sigma^{(v)}\right) = \Psi_\mathsf{p}^{(v)} - \phi_0\Psi_\mathsf{A}^{(v)}, \tag{15}$$

$$\Psi_\mathsf{E}^{(v)} = \mathcal{E}_0\left(\frac{3}{2}\Psi_\Sigma^{(v)} + \frac{\Psi_\mathsf{p}^{(v)}}{f'(\mu_0)(\mu_0 + p_0)}\right) - \frac{1}{2}(\mu_0 + p_0)\Psi_\Sigma^{(v)}, \tag{16}$$

$$\Psi_\mathsf{F}^{(v)} = \left(\frac{1}{2}\phi_0 - \mathcal{A}_0\right)\left(\frac{2\Psi_\mathsf{p}^{(v)}}{3f'(\mu_0)(\mu_0 + p_0)} + \Psi_\Sigma^{(v)}\right), \tag{17}$$

$$\Psi_\mathsf{m}^{(v)} = -(\mu_0 + p_0)\Psi_\theta^{(v)}, \tag{18}$$

$$\Psi_\mathsf{p}^{(v)} = f'(\mu_0)\Psi_\mathsf{m}^{(v)}, \tag{19}$$

where Eq. (19) follows from Eq. (11). The constraint equation (15) is not propagated; therefore, it cannot be used to reduce the size of the system of differential equations (12)–(14). In fact, it is straightforward to show that Eqs. (12)–(15) imply Eq. (8).

To select the physically acceptable solutions and formalize the boundary value problem, we impose the following boundary conditions:

(i) the energy density and the pressure perturbations at the center of the star, $r = 0$, must be finite in a neighborhood of the initial instant; and
(ii) the interior perturbed spacetime can be smoothly matched to an exterior vacuum Schwarzschild spacetime at a timelike hypersurface, the boundary of the star.

From the point of view of the comoving observer, the boundary condition (ii) implies that the pressure of the perturbed fluid is identically zero at all times at the hypersurface and so p, hence $\Psi_{\mathsf{p}}^{(v)}$, is also identically zero at the boundary, that is,

$$\Psi_{\mathsf{p}}^{(v)}(r_b) = 0. \tag{20}$$

Before proceeding, we remark that, in general, the coordinate system in the background spacetime and that of the perturbed spacetime are not necessarily the same: any smooth mapping can be considered. Since the perturbation variables are gauge invariant, that choice does not affect the results. In the particular case of isotropic perturbations, the Schwarzschild coordinate system can always be defined since there is no gravitational wave emission, and the spacetime is asymptotically flat; hence, the time coordinate t and circumferential radius r are equally defined by an observer at spatial infinity in both spacetimes. In particular, the adoption of this coordinate system is useful to compare our approach with the classical results of Chandrasekhar, found from metric-based perturbation theory [1,2]. In Appendix C we show explicitly that Chandrasekhar's second-order radial pulsation equation follows from the system above by relating the kinematical quantities with the radial displacement parameter and its derivatives.

D. Analytic solutions

The system (12)–(19) with boundary conditions (i) and (ii) completely characterizes adiabatic, radial perturbations of a star composed of a perfect fluid. Specifying the background spacetime and the equation of state of the perturbed fluid, numerical methods can be used to find approximate solutions. Nonetheless, contrary to the original form of the second-order Chandrasekhar's pulsation equation [1,2], or the associated first-order realizations of Refs. [4,6], it is possible, under rather general conditions, to find analytic solutions for the perturbations using standard theory of systems of linear ordinary differential equations.

To find analytic solutions for the system (12)–(19), we will further impose the following regularity constraints:

(iii) the weak energy condition holds for the equilibrium fluid;
(iv) as background spacetime, we consider a solution of the Tolman-Oppenheimer-Volkoff (TOV) equation for which the energy density and pressure functions are real analytic for the whole range within the equilibrium star; and
(v) in the interior and at the boundary of the star, f', i.e., the square of the speed of sound of the perturbed matter fluid, is positive and real analytic.

The stronger constraint among the above is certainly the real-analytical character of the source thermodynamical potentials, especially in light of the complexities associated with a realistic description of matter in relativistic stars. On the other hand, every classical solution of the TOV equations verifies this hypothesis, at least in some open neighborhood of the center of the star ($r = 0$). Hence, we can safely assume that the treatment below holds for all known exact solutions of the TOV equations.

Imposing the above conditions, we can express the quantities ϕ_0, $\mathcal{E}_0$, and $\mathcal{A}_0$ as

$$\begin{aligned} \phi_0 &= \frac{2}{r}\sqrt{1 - \frac{2M(r)}{r}}, \\ \mathcal{E}_0 &= \frac{1}{3}\mu_0 - \frac{2M(r)}{r^3}, \\ \mathcal{A}_0\phi_0 &= p_0 + \frac{2M(r)}{r^3}, \end{aligned} \tag{21}$$

where

$$M(r) \coloneqq \frac{1}{2}\int_0^r \mu_0 x^2 dx \tag{22}$$

is usually called the mass function. Thus, if the functions μ_0 satisfy the regularity conditions above, the functions $\mathcal{A}_0$ and $\mathcal{E}_0$ will be real analytic within the star, with the exception of ϕ_0 which will present a simple pole at the center, $r = 0$. Consequently, it is possible to find solutions of the system Eqs. (12)–(14) as power series around the singular point $r = 0$ using the method in Ref. [25].

As a first step, we can recast the system (12)–(14) in matrix form:

$$\frac{d\mathbb{W}}{dr} = (r^{-1}\mathbb{R} + \Theta)\mathbb{W}, \tag{23}$$

where

$$\mathbb{W} = \begin{bmatrix} \Psi_{\mathsf{p}}^{(\lambda)} \\ \Psi_{\mathsf{A}}^{(\lambda)} \\ \Psi_{\Sigma}^{(\lambda)} \end{bmatrix}, \qquad \mathbb{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \tag{24}$$

$$\Theta = \frac{2}{r\phi_0}\begin{bmatrix} -2\mathcal{A}_0\left(1+\frac{1}{3f'(\mu_0)}\right) & -(\mu_0+p_0) & -(\mu_0+p_0)\mathcal{A}_0 \\ \frac{\mathcal{E}_0}{f'(\mu_0)(\mu_0+p_0)} & -3\mathcal{A}_0 & -\frac{3}{2}\left(v^2+\mathcal{A}_0^2+\frac{1}{3}\mu_0-2\mathcal{E}_0\right) \\ \frac{3f''(\mu_0)r\phi_0\partial_r\mu_0+4\mathcal{A}_0}{9(\mu_0+p_0)[f'(\mu_0)]^2}+\frac{r\phi_0\partial_r\mu_0+2\mathcal{A}_0(\mu_0+p_0)}{3(\mu_0+p_0)^2f'(\mu_0)} & \frac{2}{3f'(\mu_0)} & \frac{2\mathcal{A}_0}{3f'(\mu_0)} \end{bmatrix}. \tag{25}$$

The real-analytic character of the matrix Θ at $r=0$ is ensured by the regularity conditions above. More specifically $r\phi_0$ does not vanish in the interior of the star, and Eq. (23) allows us to conclude that $r=0$ is a regular singular point of the system. However, this does not imply that all solutions must be singular at the center. To select only the physically acceptable solutions, we will impose the boundary conditions (i) and (ii).

Since the Θ matrix is real analytic at the center of the star, it can be expanded in a convergent power series of the form[1]

$$\Theta(r) = \sum_{n=0}^{+\infty}\Theta_n r^n. \tag{26}$$

As a result, we can also write the solution $\mathbb{W}$ of the system in Eqs. (23)–(25) in the form of a power series, and this series will converge to the solution in a neighborhood of $r=0$ which is equal (except maybe in $r=0$) to the radius of the series in Eq. (26).

Now, the general family of solutions of the system (23)–(25) is quite complex. However, using the TOV equations and considering the previous regularity constraints, it can be shown that some entries of the coefficient matrices $\{\Theta_0,\Theta_1,\Theta_2,\Theta_3\}$ must vanish identically, such that the physically relevant family of solutions of the system are significantly simpler. Indeed, the general physical solutions are given by

$$\begin{bmatrix}\Psi_{\mathsf{p}}^{(\lambda)}\\ \Psi_{\mathsf{A}}^{(\lambda)}\\ \Psi_{\Sigma}^{(\lambda)}\end{bmatrix} = \begin{bmatrix} -1 & \frac{12}{r}[(\Theta_0)_{12}(\Theta_0)_{23}-3(\Theta_1)_{13}] & 0 \\ 0 & 12(\Theta_0)_{23}[(\Theta_1)_{22}-(\Theta_1)_{33}]-18(\Theta_2)_{23}+4[(\Theta_0)_{23}]^2(\Theta_0)_{32}-\frac{12}{r^2}(\Theta_0)_{23} & r \\ 0 & \frac{36}{r^3} & 0\end{bmatrix}\mathbb{P}_{\mathsf{W}}\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}, \tag{27}$$

where c_1, c_2, and c_3 are λ-dependent integration constants. Here $(\Theta_n)_{ij}$ represents the ij entry of the nth-order matrix coefficient of the power series expansion of Θ, and $\mathbb{P}_{\mathsf{W}}$ is defined as

$$\mathbb{P}_{\mathbb{W}}(r) = \sum_{n=0}^{+\infty}\mathbb{P}_n r^n,$$
$$\mathbb{P}_0 = \mathbb{I}_3,$$
$$\mathbb{P}_k = \frac{1}{k}\sum_{j=0}^{k-1}\mathbb{A}_{k-1-j}\mathbb{P}_j, \quad \text{for } k\geq 1, \tag{28}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, the matrix $\mathbb{A}$ is real analytic with the same radius of convergence as Θ, and $\mathbb{A}_n$ represents the nth-order matrix coefficient of its power series expansion, that is, $\mathbb{A}(r)=\sum_{n=0}^{+\infty}\mathbb{A}_n r^n$. Due to its size, the matrix $\mathbb{A}$ is presented in Appendix D.

Using $\mathbb{P}_0=\mathbb{I}_3$ in Eq. (27), we can immediately compute the lower-order coefficients of the power series expansion of $\mathbb{W}$. Imposing the boundary condition at the center sets the coefficient c_2 to be zero; otherwise, the pressure would diverge at $r=0$ at all times. Then, we find

$$\begin{bmatrix}\Psi_{\mathsf{p}}^{(\lambda)}\\ \Psi_{\mathsf{A}}^{(\lambda)}\\ \Psi_{\Sigma}^{(\lambda)}\end{bmatrix} = \begin{bmatrix}-c_1+\mathcal{O}(r^2)\\ c_3 r+\mathcal{O}(r^3)\\ \mathcal{O}(r^2)\end{bmatrix}. \tag{29}$$

We see that the coefficient c_1 directly characterizes the behavior of $\Psi_{\mathsf{p}}^{(\lambda)}$ at $r=0$ and that both $\Psi_{\mathsf{A}}^{(\lambda)}$ and $\Psi_{\Sigma}^{(\lambda)}$ must vanish at the center. On the other hand, the coefficients c_1 and c_3 are not independent: considering the regularity of the background spacetime and imposing the constraints (15), (18), and (19) leads to

$$c_3 \overset{r=0}{=} -\frac{c_1}{3(\mu_0+p_0)f'(\mu_0)} \times\left[\frac{\lambda^2}{(g_0)_{tt}}+\frac{1}{3}\mu_0+\frac{3}{2}(\mu_0+p_0)f'(\mu_0)\right], \tag{30}$$

[1] We should stress, at this point, that the radius of convergence of the above power series, as well as the ones shown below, may be smaller than the radius of the equilibrium star. In such a case, the method we proposed will give valid results only within such a radius of convergence, and the solution must be completed with numerical methods.

where all quantities on the right-hand side are to be evaluated at $r = 0$. Therefore, for each value of λ^2 there is a single arbitrary parameter, either c_1 or c_3, to be characterized by the initial perturbation at the center of the star. That is, the coefficients of the Fourier transform of the initial perturbation provide the values of the independent parameters, setting which eigenmodes are excited and the respective initial magnitude. In what follows we will consider c_1 as the independent parameter. Specifying the background spacetime, the equation of state of the perturbed fluid and the values of the eigenfrequencies λ, these results allow us to find regular analytic solutions for the perturbations that verify the boundary conditions.

In addition to the analytical results for the eigenfunctions, in Ref. [9] it was possible to establish lower bounds for the square of the eigenfrequencies, λ^2. In particular, if the regularity conditions in the beginning of this subsection and the boundary conditions (i) and (ii) hold, nontrivial $\mathcal{C}^1$ solutions of the system (12)–(19) exist only if

$$\lambda^2 \max_{r\in]0,r_b[} (g_0)_{tt} > - \max_{r\in]0,r_b[} \left[\frac{\mu_0 + p_0}{\phi_0} \left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0 \right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{r\phi_0 \mathcal{A}_0}{2(\mu_0 + p_0)} \frac{d\mu_0}{dr} \right]. \tag{31}$$

This result can be useful to determine numerically the eigenfrequencies of the system, offering a baseline to search for their values.

III. PERTURBATIONS AND FUNDAMENTAL EIGENFREQUENCIES OF CLASSIC EXACT SOLUTIONS

As illustrative examples of the general results in the previous section, we will study the properties of adiabatic, radial perturbations of some classical solutions of perfect fluid stars within general relativity. We consider spacetimes that verify the four regularity criteria presented in Ref. [22]. Namely, we will study the stability of specific Interior Schwarzschild, Tolman IV, Kuchowicz 2-III, and Heintzmann IIa solutions. Since the independent parameter c_1 simply characterizes the magnitude of a specific eigenfunction at $r = 0$, without loss of generality, in this section we will set $c_1 = -1$.

In Table I, considering a line element of the form (1), we present the nontrivial metric coefficients. In Table II we present the absolute value of the first three eigenfrequencies for specific models for the equilibrium background spacetime. For the interior Schwarzschild solution, at linear order, the perturbed fluid is completely characterized by a constant speed of sound, which must be provided as an extra parameter. For the other models, we assume that the equation of state of the perturbed fluid is the same as that of the equilibrium setup. For all considered models, the eigenfrequencies take real values; therefore, all equilibrium spacetimes represent stable configurations under adiabatic, radial perturbations.

In Figs. 1–4 we present the radial profile of the Fourier coefficients of the functions p, A, and Σ, associated with the eigenfrequencies presented in Table II for the various background spacetimes. Figures 1–4 highlight the expected behavior for the eigenfunctions. For a real-analytic background spacetime, in Ref. [9] it was shown that the perturbation equations can be cast in the form of a Sturm-Liouville eigenvalue problem. Therefore, in particular, the number of roots of the eigenfunctions is associated with the order of the associated eigenvalue in the sequence $(\lambda_n^2)_{n\in\mathbb{N}}$.

Except for the interior Schwarzschild solution, the results in Table II were compared with the predictions of the systems in Refs. [4,6]. Implementing a shooting method to solve numerically each of those systems for each equilibrium spacetime, all values for the fundamental

TABLE I. Metric coefficients of classical solutions of the Einstein field equations. The spacetimes are assumed to be characterized by a line element of the form of Eq. (1). We follow the naming conventions for the solutions of Ref. [22].

Spacetime	Nontrivial metric components
Interior Schwarzschild	$(g_0)_{tt} = \left(3\sqrt{1-\frac{2M}{r_b}} - \sqrt{1-\frac{2Mr^2}{r_b^3}}\right)^2$ $(g_0)_{rr} = \left(1-\frac{2Mr^2}{r_b^3}\right)^{-1}$
Tolman IV	$(g_0)_{tt} = B^2\left(\frac{r^2}{A^2}+1\right)$ $(g_0)_{rr} = \frac{\frac{2r^2}{A^2}+1}{\left(1+\frac{r^2}{A^2}\right)\left(1-\frac{r^2}{R^2}\right)}$
Kuch2-III	$(g_0)_{tt} = Be^{\frac{Ar^2}{2}}$ $(g_0)_{rr} = (r^2 e^{-\frac{1}{2}Ar^2}[C - \frac{A}{2e}\mathrm{Ei}(\frac{Ar^2}{2}+1)]+1)^{-1}$
Heint IIa	$(g_0)_{tt} = A^2(ar^2+1)^3$ $(g_0)_{rr} = \left(1-\frac{3ar^2[c(4ar^2+1)^{-\frac{1}{2}}+1]}{2(ar^2+1)}\right)^{-1}$

TABLE II. First absolute values of the eigenfrequencies, λ, rounded to three decimal places, for the equilibrium solutions in Table I for specific values of the spacetime parameters. In all examples, the eigenfrequencies are real; hence, all spacetimes are stable under adiabatic, radial perturbations. We follow the naming conventions for the solutions of Ref. [22].

Spacetime	Parameters	$\lvert\lambda_0\rvert$	$\lvert\lambda_1\rvert$	$\lvert\lambda_2\rvert$
Interior Schwarzschild	$(M, r_b, c_s^2) = (0.1, 1, 0.1)$	0.108	0.305	0.478
Tolman IV	$(A, B, R) = (1, 1, 1.5)$	1.533	4.281	6.723
Kuch2-III	$(A, B, C) = (5, 1, -3)$	20.214	41.085	61.808
Heint IIa	$(a, A, C) = (1, 1, 1.5)$	4.004	10.262	15.939

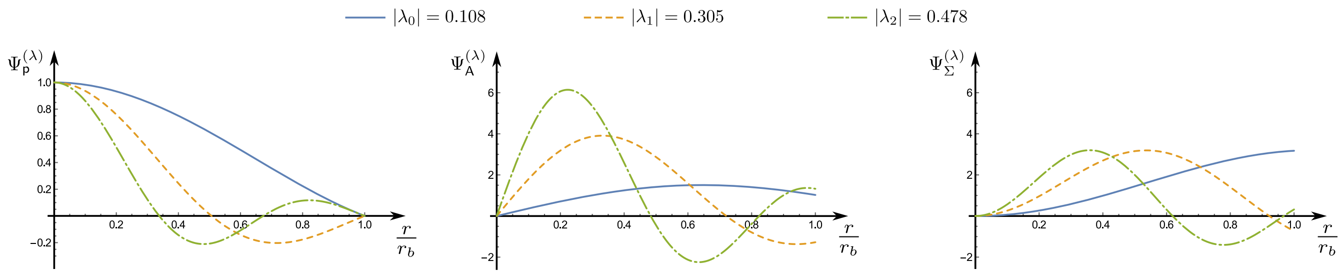

FIG. 1. Radial profile of the Fourier coefficients of the functions p, A, and Σ, associated with the eigenfrequencies presented in Table II for the interior Schwarzschild spacetime. For all oscillation modes, it was assumed $c_1 = -1$.

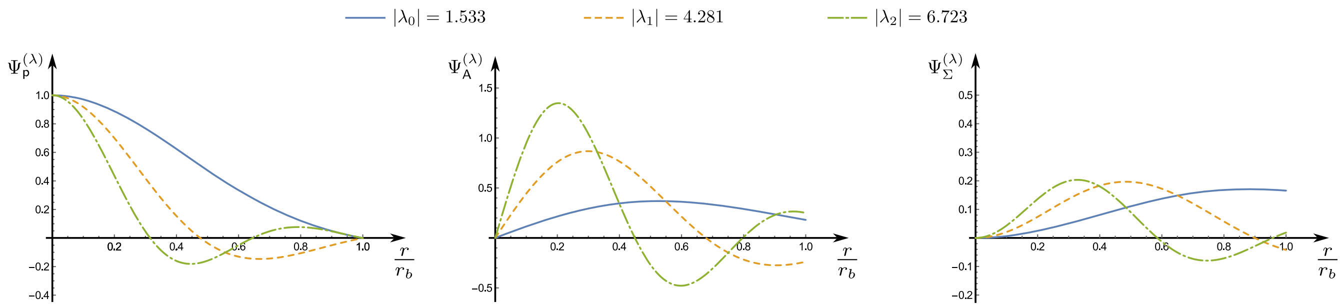

FIG. 2. Radial profile of the Fourier coefficients of the functions p, A, and Σ, associated with the eigenfrequencies presented in Table II for the Tolman IV spacetime. For all oscillation modes, it was assumed $c_1 = -1$.

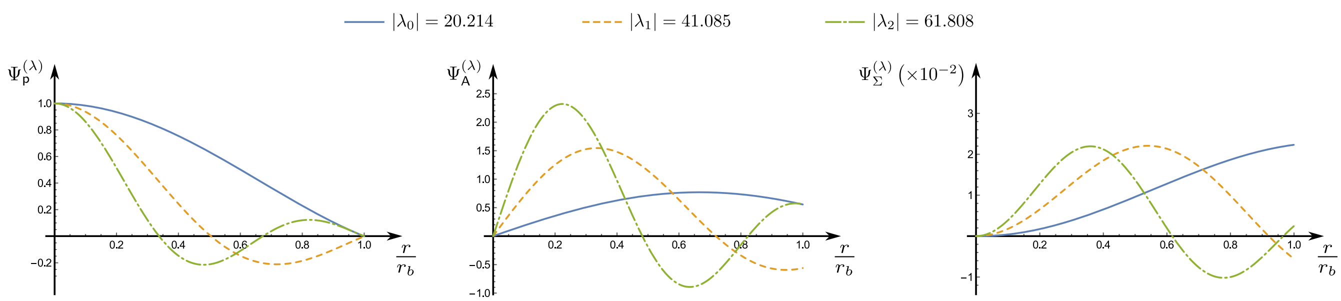

FIG. 3. Radial profile of the Fourier coefficients of the functions p, A, and Σ, associated with the eigenfrequencies presented in Table II for the Kuchowicz 2-III spacetime. For all oscillation modes, it was assumed $c_1 = -1$.

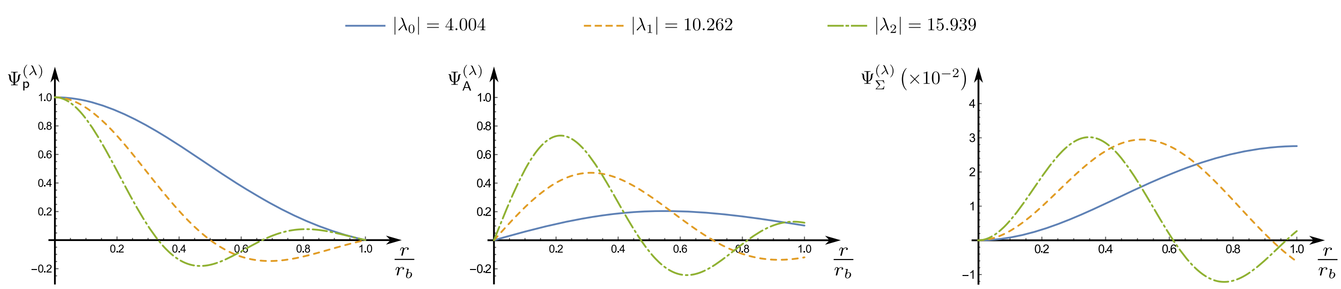

FIG. 4. Radial profile of the Fourier coefficients of the functions p, A, and Σ, associated with the eigenfrequencies presented in Table II for the Heintzmann IIa spacetime. For all oscillation modes, it was assumed $c_1 = -1$.

eigenfrequencies matched to the considered numerical accuracy. Moreover, the general solutions found in Sec. II are regular around the center, and for the spacetime parameters in Table II, the solutions are exact and the radius of convergence of the power series is greater than the radius of the star. Hence, we can explicitly evaluate the boundary conditions at any point within the star, in particular at the center and the surface.

IV. BOUND ON MAXIMUM COMPACTNESS

The dynamical stability of an equilibrium static, self-gravitating fluid is a crucial problem in astrophysics. Finding that a given solution of the Einstein field equations is dynamically unstable for some type of perturbation implies that such a solution might not be suitable to describe massive compact objects that are expected to be empirically observable in the current universe. In that regard, considerable effort has been devoted to finding an upper bound for the maximum compactness of a star: the ratio between the gravitational mass and the circumferential radius of a star, beyond which it is unstable (cf., e.g., Refs. [26,27] for a review for perfect and nonperfect fluids, and Ref. [28] and references therein for the case of electrically charged self-gravitating fluids). Nonetheless, the results in the literature are fundamentally connected to the equation of state of the equilibrium fluid and, to our knowledge, no universal upper bound has been found for the maximum compactness of these types of objects. In the particular case of perfect fluid stars, we can use the previous results and the interior Schwarzschild solution to conjecture an equation of state agnostic upper bound for the dynamical stability of this type of objects.

The interior Schwarzschild spacetime represents a solution of general relativity describing the interior of a star composed of a perfect fluid with a constant energy density that can be smoothly matched with an exterior vacuum Schwarzschild spacetime. This solution can be characterized by a line element of the form (1), with

$$(g_0)_{tt} = \left(3\sqrt{1-\frac{2M}{r_b}} - \sqrt{1-\frac{2Mr^2}{r_b^3}}\right)^2,$$
$$(g_0)_{rr} = \left(1-\frac{2Mr^2}{r_b^3}\right)^{-1}, \tag{32}$$

where M represents the Arnowitt–Deser– Misner mass and r_b the value of the circumferential radius at the boundary of the star. Although the interior Schwarzschild solution does not represent a physically reasonable configuration, this solution is important from a conceptual point of view since it saturates the Buchdahl bound for the maximum compactness:

$$\frac{M}{r_b} \leq \frac{4}{9} = 0.\bar{4}, \tag{33}$$

hence, it can be thought of as the extreme scenario for a static, self-gravitating perfect fluid. On this basis, this solution can be used to conjecture the maximum compactness of a star composed of a perfect fluid, beyond which it becomes dynamically unstable.

Considering the results of the previous section, we can study the dynamical stability of the interior Schwarzschild solution by computing the fundamental eigenfrequency for various values of the compactness parameter $\frac{M}{r_b}$. Notice that, for an interior Schwarzschild background spacetime with a specific value of the compactness parameter, the value of r_b does not affect the values of the eigenfrequencies, but only the radial profile of the perturbation variables. Since the energy density of the background is constant, at a linear level, the square of the speed of sound, $f'(\mu_0)$, is a constant. Then, to infer the maximum compactness of a physical star described by the interior Schwarzschild solution, we can impose the causality condition and consider the extreme scenario where $f'(\mu_0) = c_s^2 = 1$; that is, the square of the speed of sound of the perturbed star is the vacuum speed of light. Applying the previous results, we find that

$$M/r_b = 0.367 \Rightarrow \lambda_0^2 > 0, \qquad M/r_b = 0.368 \Rightarrow \lambda_0^2 < 0. \tag{34}$$

Following this reasoning, we extrapolate that a static, spherically symmetric solution of the Einstein field equations with a perfect fluid source is dynamically unstable if

$$\frac{M}{r_b} \gtrsim 0.368. \tag{35}$$

The accuracy of the above estimate, of course, can be increased, but this level of accuracy is unlikely to be achieved experimentally. The result in Eq. (35) is significantly smaller than the Buchdahl bound, Eq. (33). Moreover, it is slightly higher, but in line with the estimates of Refs. [29,30] for the maximum compactness, found by considering an affine fluid model and imposing the hypothesis that such a model yields the most compact star composed of a perfect fluid verifying a barotropic equation of state. The analysis in this section, however, is independent of the equation of state of the equilibrium star and follows from simply considering the extreme case of the causality condition for the perturbed fluid. Consequently, this upper bound for instability is universal, and it is not expected to be saturated by any perfect fluid star solution verifying a physically meaningful equation of state.

V. CONCLUSION

We have presented a system of first-order differential equations to describe general linear adiabatic, radial perturbations of spatially compact, static, spherically symmetric solutions of general relativity with a perfect fluid source. The new results do not rely on auxiliary test functions nor on the introduction of a singular point at the boundary of the star. Contrary to previous approaches, assuming some regularity conditions for the equilibrium spacetime, the system can be solved analytically, finding general solutions for the perturbation variables.

The results were then used to study adiabatic, radial perturbations of classical exact solutions of the Einstein field equations, computing the first eigenfrequencies for particular values of the spacetime parameters. For the considered models, the equilibria proved to be stable. We have also plotted the eigenfunctions associated with those eigenfrequencies, illustrating their radial profile. At first glance, the plots in Figs. 1–4 look remarkably similar. However, this is a product of the very nature of the Sturm-Liouville problem, the constraints imposed by the boundary conditions, and the normalized radial coordinate we have used.

We have considered only a small subset of physically relevant exact solutions, leaving aside some important solutions that have been used in the past to study physically meaningful scenarios. As was discussed in Ref. [9], although the thermodynamical description of the perturbed matter fluid is simpler in a comoving frame, this leads to extra complexity to describe its dynamics such that we end up with a three-by-three system of differential equations with a constraint, Eqs. (12)–(15). Indeed, the formulation of the problem can be made more computationally efficient. By changing the frame of reference, we can significantly simplify the problem and efficiently study adiabatic, radial perturbations of more complex background solutions. In other frames, however, the stress-energy tensor is non-diagonal, containing fluxes and anisotropic components. Therefore, great care has to be taken to ascertain what is meant by adiabatic perturbations in the new frame. This will be done elsewhere.

The newly found system of equations also allowed us to conjecture, independently of the information on the equation of state of the equilibrium fluid, the upper bound $M/r_b \approx 0.368$ for the maximum compactness of a stable, static, self-gravitating perfect fluid. The result relies on the hypothesis that the interior Schwarzschild solution represents the extreme case for a perfect fluid star. Hence, the maximum bound for the compactness should be absolute, in the sense that no physically meaningful solution of general relativity is expected to saturate it. Nonetheless, the result is a tighter upper bound compared to the previously suggested bounds in the literature, found by considering a specific barotropic equation of state. We remark, however, that the reasoning for this result omits various important issues on the stability of compact stellar objects. One example concerns the evolutionary timescales. The eigenfrequencies $\{\lambda_n\}_{n\in\mathbb{N}}$ can be used to determine the oscillation period or the period of e-folds of each mode. In the case of $\lambda_0 \in \mathbb{R}$, the star will oscillate, and the period of oscillation for the fundamental mode is given by $T = 2\pi/|\lambda_0|$. On the other hand, if λ_0 is purely imaginary, the perturbations will not be bounded and the quantity $T = 2\pi/|i\lambda_0|$ represents the period of e-folds. Since the fundamental mode has the shortest growth timescale, we can assess the growth timescale of the radial instabilities in the considered idealized setup, considering this mode. Nonetheless, the evolutionary timescale of compact stellar objects depends on various factors omitted in our analysis, where we have assumed the idealized scenario of an adiabatic perturbation. Indeed, the thermodynamic evolution of the matter fields plays a predominant role in the evolution of those objects, such that even if the star is pulsationally unstable, the radial modes might not have time to grow and affect the structure. Moreover, the previous discussion does not consider the issue of quasistability. In principle, a dynamically unstable object, but such that its evolution would have a characteristic time comparable to the timescale of the universe, might be effectively considered stable. A realistic analysis of the topic should take into consideration all these aspects. The presented upper bound for dynamical stability should then be valid in an idealized scenario, disregarding any other effects.

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APPENDIX A: THE 1+1+2 DECOMPOSITION

1. Projectors and the Levi-Civita volume form

Let us consider, in a Lorentzian manifold $(\mathcal{M}, g)$ of dimension four endowed with a metric g, two local congruences, one composed of timelike curves with tangent vector field u, and one composed by spacelike curves with tangent vector field e. Let the curves of each congruence be affinely parametrized and assume $u_\alpha u^\alpha = -1$ and $e_\alpha e^\alpha = 1$.

If we foliate the manifold by 3-surfaces, V, pointwise orthogonal to the curves of the timelike congruence, all tensorial quantities that characterize $(\mathcal{M}, g)$ are defined by their behavior along the direction of u and in V. These quantities can be proven to satisfy a set of equations which is equivalent to the Einstein equations. Such an approach is called $1+3$ spacetime covariant decomposition. This decomposition is performed by means of a projector tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \tag{A1}$$

with the following properties:

$$h_{\alpha\beta} = h_{\beta\alpha}, \qquad h_{\alpha\beta}h^{\beta\gamma} = h_\alpha{}^\gamma,$$
$$h_{\alpha\beta}u^\alpha = 0, \qquad h_\alpha{}^\alpha = 3. \tag{A2}$$

Using the spacelike congruence we can perform a further foliation so that a given tensorial quantity in V is characterized by its behavior along e and the 2-surfaces W, pointwise orthogonal to the curves of the spacelike congruence. As before, such decomposition is performed by means of a projector tensor defined, in this case, as

$$N_{\alpha\beta} = h_{\alpha\beta} - e_\alpha e_\beta, \tag{A3}$$

verifying

$$N_{\alpha\beta} = N_{\beta\alpha}, \qquad N_{\alpha\beta}N^{\beta\gamma} = N_\alpha{}^\gamma,$$
$$N_{\alpha\beta}u^\alpha = N_{\alpha\beta}e^\alpha = 0, \qquad N_\alpha{}^\alpha = 2. \tag{A4}$$

The characterization of all the tensorial quantities on V and W requires the introduction of the skew-symmetric tensors

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\sigma}u^\sigma,$$
$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}e^\gamma, \tag{A5}$$

where $\varepsilon_{\alpha\beta\gamma\sigma}$ is the covariant Levi-Civita tensor.

In what follows, we will adopt the convention to indicate the symmetric and antisymmetric parts of a tensor using parentheses and brackets, such that for a 2-tensor χ

$$\chi_{(\alpha\beta)} = \frac{1}{2}(\chi_{\alpha\beta} + \chi_{\beta\alpha}), \qquad \chi_{[\alpha\beta]} = \frac{1}{2}(\chi_{\alpha\beta} - \chi_{\beta\alpha}). \tag{A6}$$

2. Covariant derivatives of *u* and *e*

Using the projector operators h and N, the covariant derivatives of the tangent vector fields u and e can be uniquely decomposed at each point in their components along u, along e, and in W. This decomposition yields the so-called kinematical quantities of the congruences formed by the integral curves of u and e, providing a clear geometric and physical interpretation of the behavior of the congruence.

To simplify the readability of the formulas in this paper we will introduce a compact notation for the derivatives along the integral curves of the vector fields u and e. Given a tensor quantity χ, we set

$$\dot{\chi} \coloneqq u^\mu \nabla_\mu \chi, \qquad \hat{\chi} \coloneqq e^\mu \nabla_\mu \chi. \tag{A7}$$

In the $1+1+2$ formalism, the covariant derivative of the tensor field u can be decomposed as

$$\nabla_\alpha u_\beta = -u_\alpha(\mathcal{A}e_\beta + \mathcal{A}_\beta) + \frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \tag{A8}$$

where

$$\mathcal{A} = -u_\mu u^\nu \nabla_\nu e^\mu, \qquad \mathcal{A}_\alpha = N_{\alpha\mu}\dot{u}^\mu, \tag{A9}$$

and the quantities θ, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$ are the kinematical quantities of the congruence of the integral curves of u. Namely, θ is the expansion scalar, $\sigma_{\alpha\beta}$ is the shear tensor, and $\omega_{\alpha\beta}$ is the vorticity tensor, defined as

$$\theta = h^{\mu\nu}\nabla_\mu u_\nu,$$
$$\sigma_{\alpha\beta} = \left(\frac{h_\alpha{}^\mu h_\beta{}^\nu + h_\alpha{}^\nu h_\beta{}^\mu}{2} - \frac{1}{3}h_{\alpha\beta}h^{\mu\nu}\right)\nabla_\mu u_\nu$$
$$= \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha}e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right),$$
$$\omega_{\alpha\beta} = \frac{1}{2}(h_\alpha{}^\mu h_\beta{}^\nu - h_\alpha{}^\nu h_\beta{}^\mu)\nabla_\mu u_\nu = \varepsilon_{\alpha\beta\mu}(\Omega e^\mu + \Omega^\mu), \tag{A10}$$

with

$$\Sigma_{\alpha\beta} = \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)\sigma_{\mu\nu},$$
$$\Sigma_\alpha = N_\alpha{}^\mu e^\nu \sigma_{\mu\nu}, \qquad \Sigma = e^\mu e^\nu \sigma_{\mu\nu}, \tag{A11}$$

and

$$\Omega^\alpha = \frac{1}{2}N_\gamma{}^\alpha \varepsilon^{\mu\nu\gamma}\nabla_\mu u_\nu, \qquad \Omega = \frac{1}{2}\varepsilon^{\mu\nu}\nabla_\mu u_\nu. \tag{A12}$$

From their definitions, it is immediate to conclude that the covariantly defined vector and 2-tensor quantities characterize the behavior of the kinematical quantities on the surfaces W, whereas the scalars characterize the behavior along u or e.

Similarly, we can decompose the covariant derivative of e along u, along e, and onto W as

$$\nabla_\alpha e_\beta = \frac{1}{2}N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi + e_\alpha a_\beta - u_\alpha \alpha_\beta - \mathcal{A}u_\alpha u_\beta$$
$$+ \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha u_\beta + (\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu)u_\beta, \tag{A13}$$

where

$$\phi = N^{\mu\nu}\nabla_\mu e_\nu,$$
$$\zeta_{\alpha\beta} = \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)\nabla_\mu e_\nu,$$
$$\xi = \frac{1}{2}\varepsilon^{\mu\nu}\nabla_\mu e_\nu \tag{A14}$$

represent, respectively, the expansion scalar, the shear tensor, and the twist of the congruence of the integral curves of e projected on W, and

$$a_\alpha = e^\mu h_\alpha{}^\nu \nabla_\mu e_\nu, \qquad \alpha_\alpha = u^\mu h_\alpha{}^\nu \nabla_\mu e_\nu. \tag{A15}$$

3. Weyl and stress-energy tensors

In the covariant formalism the Weyl tensor has a very important role. In four spacetime dimensions, this tensor is defined by decomposing the Riemann curvature tensor, with components $R_{\alpha\beta\gamma\delta}$, as

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + R_{\alpha[\gamma}g_{\delta]\beta} - R_{\beta[\gamma}g_{\delta]\alpha} - \frac{1}{3}Rg_{\alpha[\gamma}g_{\delta]\beta}, \tag{A16}$$

where $C_{\alpha\beta\gamma\delta}$ represent the components of the Weyl tensor in a local coordinate system. In general, $C_{\alpha\beta\gamma\delta}$ can be completely characterized by its "electric" and "magnetic" parts which are both symmetric and traceless tensors on V:

$$C_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}u_{\delta]} + 2u_\beta E_{\alpha[\gamma}u_{\delta]} - 2\varepsilon_{\alpha\beta\mu}H^\mu{}_{[\gamma}u_{\delta]} - 2\varepsilon_{\mu\gamma\delta}H^\mu{}_{[\alpha}u_{\beta]}. \tag{A17}$$

In the $1+1+2$ covariant formalism, $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are further decomposed as follows:

$$\begin{aligned} E_{\alpha\beta} &= \mathcal{E}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{E}_\alpha e_\beta + e_\alpha\mathcal{E}_\beta + \mathcal{E}_{\alpha\beta}, \\ H_{\alpha\beta} &= \mathcal{H}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{H}_\alpha e_\beta + e_\alpha\mathcal{H}_\beta + \mathcal{H}_{\alpha\beta}, \end{aligned} \tag{A18}$$

where

$$\begin{aligned} \mathcal{E} &= E_{\mu\nu}e^\mu e^\nu = -N^{\mu\nu}E_{\mu\nu}, & \mathcal{H} &= e^\mu e^\nu H_{\mu\nu} = -N^{\mu\nu}H_{\mu\nu}, \\ \mathcal{E}_\alpha &= N_\alpha{}^\mu e^\nu E_{\mu\nu} = e^\mu N_\alpha{}^\nu E_{\mu\nu}, & \mathcal{H}_\alpha &= N_\alpha{}^\mu e^\nu H_{\mu\nu} = e^\mu N_\alpha{}^\nu H_{\mu\nu}, \\ \mathcal{E}_{\alpha\beta} &= E_{\{\alpha\beta\}}, & \mathcal{H}_{\alpha\beta} &= H_{\{\alpha\beta\}}. \end{aligned} \tag{A19}$$

Last, to write the Einstein field equations in the language of the $1+1+2$ formalism, we need the covariant decomposition of the metric stress-energy tensor $T_{\alpha\beta}$, which reads

$$T_{\alpha\beta} = \mu u_\alpha u_\beta + (p+\Pi)e_\alpha e_\beta + \left(p - \frac{1}{2}\Pi\right)N_{\alpha\beta} + 2Qe_{(\alpha}u_{\beta)} + 2Q_{(\alpha}u_{\beta)} + 2\Pi_{(\alpha}e_{\beta)} + \Pi_{\alpha\beta}, \tag{A20}$$

with

$$\begin{aligned} \mu &= u^\mu u^\nu T_{\mu\nu}, & Q_\alpha &= -N_\alpha{}^\mu u^\nu T_{\mu\nu}, \\ p &= \frac{1}{3}(e^\mu e^\nu + N^{\mu\nu})T_{\mu\nu}, & \Pi_\alpha &= N_\alpha{}^\mu e^\nu T_{\mu\nu}, \\ \Pi &= \frac{1}{3}(2e^\mu e^\nu - N^{\mu\nu})T_{\mu\nu}, & \Pi_{\alpha\beta} &= \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)T_{\mu\nu}. \\ Q &= -e^\mu u^\nu T_{\mu\nu}, \end{aligned} \tag{A21}$$

The various contributions in the covariant decomposition of the stress-energy tensor in Eq. (A21) have direct physical meaning. Given an observer with 4-velocity u, μ represents the mass-energy density of the fluid; p is the isotropic pressure; Q and Q_α represent, respectively, heat and momentum flows along e and in W; and Π, Π_α, and $\Pi_{\alpha\beta}$ characterize the anisotropic pressure within the fluid.

APPENDIX B: 1+1+2 SCALAR PERTURBATION EQUATIONS

Using the $1+1+2$ formalism, briefly introduced in Appendix A, we list here the covariant version of the equations that describe the scalar adiabatic perturbations of a fluid distribution in a static, LRS II background. These equations are written in terms of the variables in Eq. (5) and characterize the perturbation from the point of view of an observer locally comoving with the matter composing the relativistic compact stellar object. They read

$$\begin{aligned} \hat{\mathsf{A}} - \ddot{\theta} = \frac{1}{2}(\mathsf{m} + 3\mathsf{p}) + \hat{\mathcal{A}}_0\left(\frac{1}{3}\theta + \Sigma\right) \\ - (3\mathcal{A}_0 + \phi_0)\mathsf{A} - \mathcal{A}_0\mathsf{F}, \end{aligned} \tag{B1}$$

$$\hat{\mathsf{p}} = \left(\frac{1}{3}\theta + \Sigma\right)\hat{p}_0 - \mathcal{A}_0(\mathsf{m} + 2\mathsf{p}) - (\mu_0 + p_0)\mathsf{A}, \tag{B2}$$

$$\ddot{\Sigma} - \frac{2}{3}\ddot{\theta} = \frac{1}{3}(\mathsf{m} + 3\mathsf{p}) - \mathsf{E} - \mathcal{A}_0\mathsf{F} - \phi_0\mathsf{A}, \tag{B3}$$

$$\frac{2}{3}\hat{\theta} - \hat{\Sigma} = \frac{3}{2}\phi_0\Sigma. \tag{B4}$$

$$\mathsf{E} = \mathcal{E}_0\left(\frac{3}{2}\Sigma - \theta\right) - \frac{1}{2}(\mu_0 + p_0)\Sigma, \tag{B5}$$

$$\mathsf{F} = (2\mathcal{A}_0 - \phi_0)\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right), \tag{B6}$$

$$\mathsf{m} = -(\mu_0 + p_0)\theta. \tag{B7}$$

Upon reorganization, using the background gravitational field equations and applying a harmonic decomposition of the perturbation variables, the above equations can be recast into the system in Sec. II C.

The perturbation equations become increasingly more complicated as we relax the constraints that we have assumed before. The general set of perturbation equations for the variables in Eq. (5), together with additional perturbation variables useful in other specific frames, can be found in [9].

APPENDIX C: DERIVATION OF CHANDRASEKHAR'S RADIAL PULSATION EQUATION

In this appendix, we show how the system of equations for the comoving observer in Appendix B returns the well-known Chandrasekhar radial pulsation equation [1,2]. We will then start by recalling the classical derivation of the Chandrasekhar equation. Then, we will break covariance and gauge invariance in the $1+1+2$ perturbation equations to prove that they lead to the same result.

Let us start by constructing the Chandrasekhar pulsation equation considering perturbations of the metric directly. Consider the Schwarzschild coordinate system (t, r, ψ, φ) defined by an observer at spatial infinity, such that events in both the equilibrium and the perturbed spacetime can be identified unambiguously. Let the equilibrium spacetime be characterized by a line element of the form

$$ds^2 = (g_0)_{\alpha\beta}dx^\alpha dx^\beta = -e^{2\Phi_0(r)}dt^2 + e^{2\Lambda_0(r)}dr^2 + r^2 d\Omega^2, \tag{C1}$$

where $d\Omega^2 = d\psi^2 + \sin^2\psi d\varphi^2$ represents the natural line element of the unit 2-sphere. We will adopt the nomenclature of the body of the text and indicate quantities in the background spacetime by a subscript "0."

Assuming the equilibrium spacetime to be permeated by a perfect fluid, such that, from the point of view of an observer at rest with matter, the energy-momentum tensor will be written as in Eq. (3)

$$T_{\alpha\beta} = (\mu_0 + p_0)(u_0)_\alpha(u_0)_\beta + p_0(g_0)_{\alpha\beta}, \tag{C2}$$

where

$$(u_0)_\alpha dx^\alpha = -e^{\Phi_0}dt, \tag{C3}$$

the Einstein field equations are

$$\Lambda_0' = \frac{1}{2}e^{2\Lambda_0}\mu_0 r - \frac{e^{2\Lambda_0}}{2r} + \frac{1}{2r}, \tag{C4}$$

$$\Phi_0' = \frac{1}{2}e^{2\Lambda_0}p_0 r + \frac{e^{2\Lambda_0}}{2r} - \frac{1}{2r}, \tag{C5}$$

$$\Phi_0'' = \frac{1}{4}e^{2\Lambda_0}\mu_0 + \frac{1}{4}e^{4\Lambda_0}\mu_0 + \frac{5}{4}e^{2\Lambda_0}p_0 - \frac{3}{4}e^{4\Lambda_0}p_0 + \frac{1}{4}e^{4\Lambda_0}\mu_0 p_0 r^2 - \frac{1}{4}e^{4\Lambda_0}p_0^2 r^2 - \frac{e^{4\Lambda_0}}{2r^2} + \frac{1}{2r^2}, \tag{C6}$$

and the Bianchi identity

$$p_0' = -\Phi_0(\mu_0 + p_0). \tag{C7}$$

Now, radially perturbing the equilibrium configuration leads to a new metric associated with the line element

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -e^{2\Phi(t,r)}dt^2 + e^{2\Lambda(t,r)}dr^2 + r^2 d\Omega^2, \tag{C8}$$

and the energy-momentum tensor for an observer comoving with matter will be described by

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta}, \tag{C9}$$

where u_α represent, in a local coordinate system, the components of the 4-velocity form of an observer comoving with the perturbed fluid. The perturbation will induce a change in a given scalar variable X, which can be characterized by the quantity

$$\delta X = X(t, r) - X_0(r). \tag{C10}$$

This expression constitutes the Eulerian representation of perturbations, i.e., the description of the perturbation from the point of view of an observer in a position with a constant r coordinate.

In addition to the δ-variations, we need to introduce an extra parameter to describe the radial displacement of a fluid element. A fluid element at the coordinate r in the unperturbed spacetime is moved to a new position $\tilde{r}$, in the perturbed spacetime, such that

$$\tilde{r} = r + \eta(t, r). \tag{C11}$$

The parameter η is usually called "radial displacement." We need to be especially careful in understanding the meaning of η. Even if this function is connected to the choice of a local coordinate system, η effectively also characterizes the mapping between the unperturbed and perturbed spacetimes, as it is relabeling events. Therefore, it cannot be just considered a product of a coordinate transformation, but rather a gauge. For this reason, η is often called "gauge parameter." In this appendix, for example, in our construction, we have tacitly considered the Schwarzschild coordinate system to write the components of the perturbed metric and those of the background metric. Therefore, we will find that the perturbations depend directly on the gauge parameter. This, of course, is not a necessity, but it is useful in the treatment of this type of perturbations. We stress, however, that this approach should be used with caution. In the case of adiabatic, radial perturbations, it is indeed possible to define the same coordinate system in both the equilibrium and the perturbed spacetimes. Therefore, the conclusions are not ambiguous. However, in general this is not the case, and attributing physical observable effects to active coordinate transformations might lead to gauge-dependent, physically meaningless conclusions. In addition, choosing from the start the coordinate system of the equilibrium and the perturbed spacetimes will force the analysis to a specific gauge, which might lead to spurious complexity of the equations.

In terms of Eulerian perturbations, at first perturbative order, that is, disregarding quadratic and higher-order terms of the δ-variations, we have

$$u_\alpha dx^\alpha = -e^{\Phi_0}(1+\delta\Phi)dt + e^{2\Lambda_0-\Phi_0}\eta\eta^* dr, \tag{C12}$$

where we have indicated the derivative with respect to t with an "asterisk." Then the perturbed gravitational field equations associated with the metric Eq. (C8) read

$$(\delta\Lambda)^* = -(\Lambda_0' + \Phi_0')\eta^*, \tag{C13}$$

$$(\delta\Lambda)' = \left(2r\Lambda_0' - \frac{1}{r}\right)\delta\Lambda + \frac{1}{2}re^{2\Lambda_0}\delta\mu, \tag{C14}$$

$$(\delta\Phi)' = \left(2\Phi_0' + \frac{1}{r}\right)\delta\Lambda + \frac{1}{2}re^{2\Lambda_0}\delta p, \tag{C15}$$

$$\begin{aligned}(\delta\Phi)'' = {}& e^{2(\Lambda_0-\Phi_0)}\delta\Lambda^{**} + \frac{1}{2}e^{2\Lambda_0}(1+r\Phi_0')\delta\mu \\ &+ \frac{1}{2}e^{4\Lambda_0}(1+r\Lambda_0' - 2r\Phi_0')\delta p \\ &- \frac{e^{2\Lambda_0}}{r^2}[2e^{2\Lambda_0} - r\Lambda_0'(3+4r\Phi_0') \\ &+ r\Phi_0'(1+4r\Phi_0')]\delta\Lambda,\end{aligned} \tag{C16}$$

and the Bianchi identities

$$\delta\mu^* = \frac{1}{r}(-2\eta^* + r\eta^*\Phi_0' - r\eta'^*)(\mu_0+p_0) - \eta^*\mu_0', \tag{C17}$$

$$\begin{aligned}&(\delta p)' + \eta^{**}e^{2(\Lambda_0-\Phi_0)}(\mu_0+p_0) + (\delta\Phi)'(\mu_0+p_0) \\ &\quad + (\delta\mu+\delta p)\Phi_0' = 0,\end{aligned} \tag{C18}$$

where we have indicated the derivative with respect to r with a "prime." Integrating Eq. (C17) with respect to t leads to

$$\delta\mu = \frac{1}{r}(-2\eta + r\eta\Phi_0' - r\eta')(\mu_0+p_0) - \eta\mu_0', \tag{C19}$$

where we have set the integration constant to zero as we assume $\delta\mu = 0$ when $\eta = 0$. As we have mentioned before, this choice of integration constant follows from encoding the perturbations in the gauge parameter. Consistently, in the above expressions, if $\eta = 0$, we recover the background spacetime.

Continuing, the system above is closed by providing an equation of state that relates the pressure to the energy density. Following Chandrasekhar's original derivation, we will consider a barotropic equation of state, such that $p = f(\mu)$. Using the equation of state, Eq. (C19) allows us to write δp in terms of η and its derivatives:

$$\delta p = \frac{\Gamma_1 p_0}{r}(-2\eta + r\eta\Phi_0' - r\eta') - \eta p_0', \tag{C20}$$

where the quantity

$$\Gamma_1 = \frac{\mu_0+p_0}{p_0}f'(\mu_0) \tag{C21}$$

is the adiabatic index of the fluid. Integrating Eq. (C13) with respect to t, choosing the integration constant such that $\delta\Lambda = 0$ when $\eta = 0$, yields

$$\delta\Lambda = -(\Lambda_0' + \Phi_0')\eta. \tag{C22}$$

Substituting the above relations and Eq. (C15) in the momentum conservation equation (C18), after a laborious yet straightforward simplification, we obtain an equation for η, the so-called Chandrasekhar radial pulsation equation.

Alternatively to the Eulerian description, we can define a Lagrangian description of the perturbations. In this picture, an observer located at some point with radial coordinate r in the background spacetime is moved together with matter to $\tilde{r}$ in the perturbed spacetime. We then define a Lagrangian perturbation of a scalar quantity X as

$$\Delta X = X(t,\tilde{r}) - X_0(r). \tag{C23}$$

The $1+1+2$ perturbation variables we have used in this paper correspond, by definition, to Lagrangian perturbation variables. Indeed, since all the quantities in Sec. II C are defined from the point of view of a comoving observer, they are naturally Lagrangian.

Thus, to prove that the equations following from the covariant approach imply the Chandrasekhar equation, we have to break covariance and gauge invariance and show that the perturbation equations (B1)–(B4) reduce to the perturbed field equations or to the Bianchi identities (C13)–(C18). In that regard, we have to relate the Lagrangian variables $\{\mathcal{A}, \phi, \theta, \Sigma\}$ and $\{\mathsf{m}, \mathsf{p}, \mathsf{A}, \mathsf{F}, \mathsf{E}, \theta, \Sigma\}$ in terms of Eulerian variables to linear perturbation order.

We already know how to write, at first order, the 1-form associated with the vector field u in terms of the metric coefficients of the equilibrium and the perturbed spacetime, Eq. (C12). For the e congruence, we have

$$(e_0)_\alpha dx^\alpha = e^{\Lambda_0} dr,$$
$$e_\alpha dx^\alpha = -e^{\Lambda_0}\eta^* dt + e^{\Lambda_0}(1+\delta\Lambda) dr. \tag{C24}$$

Then, from the definitions in Appendix A and Eqs. (C3), (C12), and (C24) we obtain

$$\mathcal{A}_0 = e^{-\Lambda_0}\Phi_0', \tag{C25}$$

$$\phi_0 = \frac{2e^{-\Lambda_0}}{r}, \tag{C26}$$

$$\theta_0 = 0, \tag{C27}$$

$$\Sigma_0 = 0, \tag{C28}$$

and

$$\mathcal{A} = e^{-\Lambda_0}\Phi_0' + e^{-\Lambda_0}\delta\Phi' - e^{-\Lambda_0}\Phi_0'\delta\Lambda + \eta^{**}e^{\Lambda_0 - 2\Phi_0}, \tag{C29}$$

$$\phi = \frac{2e^{-\Lambda_0}}{r}(1-\delta\Lambda), \tag{C30}$$

$$\theta = \frac{e^{-\Phi_0}}{r}(r\delta\Lambda^* + 2\eta^* + r\eta^*\Lambda_0' + r\eta'^*), \tag{C31}$$

$$\Sigma = \frac{2}{3}\frac{e^{-\Phi_0}}{r}(r\delta\Lambda^* + \eta^* + r\eta^*\Lambda_0' + r\eta'^*). \tag{C32}$$

Now, the following relation holds between the Lagrangian and Eulerian perturbations of a scalar quantity X:

$$\Delta X = \delta X - X_0'\eta. \tag{C33}$$

This equation can be used to relate the Lagrangian variables $\{\mathsf{m}, \mathsf{p}, \mathsf{A}\}$ with their Eulerian counterparts. For the Lagrangian perturbation of the time derivative of the pressure, p, we find

$$\mathsf{p} = \dot{p} - \mathring{p}_0, \tag{C34}$$

where we have defined

$$\mathring{p} = (u_0)^\alpha \partial_\alpha p. \tag{C35}$$

Expressing the right-hand side of Eq. (C34) in terms of Eulerian perturbations yields

$$\mathsf{p} = (\delta p)^{\cdot} + p_0'\dot{\eta} = e^{-\Phi_0}[(\delta p)^* + p_0'\eta^*]. \tag{C36}$$

It is also useful to find the expression for the hat derivative of p in terms of the metric perturbations:

$$\hat{\mathsf{p}} = e^{-\Lambda_0}\Phi_0'\mathsf{p} - e^{-(\Phi_0+\Lambda_0)}[(\delta p)'^* + p_0''\eta^* + p_0'\eta'^*]. \tag{C37}$$

Similarly, we obtain the following expressions for the m and A quantities:

$$\mathsf{m} = e^{-\Phi_0}[(\delta\mu)^* + \mu_0'\eta^*], \tag{C38}$$

$$\mathsf{A} = -e^{-\Lambda_0 - 3\Phi_0}[e^{2\Phi_0}\Phi_0'\delta\Lambda^* - e^{2\Phi_0}(\delta\Phi)'^* + e^{2\Phi_0}(\Lambda_0'\Phi_0' - \Phi_0'')\eta^* - e^{2\Lambda_0}\eta^{***}]. \tag{C39}$$

Finally, using the above results, Eqs. (4), (B5), and (B6), we can obtain algebraically the expressions for $\mathcal{E}_0$, F, and E in terms of the coefficients of the metric in Eq. (C8) and their perturbations.

Gathering all the above results and substituting them in the perturbed momentum conservation equation for the comoving frame, Eq. (B2), yields, after some fairly long calculations, the momentum conservation equation (C18). Applying the same substitutions in the remaining equations of Appendix B yield a system of equations equivalent to combinations of the perturbed Einstein equations (C13)–(C22). Thus, we conclude that in the comoving frame, the covariant, gauge-invariant equations for adiabatic, radial perturbations are equivalent to the ones derived from the metric, coordinate-based approach.

APPENDIX D: MATRIX $\mathbb{A}$

Consider the matrix Θ defined in Eq. (25). To shorten the expressions, let Θ_{ij} represent the ij-entry of Θ, and $(\Theta_n)_{ij}$ be interpreted as the ij-entry of the nth-order coefficient of the power expansion of Θ at $r = 0$, Eq. (26). Then, matrix $\mathbb{A}$ in Eq. (28) is given by

$$\mathbb{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where

$$A_{11} = \Theta_{11} - \frac{1}{3}r^2\Theta_{31}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}], \tag{D1}$$

$$A_{12} = -\frac{36\Theta_{13}}{r^3} + \frac{12}{r^2}[\Theta_{12}(\Theta_0)_{23} - (\Theta_0)_{12}(\Theta_0)_{23} + 3(\Theta_1)_{13}] + \frac{12}{r}(\Theta_{33} - \Theta_{11})[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}]$$
$$+ \frac{2}{3}r^2\Theta_{32}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}]\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} + 2(\Theta_0)_{32}[(\Theta_0)_{23}]^2\}$$
$$- 4\Theta_{32}(\Theta_0)_{23}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}] - 12\Theta_{12}(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] + 18\Theta_{12}(\Theta_2)_{23}$$
$$- 4\Theta_{12}(\Theta_0)_{32}[(\Theta_0)_{23}]^2 + 4r\Theta_{31}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}]^2, \quad \text{(D2)}$$

$$A_{13} = \frac{1}{3}r^3\Theta_{32}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}] - r\Theta_{12}, \quad \text{(D3)}$$

$$A_{21} = -\frac{1}{36}r^3\Theta_{31}, \quad \text{(D4)}$$

$$A_{22} = \frac{1}{9}r^3\Theta_{32}\left\{3(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - \frac{9}{2}(\Theta_2)_{23} + [(\Theta_0)_{23}]^2(\Theta_0)_{32}\right\}$$
$$+ \frac{1}{3}r^2\Theta_{31}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}] - \frac{r}{3}(\Theta_0)_{23}\Theta_{32} + \Theta_{33}, \quad \text{(D5)}$$

$$A_{23} = \frac{1}{36}r^4\Theta_{32}, \quad \text{(D6)}$$

$$A_{31} = \frac{1}{9}\Theta_{31}\{3(\Theta_0)_{23}[r^2(\Theta_1)_{22} - r^2(\Theta_1)_{33} - 1] - \frac{9}{2}r^2(\Theta_2)_{23} + r^2(\Theta_0)_{32}[(\Theta_0)_{23}]^2\} - \frac{\Theta_{21}}{r}, \quad \text{(D7)}$$

$$A_{32} = -\frac{2r}{3}\Theta_{31}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}]\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} + 2[(\Theta_0)_{23}]^2(\Theta_0)_{32}\}$$
$$- r^2\Theta_{32}(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}]\left\{4(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 6(\Theta_2)_{23} + \frac{4}{3}[(\Theta_0)_{23}]^2(\Theta_0)_{32}\right\}$$
$$- r^2\Theta_{32}[(\Theta_0)_{23}]^2(\Theta_0)_{32}\left\{\frac{4}{3}(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 2(\Theta_2)_{23} + \frac{4}{9}[(\Theta_0)_{23}]^2(\Theta_0)_{32}\right\}$$
$$+ \frac{36}{r^4}[\Theta_{23} - (\Theta_0)_{23}] - \frac{12}{r^3}(\Theta_{22} - \Theta_{33})(\Theta_0)_{23} + \frac{12}{r^2}\Theta_{21}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}]$$
$$+ \frac{2}{r}(\Theta_{22} - \Theta_{33})\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} + 2[(\Theta_0)_{23}]^2(\Theta_0)_{32}\}$$
$$+ \frac{4}{3}(\Theta_0)_{23}\Theta_{32}\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} + 2[(\Theta_0)_{23}]^2(\Theta_0)_{32}\}$$
$$+ r^2\Theta_{32}(\Theta_2)_{23}\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} + 2[(\Theta_0)_{23}]^2(\Theta_0)_{32}\}$$
$$+ \frac{2}{r^2}\{6(\Theta_0)_{23}[(\Theta_1)_{22} - (\Theta_1)_{33}] - 9(\Theta_2)_{23} - 2[(\Theta_0)_{23}]^2[\Theta_{32} - (\Theta_0)_{32}]\}$$
$$+ \frac{4}{r}\Theta_{31}(\Theta_0)_{23}[(\Theta_0)_{12}(\Theta_0)_{23} - 3(\Theta_1)_{13}], \quad \text{(D8)}$$

$$A_{33} = \frac{r}{9}\Theta_{32}\left\{3(\Theta_0)_{23}(1 - r^2[(\Theta_1)_{22} - (\Theta_1)_{33}]) + \frac{9}{2}r^2(\Theta_2)_{23} - [(\Theta_0)_{23}]^2(\Theta_0)_{32}r^2\right\} + \Theta_{22}. \quad \text{(D9)}$$

At first glance, it might seem that the $\mathbb{A}$ matrix is singular at $r = 0$. However, after direct substitution and simplification, one can verify that $\mathbb{A}$ is real and analytic, and it has the same radius of convergence as the matrix Θ.

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