Noncomoving description of adiabatic radial perturbations of relativistic stars

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We study adiabatic, radial perturbations of static, self-gravitating perfect fluids within the theory of general relativity employing a new perturbative formalism. We show that by considering a radially static observer, the description of the perturbations can be greatly simplified with respect to the standard comoving treatment. The new perturbation equations can be solved to derive analytic solutions to the problem for a general class of equilibrium solutions. We discuss the thermodynamic description of the fluid under isotropic frame transformations, showing how, in the radially static, noninertial frame, the stress-energy tensor of the fluid must contain momentum transfer terms. As illustrative examples of the new approach, we study perturbations of equilibrium spacetimes characterized by the Buchdahl I, Heintzmann IIa, Patwardhan-Vaidya IIa, and Tolman VII solutions, computing the first oscillation eigenfrequencies and the associated eigenfunctions. We also analyze the properties of the perturbations of cold neutron stars composed of a perfect fluid verifying the Bethe-Johnson model I equation of state, computing the oscillation eigenfrequencies and the e-folding time.

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I. INTRODUCTION

Perturbative analysis, when possible, is one of the most important tools for exploring the intricacies of complex physical phenomena. In general relativity (GR), the study of perturbative solutions is crucial in several different contexts. Nonetheless, a fully mature theory of perturbation for relativistic gravitation came quite late compared to other research sectors. This delay can be ascribed to two specific issues in developing the perturbation theory: the so-called gauge problem and the consistent formulation of relativistic thermodynamics. Indeed, the fundamental dependence of a relativistic perturbation theory on the gauge makes it challenging to understand the behavior of the perturbed spacetime, possibly leading to ambiguous conclusions [1].

In the study of black hole perturbations, the first gauge-invariant approach was proposed by Moncrief [2], and many works have further developed the subject; in particular, the authors of Ref. [3] developed a gauge-invariant and covariant framework to study linear perturbations of static, spherically symmetric black holes. In the context of cosmology, Bardeen pioneered the development of a gauge-independent approach to perturbations [4], and Ellis and Bruni have constructed a fully covariant, gauge-invariant theory of cosmological perturbations [5]. In relativistic astrophysics, however, until recently, there was no covariant, gauge-invariant perturbation theory applicable to those types of solutions.

In the case of relativistic compact stellar objects, all studies on the subject rely on one specific gauge. Moreover, in many instances, the gauge is chosen to itself encode completely the degrees of freedom of the perturbations. This approach was first introduced by Chandrasekhar while studying adiabatic, radial perturbations of stellar compact objects [6,7], and all subsequent studies and reformulations have adopted this idea [8–10]. Although undeniably pioneering, the Chandrasekhar approach presents many issues and limitations. One such limitation is the focusing of the description of adiabatic, radial perturbations to the point of view of the frame locally comoving with the fluid, hindering the possibility of exploiting covariance to simplify the perturbation problem.

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Given the revolution that multimessenger astronomy has brought to relativistic astrophysics, it becomes increasingly important to develop perturbation schemes that can perform better than the ones currently in use. In Ref. [11], we have constructed such a scheme by developing a fully covariant, gauge-invariant theory of perturbation for nonvacuum, compact, locally rotationally symmetric of class II spacetimes, which can be used to study perturbations of relativistic stars spacetimes. Specifically, we laid out the mathematical foundations of the perturbation theory, addressing the properties of the differential equations that describe general perturbations, and we proposed a method to find exact solutions in terms of power series. A second paper, Ref. [12], was dedicated to the application of the new perturbation theory considering the locally comoving frame with the perturbed fluid. We studied perturbations of classical exact solutions of the Einstein field equations, computing the first eigenfrequencies and deducing universal criteria for stability. The analysis, however, confirmed what was already discussed in Ref. [11]: that formulation of the problem is not the most computationally efficient, whereas a change to a noncomoving frame would be more advantageous.

The aim of the present paper is then to explore the potential of the static frame perturbation equations. A challenge in this new formulation will be the identification of the thermodynamic properties of the matter sources in a static, noninertial frame. However, the covariance of the formalism will allow us to characterize the effects, offering, at the same time, several interesting insights into relativistic thermodynamics.

The results found in this paper, as well as Refs. [11,12] rely on a specific formalism called the $1+1+2$ covariant approach developed in Refs. [3,13,14]. The $1+1+2$ covariant approach can be thought of as an extension of the so-called $1+3$ decomposition (cf., e.g., Refs. [15,16]). Both these formalisms can be considered semi-tetradic representations of the spacetime, where all tensor quantities are locally, partially decomposed with respect to a timelike and a spacelike vector field. The advantages of this description of gravitational systems are manifold, but probably the most important one is the ability to describe in a relatively simple, mathematically rigorous, and physically transparent very complex gravitational systems without relying on specific coordinate systems. The $1+1+2$ covariant approach, in particular, has been used to formulate a covariant version of the Tolman-Oppenheimer-Volkoff (TOV) equations, leading to the discovery of new solutions in GR [17–20], and in some extension of Einstein's theory [21,22], and used to reformulate the theory of junction conditions [23].

The paper is organized as follows. Section II describes in detail the thermodynamics of a perfect fluid in a noncomoving frame, emphasizing the first law of thermodynamics, heat transfer, and entropy transformation. Section III contains a description of the perturbation variables, the perturbation equations together with their boundary condition, and a general algorithm to find power series solutions of the perturbation equations. Section IV deals with the application of the resolution algorithm to some specific cases that are not easily solvable in the comoving frame. In Sec. V, we analyze the properties of the adiabatic, radial perturbations of cold neutron stars composed by a fluid characterized by the Bethe-Johnson model I equation of state. We then draw our conclusions in Sec. VI. The paper contains two Appendixes. In Appendix A, we introduce the general definitions for the $1+1+2$ covariant quantities, and in Appendix B, we present the general transformation associated with a change of frame and its consequences on the $1+1+2$ quantities.

Throughout the article we will consider the metric signature $(-+++)$, and, except in Sec. V, we will work in the geometrized unit system where $8\pi G = c = k_B = 1$.

II. THERMODYNAMIC DESCRIPTION OF A PERFECT FLUID IN THE STATIC FRAME

We are interested in studying perturbative solutions of static, spatially compact, spherically symmetric spacetimes with a perfect fluid source under adiabatic, radial perturbations using the new formalism constructed in Ref. [11]. In Ref. [12] this analysis was carried out considering the point of view of a comoving observer, the so-called *Lagrangian picture* of the fluid flow. In this article, we will show how the analysis can be greatly simplified if, instead, we consider the point of view of a static observer with respect to a static observer at spatial infinity, which can be considered as the gauge-invariant, covariant *Eulerian picture*. In that regard, we have to allow a general stress-energy tensor for the perturbed fluid. We will impose the perturbed fluid to remain perfect. However, as it is shown in Appendix B, from the point of view of a static observer, the fluid will not be seen as perfect, and momentum transfer terms have to be included. However, in this section, we will show that in first-order perturbation theory, those momentum fluxes do not represent energy exchange between the volume elements of the fluid as heat. This discussion will help the understanding of the results on the evolution of the perturbations contained in the next sections.

To keep the treatment in this section independent of the choice of a local coordinate system, we will consider the covariant quantities of the $1+1+2$ formalism, whose definitions we present in Appendix A. Nonetheless, the final set of perturbation equations in the static frame used in the following sections will be written immediately in the standard Schwarzschild coordinate system and, thus, no knowledge of covariant methods is required for their use.

A. Transformation of the particle number density vector field

Consider a perfect fluid permeating a spacetime and two observers: an observer locally comoving with the volume elements of the fluid and another observer whose frame is related with the comoving frame by an isotropic frame transformation (cf. Appendix B). Let u represent the 4-velocity of the comoving observer and e represent a spacelike vector field orthogonal to u, such that the integral curves of u and the integral curves of e form two local congruences in the spacetime. On the other hand, let an observer in the second frame be characterized by a 4-velocity $\bar{u}$ and $\bar{e}$ be a spacelike vector field orthogonal to $\bar{u}$, such that, similarly, the integral curves of $\bar{u}$ and the integral curves of $\bar{e}$ also form two local congruences in the spacetime. In short, we will say the dyad (u, e) is associated with the comoving frame and $(\bar{u}, \bar{e})$ is associated with the second frame. In what follows, for simplicity, we will use an overline to identify quantities defined in the second frame. In addition, without loss of generality, we will assume the integral curves of the vector fields $\{u, \bar{u}, e, \bar{e}\}$ to be parametrized by an affine parameter, such as the proper time or the proper length, and the tangent vector fields are normalized such that

$$u^\alpha u_\alpha = -1, \qquad e^\alpha e_\alpha = 1,$$
$$\bar{u}^\alpha \bar{u}_\alpha = -1, \qquad \bar{e}^\alpha \bar{e}_\alpha = 1. \tag{1}$$

In the considered setup, by definition, the particle 4-current density vector in the comoving frame is given by

$$N^\alpha = n u^\alpha, \tag{2}$$

where n represents the particle number density, whereas in the barred frame, in general, we have

$$\bar{N}^\alpha = \bar{n}\bar{u}^\alpha + \bar{n}^\alpha, \tag{3}$$

where $\bar{n}^\alpha$ is the particle number density flux vector, orthogonal to $\bar{u}$, measured by the observer associated with the dyad $(\bar{u}, \bar{e})$, such that $\bar{n}^\alpha \bar{u}_\alpha = 0$. Given an isotropic change of frame, Eq. (B2), the components of the vector field N transform as

$$N^\alpha = n u^\alpha \rightarrow \bar{N}^\alpha = \bar{u}^\alpha n \cosh\beta - \bar{e}^\alpha n \sinh\beta, \tag{4}$$

and, therefore,

$$\bar{n} = n \cosh\beta,$$
$$\bar{n}^\alpha = -\bar{e}^\alpha n \sinh\beta. \tag{5}$$

In what follows, we will impose particle number conservation in both frames, that is, $\nabla_\alpha N^\alpha = 0$ and $\nabla_\alpha \bar{N}^\alpha = 0$. This implies the following relations for the comoving and the barred frame, respectively:

$$u^\alpha \nabla_\alpha n + n\theta = 0, \tag{6}$$

$$\bar{u}^\alpha \nabla_\alpha \bar{n} + \bar{n}\,\bar{\theta} + \nabla_\alpha \bar{n}^\alpha = 0, \tag{7}$$

where θ and $\bar{\theta}$ represent the expansion scalars associated with the integral curves of u and of $\bar{u}$, respectively. Using Eqs. (B1), (B8), and (6), Eq. (7) can be rewritten in the useful form

$$\frac{\nabla_\alpha \bar{n}^\alpha}{\bar{n}} = -\sinh\beta\left[\mathcal{A} + \phi + \frac{e^\alpha \nabla_\alpha n}{n}\right] - \tanh\beta\,\bar{u}^\alpha \nabla_\alpha \beta - \bar{e}^\alpha \nabla_\alpha \beta. \tag{8}$$

B. First law of thermodynamics and energy conservation in the barred frame

To describe the thermodynamics of the perturbed fluid, we will consider the local-thermodynamic equilibrium ansatz, which presupposes that given an off-thermodynamic equilibrium system, we can still locally define volume elements that act as a thermodynamic subsystem in equilibrium, such that within these elements, the state is characterized by well-defined state variables. In our case, this assumption is reasonable, given the considered perturbative setup, and assuming that the evolution timescale of the perturbations within the fluid is much greater than a characteristic local relaxation time.

Now, assume the fluid to be composed of a single particle species with rest mass m_N. For the barred frame, let $\bar{\mu}$ represent the relativistic energy density of the fluid, $\bar{\varepsilon}$ the specific internal energy density, $\bar{n}$ the particle density, and $\bar{\mu}_N = m_N \bar{n}$ the rest mass density, such that $\bar{\mu} = \bar{\mu}_N(1 + \bar{\varepsilon})$, and hence

$$d\bar{\mu} = (1 + \bar{\varepsilon}) d\bar{\mu}_N + \bar{\mu}_N d\bar{\varepsilon}. \tag{9}$$

Defining the specific volume $\bar{v} = 1/\bar{\mu}_N$, such that

$$d\bar{v} = -\bar{v}^2 d\bar{\mu}_N = -m_N \bar{v}^2 d\bar{n}, \tag{10}$$

Equation (9) can be rewritten as

$$d\bar{\mu} = -\bar{\mu}\frac{d\bar{v}}{\bar{v}} + \bar{\mu}_N d\bar{\varepsilon}. \tag{11}$$

For a quasiequilibrium process, the first law of thermodynamics reads

$$d\bar{\varepsilon} = -\delta\bar{\mathsf{W}} + \delta\bar{\mathsf{Q}} = -\bar{p}\,d\bar{v} + \bar{T} d\bar{S}, \tag{12}$$

where δ represents the variation of a path-dependent quantity under a given thermodynamical process, $\bar{p}$ represents the local pressure, $\bar{T}$ is the local temperature, and $\bar{S}$ is the local specific entropy, all defined in the barred frame. Then,

$$d\bar{\mu} = -(\bar{\mu} + \bar{p})\frac{d\bar{v}}{\bar{v}} + \bar{\mu}_N \bar{T} d\bar{S}. \tag{13}$$

From Eq. (10) we have

$$\frac{d\bar{v}}{\bar{v}} = -\frac{d\bar{n}}{\bar{n}}. \tag{14}$$

Then, along the world lines of the barred observers and imposing the conservation of particle density (7),

$$\frac{\dot{\bar{v}}}{\bar{v}} = -\frac{\dot{\bar{n}}}{\bar{n}} = \bar{\theta} + \frac{1}{\bar{n}}\nabla_\alpha \bar{n}^\alpha, \tag{15}$$

where "dot" over a barred quantity represents the directional derivative taken with respect to $\bar{u}$, that is, a derivative taken with respect to the proper time, τ, of the barred observer. Then, from Eq. (13) we find

$$\dot{\bar{\mu}} = -(\bar{\mu}+\bar{p})\left(\bar{\theta} + \frac{1}{\bar{n}}\nabla_\alpha \bar{n}^\alpha\right) + \bar{\mu}_N \bar{T}\,\dot{\bar{S}}. \tag{16}$$

C. Perturbative analysis of the first law

So far, the discussion was kept rather general. We will now and for the remainder of the section consider the specific setup of an equilibrium perfect fluid in a static, spherically symmetric spacetime that is adiabatically, radially perturbed, such that the perturbed fluid is also a perfect fluid. The equilibrium fluid is characterized by its energy density, μ_0, and pressure, p_0. Moreover, we will assume that an observer comoving with the equilibrium fluid has 4-velocity u_0, and we can define a spacelike vector field e_0 orthogonal to u_0 at each point. Moreover, we assume that the integral curves of u_0 and the integral curves of e_0 form two local congruences in the equilibrium background spacetime.

Because of the nature of the background spacetime, observers comoving with volume elements of the equilibrium fluid are also radially static observers. However, in the perturbed spherically symmetric spacetime, these observers are distinct. Let the barred frame of the previous subsections be associated with a radially static observer in the perturbed spacetime, such that $\bar{e}$ is aligned with the outward radial gradient. Then, the particle density flux vector field $\bar{n}^\alpha$, defined in Eqs. (3)–(5), vanishes in the background spacetime, since the comoving and the static observers are the same. Therefore, $\bar{n}^\alpha$ is a gauge-invariant, first-order quantity according to the Stewart-Walker lemma [1]. Moreover, the tilting angle β associated with the isotropic frame transformation between the comoving and the barred frames also vanishes in the background and is a gauge-invariant, first-order quantity with respect to the background (cf. Appendix B). Then, at first order, that is, disregarding terms with products of multiple first-order quantities with respect to the equilibrium background spacetime, Eq. (8) reads

$$\frac{\nabla_\alpha \bar{n}^\alpha}{\bar{n}} = -\beta\left(\mathcal{A}_0 + \phi_0 + \frac{\hat{n}_0}{n_0}\right) - \bar{e}^\alpha \nabla_\alpha \beta, \tag{17}$$

where $\hat{n}_0 = e_0^\alpha \nabla_\alpha n_0$. Using Eq. (B9), we find the following relation valid at linear level:

$$\frac{\nabla_\alpha \bar{n}^\alpha}{\bar{n}} = \left(\phi_0 + 2\mathcal{A}_0 + \frac{\hat{n}_0}{n_0} - \frac{\hat{\mu}_0}{\mu_0+p_0}\right)\frac{\bar{Q}}{\mu_0+p_0} + \frac{\hat{\bar{Q}}}{\mu_0+p_0}, \tag{18}$$

where $\hat{\mu}_0 = e_0^\alpha \nabla_\alpha \mu_0$ and $\hat{\bar{Q}} = \bar{e}^\alpha \nabla_\alpha \bar{Q}$.

Last, we analyze the quantity $\hat{n}_0/n_0$ in terms of the energy density and pressure of the equilibrium fluid. Using the definition of specific volume in the case of the equilibrium fluid: $v_0 = 1/(\mu_N)_0$, for the background spacetime the total relativistic energy density is given by $\mu_0 = (\mu_N)_0(1+\varepsilon_0)$, where $d\varepsilon_0 = -p_0 dv_0$. Then,

$$\frac{\hat{n}_0}{n_0} = \frac{\hat{\mu}_0}{\mu_0+p_0}, \tag{19}$$

and Eq. (18) simplifies to

$$\frac{1}{\bar{n}}\nabla_\alpha \bar{n}^\alpha = \frac{\phi_0 + 2\mathcal{A}_0}{\mu_0+p_0}\bar{Q} + \frac{\hat{\bar{Q}}}{\mu_0+p_0}. \tag{20}$$

Substituting Eq. (20) in Eq. (16), at first order, we find

$$\dot{\bar{\mu}} = -(\mu_0+p_0)\bar{\theta} - (\phi_0 + 2\mathcal{A}_0)\bar{Q} - \hat{\bar{Q}} + \bar{\mu}_N \bar{T}\,\dot{\bar{S}}. \tag{21}$$

D. Second-order nature of the heat flow

Now, the general energy conservation law $\bar{u}_\beta \nabla_\alpha \bar{T}^{\alpha\beta} = 0$ for a spherically symmetric spacetime, in the language of the $1+1+2$ formalism reads [11]

$$\dot{\bar{\mu}} = -(\bar{\mu}+\bar{p})\bar{\theta} - (\bar{\phi} + 2\bar{\mathcal{A}})\bar{Q} - \hat{\bar{Q}} - \frac{3}{2}\bar{\Pi}\,\bar{\Sigma}. \tag{22}$$

Taking into account the discussion in Appendix B, at a linear perturbative level with respect to the background spacetime, Eq. (22) reads

$$\dot{\bar{\mu}} = -(\mu_0+p_0)\bar{\theta} - (\phi_0 + 2\mathcal{A}_0)\bar{Q} - \hat{\bar{Q}} + \text{higher-order terms}. \tag{23}$$

Comparing Eqs. (21) and (23), we see that the term $\bar{T}\,\dot{\bar{S}}$ is of higher perturbative order. Therefore, at linear level, there is no entropy change of the volume elements of the fluid along the world lines of the static observer: $\dot{\bar{S}} = 0$. This result and the assumption of a quasistatic thermodynamic evolution implies that the processes within the volume elements of the fluid are reversible. Therefore, at linear level, heat transfer verifies

$$\int_c \delta\bar{\mathsf{Q}} = \int_{\tau_i}^{\tau_f} \bar{T}\,\dot{\bar{S}}\, d\tau = 0, \tag{24}$$

leading us to conclude that along the world line c of a static observer there is no exchange of energy as heat of an infinitesimal volume element. Therefore, a reversible adiabatic process in the comoving frame is also reversible and adiabatic in the static frame if the two frames are related by an isotropic frame transformation that is first order with respect to a background spacetime. This conclusion is in line with the discussion in Refs. [6,24], confirming that if the thermodynamic processes are carried out quasistatically, the correction to the thermodynamic behavior due to the motion between two frames is a higher than linear-order effect.

In addition, Eq. (20) explicitly characterizes the origin of the momentum flux term in the stress-energy tensor in the static frame, Eq. (B10): $\bar{Q}$ and its derivative account for the fractional divergence of the particle density flux vector along the $\bar{e}$ direction; that is, the momentum flux terms account for the net particle flux entering and leaving the volume element associated with the static observer. However, Eq. (24) asserts that at first order, there is no change along $\bar{u}$ of the energy of the system as heat. Hence, these terms do not represent dissipative heat fluxes within the fluid. To clarify, the Q terms in the stress-energy tensor of a fluid represent only dissipative effects in the fluid's rest frame. In other frames, these terms also characterize energy carried by matter fluxes between volume elements of fluid, even if energy is exchanged as work.

E. Entropy and temperature of the perturbed fluid

In the previous subsection, we have discussed that for an adiabatic, quasistatic process in the comoving frame, a radially static observer will still measure momentum fluxes within the fluid. However, we have proved that in first-order perturbation theory with respect to a background spacetime, those fluxes do not represent dissipative effects, and the process is reversible. This implies that, in the considered perturbative regime, the momentum fluxes do not lead to a change in the total entropy of the system. Nonetheless, it is natural to ask how these fluxes are related with a perceived entropy flux between infinitesimal elements of volume of the fluid and if it is possible to find the local value for the temperature of the fluid in the static, noninertial frame. Considering the local-thermodynamic equilibrium ansatz, we can understand these effects by evaluating how an isotropic frame transformation of the type described in Appendix B, which are first order with respect to a static background, affects the description of the "entropy flow" in and out of the infinitesimal volume elements of the perturbed fluid defined by the radially static observers. Then, let S^α represent, formally, the entropy flow density 4-vector in the frame comoving with the volume elements of the matter fluid, with 4-velocity u, such that

$$S^\alpha = m_N S N^\alpha = m_N S n u^\alpha, \tag{25}$$

where S denotes the specific entropy and N^α represents the particle 4-current density vector with respect to a comoving observer, as defined in Eq. (2). Consider an isotropic frame transformation (B2), and let the barred frame be associated with a radially static observer with 4-velocity $\bar{u}$. Using Eqs. (3) and (5), the components of the entropy flow density 4-vector transform as

$$\begin{aligned} S^\alpha = m_N S n u^\alpha \to \bar{S}^\alpha \\ = m_N S(n\bar{u}^\alpha \cosh\beta) - m_N S(n\bar{e}^\alpha \sinh\beta). \end{aligned} \tag{26}$$

In particular, as expected, $\bar{S}^\alpha = m_N S \bar{N}^\alpha$. The first term in the right-hand side of Eq. (26) is called the "convection term," and it accounts for the entropy density flow carried along the direction of $\bar{u}$. The second term, called the "conduction and diffusion term," accounts for the entropy carried by the matter flux entering and leaving the volume element along the $\bar{e}$ direction. Indeed, to confirm this interpretation, we will explicitly relate the momentum transfer term $\bar{Q}$ with the diffusion term in the considered perturbative setup.

For simplicity, let

$$\bar{s}^\alpha \coloneqq -m_N S(n\bar{e}^\alpha \sinh\beta) = -m_N S\bar{n}^\alpha \tag{27}$$

represent the conduction and diffusion term in the static frame. As explained previously, in the equilibrium background spacetime, observers comoving with the fluid are also radially static; therefore, $\bar{s}^\alpha$ is zero in the background and is a gauge-invariant quantity according to the Stewart-Walker lemma. Then, using Eq. (B9), valid in first-order perturbation theory, Eq. (27) reduces to

$$\bar{s}^\alpha = \frac{m_N n_0 S_0}{\mu_0 + p_0} \bar{Q} \bar{e}^\alpha. \tag{28}$$

On the other hand, a general diffusion term that depends linearly on the heat flux density can be written as

$$\bar{s}^\alpha = \frac{\eta \bar{Q} \bar{e}^\alpha}{\bar{T}}, \tag{29}$$

where η is a thermodynamic coefficient and, as in the previous section, $\bar{T}$ is the local temperature. Comparing Eqs. (28) and (29) implies

$$\frac{\eta}{\bar{T}} = \frac{m_N n_0 S_0}{\mu_0 + p_0}. \tag{30}$$

To interpret this result, notice that, in the geometrized units system, we can set $\eta = 1$, and Eq. (30) takes the familiar form for the inverse temperature found in equilibrium thermodynamics of relativistic continuous media for fluids composed of a single species (cf., e.g., Ref. [25]). Therefore, by imposing the local-thermodynamic equilibrium ansatz, Eq. (30) asserts that, up to linear order, the entropy flux density, $\bar{s}^\alpha$, and the momentum flux density,

$\bar{Q}\bar{e}^{\alpha}$, are related simply by the value of the local temperature of the equilibrium fluid.

III. ADIABATIC RADIAL PERTURBATIONS

In the previous section, we have established that to characterize a perfect fluid in frames other than in a frame locally comoving with the fluid, the stress-energy density has to contain nondiagonal terms to account for the momentum transfer in and out of the infinitesimal volume elements of the matter fluid. This implies that the description of the fluid becomes more complex in those other frames. However, as we will see, this freedom allows us to pick a noninertial frame in which the dynamical description of the fluid and the geometry of the spacetime can be greatly simplified, such that under certain general conditions, the problem of the determination of the behavior of the perturbations can be treated using standard analytic methods efficiently.

In Ref. [11], a general perturbation theory was developed that is manifestly covariant and identification gauge invariant. This theory can then be applied to study perturbations of static, compact, spherically symmetric solutions of the theory of general relativity. For simplicity, we call these types of solutions "stars," although these are suitable to model any static, self-gravitating relativistic matter distributions. The new perturbation scheme was constructed using the so-called $1+1+2$ covariant formalism, introduced in Ref. [14]. However, for clarity, in this and subsequent sections, we will consider a particular coordinate system and write the covariant variables in terms of the metric and its derivatives.

A. The equilibrium spacetime and the perturbation variables

Consider the Einstein field equations (EFE)

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}, \tag{31}$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor, R is the Ricci scalar, and $T_{\alpha\beta}$ represents the components of the metric stress-energy tensor. For simplicity, we have set the cosmological constant to zero.

Let the equilibrium background spacetime be a spatially compact, static, spherically symmetric solution of the EFE with a perfect fluid source, such that

$$T_{\alpha\beta} = (\mu_0 + p_0)(u_0)_\alpha (u_0)_\alpha + p_0 (g_0)_{\alpha\beta}, \tag{32}$$

where u_0 is the 4-velocity of an observer locally comoving with the volume elements of the fluid, $(g_0)_{\alpha\beta}$ are the components of the metric tensor in some local coordinate system, μ_0 represents the energy density, and p_0 is the isotropic pressure of the matter fluid. As we have done in the previous section and from here on, we will use the "0" to explicitly refer to quantities of the equilibrium spacetime. Generically, in the coordinates (t, r, ψ, φ) defined by a static observer at spatial infinity, this type of solutions can be characterized by a line element of the form

$$ds_0^2 = -(g_0)_{tt} dt^2 + (g_0)_{rr} dr^2 + r^2 (d\psi^2 + \sin^2 \psi \, d\varphi^2), \tag{33}$$

where the metric coefficients $(g_0)_{tt}$ and $(g_0)_{rr}$ are assumed to be functions solely of r.

We will consider the equilibrium spacetime manifold to be composed of two solutions of the EFE of the type (33). The first one, representing the exterior of the star, is a regular branch of the Schwarzschild solution, the second one, modeling the interior of the star, is a static spatially compact solution with a perfect fluid matter source. The two solutions are smoothly matched at $r = r_b$ (the boundary of the star) to each other at a common timelike hypersurface via the standard Israel-Darmois junction formalism.

For the line element (33), we can write the $1+1+2$ covariant scalar functions directly in terms of the metric components and its derivatives as

$$\phi_0 = \frac{2}{r\sqrt{(g_0)_{rr}}}, \qquad \mathcal{A}_0 = \frac{1}{2(g_0)_{tt}\sqrt{(g_0)_{rr}}} \frac{d(g_0)_{tt}}{dr}, \tag{34}$$

where ϕ_0 characterizes the spatial expansion of the normalized radial gradient vector field and $\mathcal{A}_0$ is the radial component of the 4-acceleration of an observer locally comoving with the volume elements of the matter fluid.

Now, in Ref. [1], necessary and sufficient conditions were established for a quantity to be independent of the choice of diffeomorphism between an equilibrium and a perturbed spacetime, i.e., to be identification gauge invariant. Therefore, by identifying an appropriate closed set of gauge-invariant perturbation variables, we can characterize the perturbed spacetime unambiguously. In that regard, consider the energy density, $\bar{\mu}$, and pressure, $\bar{p}$, of the perturbed fluid measured in the radially static frame. These are not gauge-invariant quantities; however, given that the background spacetime is assumed static, their proper time derivatives vanish in the background, hence, are gauge invariant, and can be used to characterize the perturbed spacetime. Indeed, to describe the perturbations, we will consider the variables

$$\mathsf{m} \coloneqq \dot{\bar{\mu}}, \quad \mathsf{p} \coloneqq \dot{\bar{p}}, \quad \mathsf{A} \coloneqq \dot{\bar{\mathcal{A}}}, \quad \mathsf{F} \coloneqq \dot{\bar{\phi}}, \quad \mathsf{E} \coloneqq \dot{\bar{\mathcal{E}}}, \tag{35}$$

where the general definitions of $\mathcal{A}$, ϕ, and $\mathcal{E}$ can be found in Appendix A, and, as in the previous section, "dot" represents derivatives along the world lines of the radially static observers, that is, derivatives taken with respect to the proper time of observers with a constant circumferential radius coordinate. In addition to the quantities in Eq. (35), we will also consider the expansion scalar, $\bar{\theta}$, and the nontrivial radial component of the shear tensor, $\bar{\Sigma}$, both associated with the local congruence formed by the world lines of the radially static observers since those quantities vanish identically in the background spacetime. For the radially static frame, as we have discussed in the previous section, we also need to include a momentum transfer term $\bar{Q}$ in the fluid's stress-energy tensor. Then, in a radially static frame, adiabatic, radial perturbations of perfect fluid stars can be, independently of the choice of gauge, fully characterized by the variables $\{\mathsf{m}, \mathsf{p}, \bar{Q}, \mathsf{A}, \mathsf{F}, \mathsf{E}, \bar{\theta}, \bar{\Sigma}\}$.

B. Harmonic decomposition

The perturbation equations are found by linearizing the EFE. The linearization procedure dramatically simplifies the field equations; however, these can be further simplified by considering the exact symmetries of the equilibrium spacetime to transform the linearized system of partial differential equations in a system of ordinary differential equations.

In this article, we will consider the background spacetime to be spatially compact, static, and with spherical symmetry. Then, at linear perturbation order, it is possible to express a covariantly defined scalar perturbation variable, χ, in terms of the eigenfunctions $e^{i\upsilon\tau}$ of the Laplace operator in $\mathbb{R}$, of the background spacetime, and the spherical harmonics, $Y_{\ell m}$:

$$\chi = \sum_{\upsilon}\left(\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\Psi_{\chi}^{(\upsilon,\ell)}Y_{\ell m}\right)e^{i\upsilon\tau}, \tag{36}$$

where τ is the proper time of the comoving observer in the background and υ parametrizes the associated eigenfrequencies, $\sum_{\upsilon}$ can stand for a discrete sum or an integral in υ, depending on the boundary conditions of the problem, and the coefficients $\Psi_{\chi}^{(\upsilon,\ell)}$ are functions of r only.

In this article, we will consider isotropic perturbations; therefore, dipole and higher-order angular multipoles are zero as those would select preferred directions in the system. Then, in the expansion (36) we can disregard all coefficients $\Psi_{\chi}^{(\upsilon,\ell)}$ with $\ell \geq 1$. Moreover, provided sufficient regularity for the background spacetime and the perturbed fluid, the squared eigenfrequencies, υ^2, are countable, are real, are simple, have a minimum, and are unbounded from above [11]. Therefore, in the considered setup, any first-order, gauge-invariant scalar quantity, χ, can be decomposed as

$$\chi = \sum_{\upsilon^2=\{\upsilon_0^2,\upsilon_1^2,\ldots\}}\Psi_{\chi}^{(\upsilon)}(r)Y_{00}e^{i\upsilon\tau}, \tag{37}$$

where we have dropped the superscript ℓ.

The expansion above was done employing the proper time, τ, of an observer locally comoving with the equilibrium fluid. However, it is more advantageous in some cases to express it by means of the time coordinate t. The eigenfrequencies associated with each of these time coordinates are connected by the relation [13]

$$\upsilon(r) = \lambda\exp\left(\int_{+\infty}^{r} -\frac{2\mathcal{A}_0}{x\phi_0}dx\right) = \frac{\lambda}{\sqrt{(g_0)_{tt}}}, \tag{38}$$

where the constant λ represents the value of an eigenfrequency measured by the observer at spatial infinity. Thus, in terms of t, a gauge-invariant, first-order scalar quantity χ can be equivalently given by

$$\chi = \sum_{\lambda^2=\{\lambda_0^2,\lambda_1^2,\ldots\}}\Psi_{\chi}^{(\lambda)}Y_{00}e^{i\lambda t}. \tag{39}$$

C. Gauge-invariant equation of state and perturbation equations

The dynamical evolution of the adiabatically, radially perturbed spacetime is characterized by a system of equations for the perturbation variables (35). Nonetheless, this system is not closed and requires additional information related to the thermodynamics of the fluid source in the form of an equation of state. In what follows, we will assume that in the local rest frame of the perturbed fluid, the pressure and the energy density are related by

$$p = f(\mu), \tag{40}$$

i.e., a barotropic equation of state. Here f is a generic twice differentiable function defined in an open neighborhood of μ_0. With these assumptions, $f'(\mu_0)$ can be associated with the square of the adiabatic speed of sound of fluid as measured by a comoving observer. In line with the prescriptions of a physically meaningful background, we will assume that $f'(\mu_0)$ is nonvanishing in the interior of the perturbed star. The equation of state above translates in the comoving frame in a similar relation for the perturbation variables m and p. However, as mentioned in Sec. II, in the static frame the fluid is no longer perceived to be perfect; therefore, the equation of state is modified, as proven in [11], into the relation

$$\mathsf{m} = \frac{1}{f'(\mu_0)}\mathsf{p} - \frac{r\phi_0}{2(\mu_0+p_0)}\left(\frac{d\mu_0}{dr} - \frac{1}{f'(\mu_0)}\frac{dp_0}{dr}\right)Q, \tag{41}$$

where we have dropped the overline for simplicity's sake. In the remainder of the article all first-order quantities are to be considered those measured in the frame of radially static observers within the star. Moreover, notice that if the equilibrium fluid is characterized by a barotropic equation of state equal to that of the perturbed fluid, the second term in the right-hand side of Eq. (41) vanishes. We also remark that we have not imposed any direct constraints on the particular type of the equation of state of the equilibrium fluid.

Gathering the previous results, we are now in a position to write the perturbation equations. In the coordinate system, (t, r, ψ, φ), and considering the harmonic decomposition (37), the nontrivial harmonic coefficients verify the following system of equations [11]:

$$\frac{d\Psi_{\mathsf{p}}^{(\upsilon)}}{dr} = \frac{2}{r\phi_0}\left[\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} + \upsilon^2\right]\Psi_{\mathcal{Q}}^{(\upsilon)} - \frac{2}{r\phi_0}\left[\frac{\mu_0+p_0}{\phi_0} + \left(2+\frac{1}{f'(\mu_0)}\right)\mathcal{A}_0\right]\Psi_{\mathsf{p}}^{(\upsilon)}, \tag{42}$$

$$\frac{d\Psi_{\mathcal{Q}}^{(\upsilon)}}{dr} = \frac{2}{r\phi_0}\left[\frac{\mathcal{A}_0}{f'(\mu_0)} + \frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} + \frac{\mu_0+p_0}{\phi_0} - \phi_0 - 2\mathcal{A}_0\right]\Psi_{\mathcal{Q}}^{(\upsilon)} - \frac{2}{r\phi_0 f'(\mu_0)}\Psi_{\mathsf{p}}^{(\upsilon)}, \tag{43}$$

and

$$\Psi_{\mathsf{m}}^{(\upsilon)} = \frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}}^{(\upsilon)} - \left(\frac{\mathcal{A}_0}{f'(\mu_0)} + \frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr}\right)\Psi_{\mathcal{Q}}^{(\upsilon)},$$
$$\Psi_{\mathsf{A}}^{(\upsilon)} = \frac{1}{\phi_0}\left[\Psi_{\mathsf{p}}^{(\upsilon)} - \left(\frac{1}{2}\phi_0+\mathcal{A}_0\right)\Psi_{\mathcal{Q}}^{(\upsilon)}\right],$$
$$\Psi_{\mathsf{E}}^{(\upsilon)} = \frac{1}{2}\phi_0\Psi_{\mathcal{Q}}^{(\upsilon)} + \frac{1}{3}\Psi_{\mathsf{m}}^{(\upsilon)},$$
$$\Psi_{\theta}^{(\upsilon)} = -\frac{1}{\phi_0}\Psi_{\mathcal{Q}}^{(\upsilon)},$$
$$\Psi_{\Sigma}^{(\upsilon)} = \frac{2}{3}\Psi_{\theta}^{(\upsilon)},$$
$$\Psi_{\mathsf{F}}^{(\upsilon)} = \Psi_{\mathcal{Q}}^{(\upsilon)}, \tag{44}$$

where the second to last equation is the kinematical requirement for the constancy of the circumferential radius coordinate of the static observer.

To solve the system of differential equations above, we must impose boundary conditions. Given the particular physical setup that we are interested in, we consider the following:

(i) at the center of the star, $r = 0$, and at the initial instant, the energy density and the pressure perturbations must be finite; and
(ii) there exists a timelike hypersurface at which the interior spacetime and an exterior vacuum Schwarzschild spacetime can be smoothly matched.

As was proven in Ref. [11], for the static observer, condition (ii) sets that at the surface of the star, at all times, we must have

$$\mathsf{p} - \mathcal{A}_0\mathcal{Q}|_{\text{boundary}} = 0, \tag{45}$$

and, therefore,

$$\Psi_{\mathsf{p}}^{(\upsilon)} - \mathcal{A}_0\Psi_{\mathcal{Q}}^{(\upsilon)}|_{\text{boundary}} = 0. \tag{46}$$

Before we discuss the solutions of the system (42)–(44), we comment on some glaring differences between this system and the one found for the comoving frame, discussed in Ref. [12]. In the comoving frame, three master perturbation variables are necessary to characterize adiabatic, radial perturbations, whereas, in the radially static frame, the description of adiabatic, radial perturbations is completely characterized by two master variables: p and $\mathcal{Q}$. Moreover, to characterize this type of perturbations in the radially static frame, only information on the square of the speed of sound of the fluid, f', is required. This is not surprising. As noted in Ref. [26], where the original Chandrasekhar second-order radial pulsation equation was recast as a first-order coupled system of differential equations, the new system also only depends on the value of the adiabatic index of the fluid and not of its derivative. Therefore, the matter model is completely characterized by the values of the adiabatic index or, equivalently, by the square of the speed of sound measured in the fluid's local rest frame.

D. Analytic solutions

Given the background spacetime and the equation of state of the perturbed fluid, Eqs. (42) and (43), together with conditions (i) and (ii), completely describe adiabatic, radial perturbations of a star composed of a perfect fluid, from the point of view of a radially static observer within the star. This type of systems of ordinary differential equations is well-suited to employ numerical methods to find approximate solutions. Notwithstanding, we have shown in [11] that it is also possible to find analytic solutions in the form of power series, using methods borrowed from the standard theory of systems of linear ordinary differential equations. In that regard, let the following extra conditions:

(a) the equilibrium fluid variables μ_0 and p_0 verify the weak energy condition;
(b) the equilibrium star is a solution of the TOV equations for real analytic, nontrivial energy density, and isotropic pressure functions; and
(c) the function f', representing the square of the adiabatic speed of sound of the perturbed fluid in the comoving frame, is positive and real analytic within the perturbed star.

Condition (b), in particular, is a very strong restriction to the type of spacetimes that can be considered, especially taking into account the nontrivial equations of state that are characteristic of nuclear matter. Serendipitously, to our knowledge, all known classical exact solutions for compact astrophysical objects have this property at least in a neighborhood of the center of the star. Indeed, if the equilibrium spacetime is a solution of the EFE with real analytic μ_0 and p_0 matter variables, such that both their power series centered at

$r = 0$ have a nonzero radius of convergence, the technique shown below can be used to characterize the perturbed spacetime, under adiabatic, radial perturbations. Nonetheless, we remark that if the radius of convergence of any of the power series is smaller than the radius of the equilibrium star, the validity of the solutions presented below is only guaranteed for points whose circumferential radius coordinate is smaller than that value of the radius of convergence.

Continuing, considering the regularity conditions above, Eq. (34) can be rewritten as

$$\phi_0 = \frac{2}{r}\sqrt{1 - \frac{2M(r)}{r}},$$
$$\mathcal{A}_0\phi_0 = p_0 + \frac{2M(r)}{r^3}, \tag{47}$$

where

$$M(r) \coloneqq \frac{1}{2}\int_0^r \mu_0 x^2 dx \tag{48}$$

is dubbed the mass function. Then, conditions (a) and (b) for the matter variables imply that the function $\mathcal{A}_0$ is real analytic at all points within the star, and the function ϕ_0 is real analytic within the star except at $r = 0$ where it has a singular point. As a consequence, Eqs. (42) and (43) constitute a system of ordinary differential equations with real analytic coefficients around $r = 0$ and a singular point at $r = 0$.

Now, we will start by recasting the system of perturbation equations in the following form:

$$\frac{d}{dr}\mathbb{W} = (r^{-1}\mathbb{R} + \Theta)\mathbb{W}, \tag{49}$$

with

$$\mathbb{W} = \begin{bmatrix} \Psi_{\mathsf{p}}^{(v)} \\ \Psi_{Q}^{(v)} \end{bmatrix}, \qquad \mathbb{R} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \tag{50}$$

and

$$\Theta = -\frac{2}{r\phi_0}\begin{bmatrix} \frac{\mu_0+p_0}{\phi_0} + 2\mathcal{A}_0 + \frac{\mathcal{A}_0}{f'(\mu_0)} & -\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) - \frac{\mathcal{A}_0^2}{f'(\mu_0)} - \frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} - v^2 \\ \frac{1}{f'(\mu_0)} & 2\mathcal{A}_0 - \frac{\mathcal{A}_0}{f'(\mu_0)} - \frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} - \frac{\mu_0+p_0}{\phi_0} \end{bmatrix}. \tag{51}$$

The regularity conditions (a)–(c) imply that $r\phi_0$ is positive in the interior of the star. Then, the matrix Θ is real analytic at $r = 0$, with power series

$$\Theta(r) = \sum_{n=0}^{+\infty}\Theta_n r^n. \tag{52}$$

In particular, we conclude that $r = 0$ is a regular singular point of the system (49), and we can employ the methods presented in Ref. [27] to find the solution $\mathbb{W}$ in the form of a convergent power series in the neighborhood of $r = 0$. Moreover, the method guarantees that this solution series has a radius of convergence equal to the series in (52) except, possibly, at the singular point $r = 0$.

Now, for a completely general Θ matrix, the solutions to the system of differential equations (49) can be quite complicated. However, assuming conditions (a)–(c), and making use of the TOV equations, it was found, in general, that some entries of the lowest-order coefficient matrices Θ_0 and Θ_1 vanish. This greatly simplifies the general family of solutions of physical interest. Following Ref. [27], and given the regularity of the background spacetime, the general solutions are given by

$$\begin{bmatrix} \Psi_{\mathsf{p}}^{(v)} \\ \Psi_{Q}^{(v)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{r}(\Theta_0)_{12} & 1 \\ \frac{1}{r^2} & 0 \end{bmatrix}\mathbb{P}_{\mathbb{W}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{53}$$

where c_1 and c_2 are integration constants. In the above expression, we have introduced the compact notation $(\Theta_n)_{ij}$ to represent the ij-element of the nth-order matrix coefficient of the power series of Θ. $\mathbb{P}_{\mathbb{W}}$ represents a real analytic matrix represented by the series

$$\mathbb{P}_{\mathbb{W}}(r) = \sum_{n=0}^{+\infty}\mathbb{P}_n r^n, \tag{54}$$

where

$$\mathbb{P}_0 = \mathbb{I}_2,$$
$$\mathbb{P}_k = \frac{1}{k}\sum_{j=0}^{k-1}\mathbb{A}_{k-1-j}\mathbb{P}_j, \quad \text{for } k \geq 1, \tag{55}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix and $\mathbb{A}_n$ is the nth-order Maclaurin coefficient of the series expansion in r of the matrix

$$\mathbb{A} = \begin{bmatrix} \Theta_{22} - r(\Theta_0)_{12}\Theta_{21} & r^2\Theta_{21} \\ \frac{\Theta_{12}-(\Theta_0)_{12}}{r^2} + \frac{(\Theta_0)_{12}(\Theta_{22}-\Theta_{11})}{r} - (\Theta_0)_{12}^2\Theta_{21} & \Theta_{11} + r(\Theta_0)_{12}\Theta_{21} \end{bmatrix}. \tag{56}$$

To select the physical solutions of Eqs. (53) and (55), we have to impose the boundary conditions (i) and (ii). In that regard, it is useful to calculate the lower-order coefficients of the power series expansion of solutions matrix $\mathbb{W}$. Considering $\mathbb{P}_0 = \mathbb{I}_2$, the boundary conditions at $r = 0$ imposes that the coefficient c_1 must be zero; otherwise, the perturbation would diverge at the center at all times, and we readily find

$$\begin{bmatrix} \Psi_{\mathsf{p}}^{(\upsilon)} \\ \Psi_{Q}^{(\upsilon)} \end{bmatrix} = \begin{bmatrix} c_2 + \mathcal{O}(r^2) \\ \mathcal{O}(r) \end{bmatrix}. \tag{57}$$

Once the equilibrium spacetime and the values of the eigenfrequencies υ, or equivalently λ, are specified, these results allow us to find real analytic solutions for the perturbation that verify the boundary conditions. The previous results, however, do not lead to a closed form expression to directly compute the values of λ^2. Yet, in Ref. [11], it was found a constraint to the minimal absolute value of the eigenvalues. Namely, if the boundary conditions (i) and (ii) are verified, the background spacetime is a $\mathcal{C}^1$ solution of the EFE, the equilibrium fluid verifies the weak energy condition, and $f'(\mu_0)$ is positive in the interior of the perturbed star; then, nontrivial $\mathcal{C}^1$ solutions of the boundary value problem (42)–(46) exist only if

$$\lambda^2 \max_{r\in]0,r_b[} (g_0)_{tt} > -\max_{r\in]0,r_b[} \left[\frac{\mu_0 + p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'(\mu_0)} + \frac{r\phi_0\mathcal{A}_0}{2(\mu_0 + p_0)}\frac{d\mu_0}{dr}\right]. \tag{58}$$

This result offers a baseline to determine the values of the eigenfrequencies of the system numerically.

IV. PERTURBATIONS AND FUNDAMENTAL EIGENFREQUENCIES OF CLASSIC EXACT SOLUTIONS

The description of adiabatic, radial perturbations in the static frame is considerably simpler than that in the comoving frame. This simplification allows us to study perturbations of a background spacetime more efficiently, when compared to the algorithm used for the comoving frame. In Ref. [12], various classical exact solutions of the theory of general relativity were considered for the equilibrium background spacetime to illustrate the new perturbation formalism. However, given the complexity of the perturbation equations, analytic analysis in the comoving frame is not computationally efficient, to the point where for selected exact solutions of the theory, the algorithm takes an unreasonable amount of time to set up the intermediate quantities on an average computer. In that regard, the algorithm used to solve the perturbation equations in the static frame is significantly faster, and we can efficiently determine the eigenfunctions and the eigenfrequencies of stellar compact objects modeled by solutions that have been used in the past to study physically meaningful scenarios.

In this section, we will study the properties of adiabatic, radial perturbations of some well-known self-gravitating equilibria of perfect fluids. We have considered background spacetimes found to be especially suitable to capture the properties of compact stellar objects. Namely, we considered solutions of the field equations that verify the four regularity criteria presented in Ref. [28]. We will then study stability of specific Buchdahl I, Heintzmann IIa, Patwardhan-Vaidya IIa, and Tolman VII solutions. Since the independent parameter c_2 simply characterizes the magnitude of a specific eigenfunction of p at $r = 0$, Eq. (57), without loss of generality in this section, we will consider $c_2 = 1$ for all eigenfunctions. Moreover, we assume that the equation of state of the perturbed fluid is the same as that of the equilibrium setup.

In Table I, we present the nontrivial metric coefficients in the line element (33) for the classical solutions of the EFE mentioned above, according to the naming conventions for the solutions of Ref. [28]. In Table II, we present the absolute values of the eigenfrequencies associated with the first three eigenmodes for specific values of the equilibrium background spacetime. Figures 1–4 show the radial profile

TABLE I. Nontrivial metric coefficients of solutions of the EFE assuming a line element of the form of Eq. (33). The naming conventions and abbreviations follow those of Ref. [28].

Spacetime	Nontrivial metric components
Buch I	$(g_0)_{tt} = A[(1 + Cr^2)^{\frac{3}{2}} + B(5 + 2Cr^2)\sqrt{2 - Cr^2}]^2 \ (g_0)_{rr} = \frac{2(1+Cr^2)}{2-Cr^2}$
Heint IIa	$(g_0)_{tt} = A^2(ar^2 + 1)^3 \ (g_0)_{rr} = \left(1 - \frac{3ar^2[c(4ar^2+1)^{-\frac{1}{2}}+1]}{2(ar^2+1)}\right)^{-1}$
P-V IIa	$(g_0)_{tt} = \left\{A\cos\left[\frac{1}{2}\mathrm{arcsinh}\left(\frac{b^2r^2-c}{\sqrt{b^2-c^2}}\right) + d\right] + B\sin\left[\frac{1}{2}\mathrm{arcsinh}\left(\frac{b^2r^2-c}{\sqrt{b^2-c^2}}\right) + d\right]\right\}^2 \ (g_0)_{rr} = (b^2r^4 - 2cr^2 + 1)^{-1}$
Tolman VII	$(g_0)_{tt} = B^2\sin^2\left[\ln\left(\sqrt{\frac{\sqrt{1-\frac{r^2}{R^2}+\frac{4r^4}{A^4}}+\frac{2r^2}{A^2}-\frac{A^2}{4R^2}}{C}}\right)\right] \ (g_0)_{rr} = (1 - \frac{r^2}{R^2} + \frac{4r^4}{A^4})^{-1}$

TABLE II. Absolute values of the eigenfrequencies associated with the first three eigenmodes rounded to three decimal places, for the equilibria in Table I assuming selected values of the spacetime parameters.

Spacetime	Parameters	$\lvert\lambda_0\rvert$	$\lvert\lambda_1\rvert$	$\lvert\lambda_2\rvert$
Buch I	$(A,B,C)=(1,0.5,1)$	16.370	36.011	54.889
Heint IIa	$(a,A,C)=(1,1,1.5)$	4.004	10.262	15.939
P-V IIa	$(A,B,b,c,d)=(1,3,2,1,1)$	8.192	18.105	27.624
Tolman VII	$(A,B,C,R)=(1,1,20,0.54)$	3.434	7.906	12.120

of the harmonic coefficients of the functions p and Q related with the eigenfrequencies for the various equilibria in Table II.

In the considered examples, all the eigenfrequencies are real. Hence, for the chosen parameters, all configurations are dynamically stable under adiabatic, radial perturbations. In Figs. 1–4 we find the expected behavior for the radial eigenmodes. For a sufficiently regular background, Eqs. (42) and (46) can be cast in the form of a Sturm-Liouville eigenvalue problem for $\Psi_Q^{(\lambda)}$ [11]. As a consequence, in particular, the number of roots of the eigenfunctions is associated with the order of the related eigenvalue in the series $(\lambda_n^2)_{n\in\mathbb{N}}$, as can be inferred from Figs. 1–4.

The adiabatic, radial perturbation of a background spacetime described by the Heintzmann IIa solution was previously analyzed in the comoving frame [12]. We repeated the analysis here, computing the eigenfunctions and the eigenfrequencies using the perturbation equations for the static frame to demonstrate the consistency between both descriptions. As expected, the computed eigenfrequencies are equal. Indeed, we have compared the values of the eigenfrequencies associated with the first three eigenmodes for all background spacetimes in Ref. [12], using the perturbation equations for the static frame, Eqs. (42)–(46), finding complete agreement up to at least 30 significant figures, confirming the equivalence between the system found for the comoving frame and the system found for the static frame. We have also compared the results in Table II

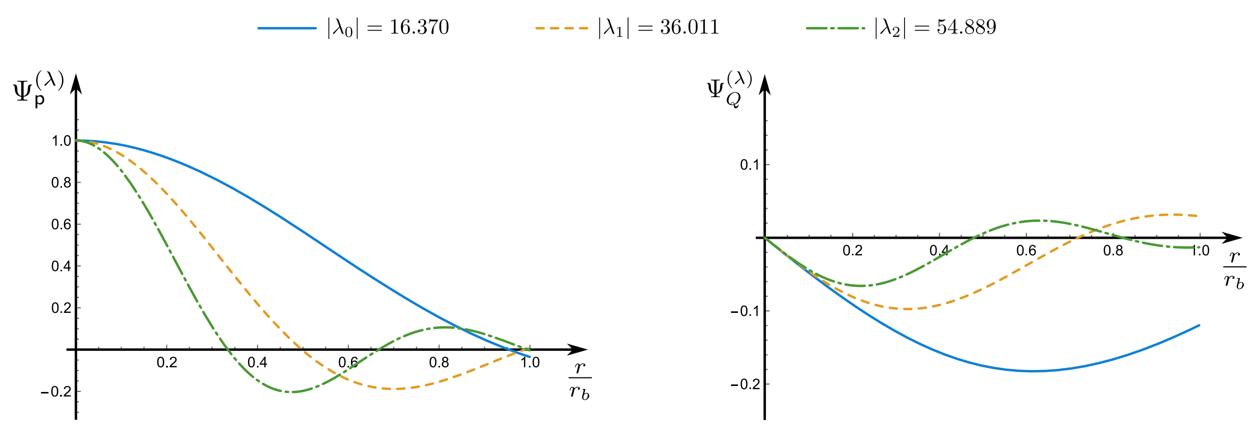

FIG. 1. Behavior of the radial harmonic coefficients of the functions p and Q, related to the eigenfrequencies in Table II for the Buch1 spacetime. It was assumed $c_2=1$ for all eigenmodes.

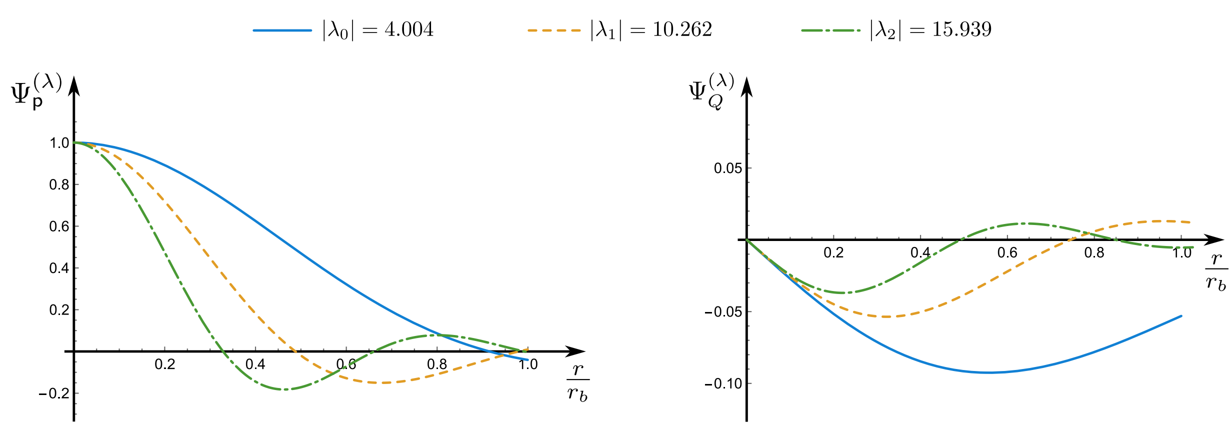

FIG. 2. Behavior of the radial harmonic coefficients of the functions p and Q, related to the eigenfrequencies in Table II for the Heint IIa spacetime. It was assumed $c_2=1$ for all eigenmodes.

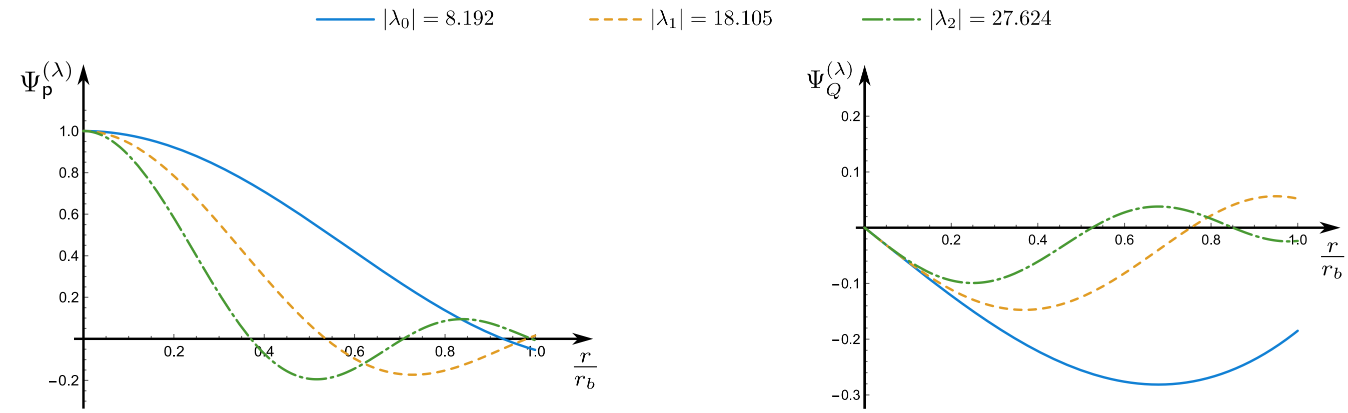

FIG. 3. Behavior of the radial harmonic coefficients of the functions p and Q, related to the eigenfrequencies in Table II for the P-V IIa spacetime. It was assumed $c_2 = 1$ for all eigenmodes.

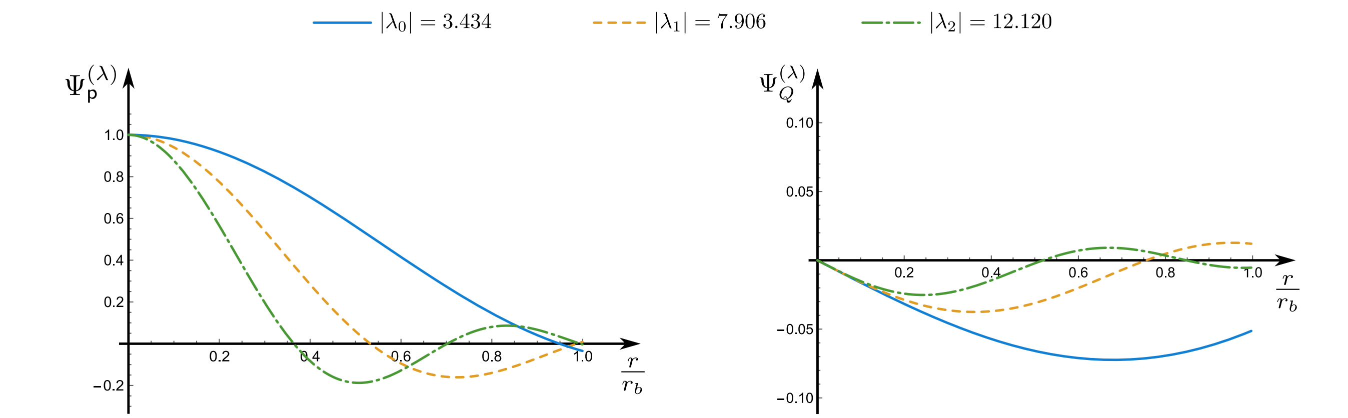

FIG. 4. Behavior of the radial harmonic coefficients of the functions p and Q, related to the eigenfrequencies in Table II for the Tolman VII spacetime. It was assumed $c_2 = 1$ for all eigenmodes.

with the predictions of the systems in Refs. [8,9]. Implementing a shooting method to numerically integrate the differential equations in those references for the various background spacetimes, the values for the fundamental eigenfrequencies agree exactly with the considered numerical accuracy, confirming the equivalence between the new perturbation formalism and the direct metric-perturbation approach.

V. PERTURBATION OF NEUTRON STARS WITH A REALISTIC EQUATION OF STATE

The spacetimes considered in the previous section have the remarkable property of being real analytic throughout the whole domain of physical interest. However, this can be different for general background solutions of the EFE. Indeed, physical solutions are only required to be twice differentiable. In some cases, even solutions in the weaker sense can be considered. For such spacetimes, the algorithm presented in Sec. III is not applicable, and other methods have to be used, in most cases numerical methods. To illustrate this approach using the new perturbation equations, we will study adiabatic, radial perturbations of equilibrium background solutions with a matter fluid source verifying a "realistic" equation of state suitable to model the interior of cold neutron stars.

Numerous equations of state have been proposed to describe matter in the high-density regime, considering different effective models for the interactions between nuclei and various particle species. Then, using the proposed equations of state, tabulated values for the matter density, baryon number, and pressure are presented. To use those values, an interpolation procedure is necessary to generate a one-parameter barotropic equation of state. This procedure, however, introduces a source of indeterminacy, such that the eigenfrequencies may diverge significantly depending on the interpolation scheme, and direct comparison with the results in the literature is not possible [10]. Alternatively, significant effort has been made to find analytical representations of unified equations of state, providing closed models for the matter fluid within a neutron star [29–31]. This is a more sensible approach, also because the various strata within a neutron star are composed of matter in different regimes. However, to our knowledge, there has been no comprehensive study of adiabatic radial perturbations for spacetimes with those

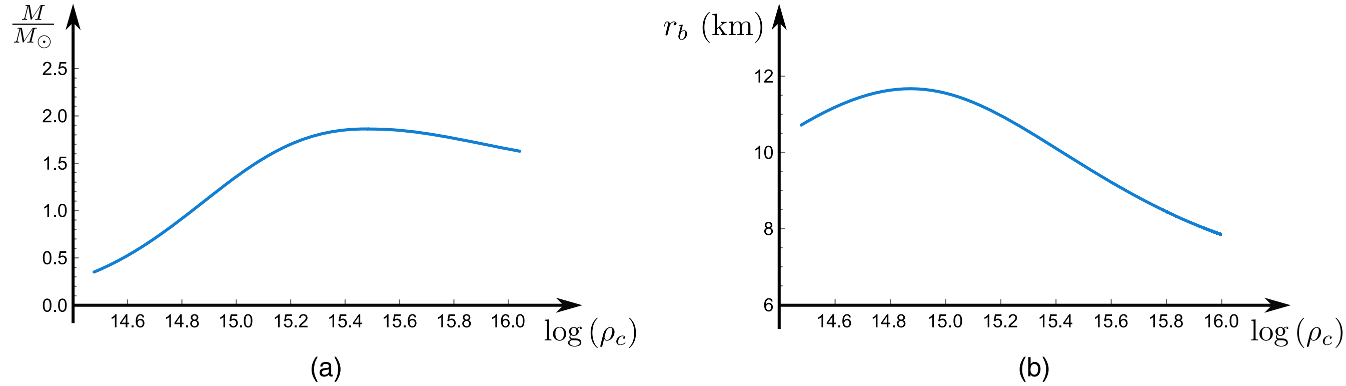

FIG. 5. Gravitational mass and radius of the neutron star as functions of the central matter density for the Bethe-Johnson model I. The central density ρ_c is measured in $\mathrm{g/cm^3}$, and log represents the logarithm base 10. (a) Gravitational mass of the neutron star as a function of the central matter density. (b) Radius of the neutron star as a function of the central matter density.

types of fluids that collect the eigenfrequencies for various equilibrium configurations. As such, we will consider the analytically tractable example of a fluid characterized by the equation of state given by the Bethe-Johnson model I within the whole neutron star, since this does not require the usage of interpolation [32].

To ease the comparison between our results and those in the literature, in this section, we will adopt a different unit system to express the various quantities.

A. The Bethe-Johnson model I and properties of the equilibrium solutions

Let E represent the energy per neutron measured in MeV, n the particle density measured in fm^{-3}, m_n the rest energy of the neutron in MeV, and p the neutron pressure measured in MeV per fm^3. Then, the Bethe-Johnson model I is characterized by the following relations:

$$E = 236n^{1.54} + m_n,$$
$$p = 363.44n^{2.54}, \tag{59}$$

valid for particle densities $0.1 \lesssim n \lesssim 3\ \mathrm{fm}^{-3}$, or mass densities,

$$1.7 \times 10^{14} \lesssim \rho \lesssim 1.1 \times 10^{16}\ \mathrm{g/cm^3}. \tag{60}$$

Notice, in particular, that in this section we do not use the geometrized unit system. Therefore, the energy density, μ, and the mass density, ρ, have to be explicitly discriminated. Considering a fluid described by this model, providing the central density as initial data, we can numerically solve the TOV equations to find the equilibrium spacetime. Since the canonical version of the TOV equations are ill-conditioned and are known to form a numerically stiff system for certain equations of state, we have verified the results using both the original version and the reformulation introduced in Ref. [33]. In Fig. 5, we plot the behavior of the total gravitational mass and the radius of the equilibrium stellar object for various values of the central matter density. Comparing the results with those in Ref. [34], we see a similar behavior for the total gravitational mass. In contrast, there is a marked discrepancy between the behavior and the values of the radius. For instance, we have found the maximum radius to be around 11.67 km, whereas in Ref. [34] the maximum radius is clearly above 12 km. This shift in the values of the radii is a consequence of the sensitivity of the model to the values considered for the fundamental constants. In particular, the radius of the neutron star strongly depends on the value of the rest energy of the neutron. By considering slightly less accurate values for m_n, we confirm that there is an overall increase in the radius, such that the maximum radius indeed increases above 12 km. On the other hand, we are not able to explain the behavior of the radius that we see in Fig. 5(b) for lower values of the central matter density, where the radius first increases to a maximum, and then monotonically decreases, whereas, in Ref. [34], the radius monotonically decreases for all considered values of the central matter density. We speculate that this is a consequence of the number of points considered in that region in Ref. [34].

B. Oscillation eigenfrequencies and *e*-folding time for adiabatic radial perturbations

Defining the background spacetime, we can numerically integrate the perturbation equations (42) and (43) imposing the boundary conditions (i) and (ii) to find the values of the eigenfrequencies.

In Table III, we present the values of the oscillation frequencies or the e-folding time associated with the first three eigenmodes for various values of the central mass density. Comparing with the results of Ref. [10], there is a disagreement between the values of the various quantities. In particular, we find that, for the same values of the central density, the neutron star is more compact than in Ref. [10] (see fourth column of Table III). However, the difference between the results is explained by the accuracy of the

TABLE III. Oscillation frequencies or *e*-folding time of adiabatic, radial perturbations of equilibrium spacetimes with a matter fluid verifying the Bethe-Johnson model I equation of state. From left to right, we list the central mass density, ρ_c, radius of the neutron star, r_b, gravitational mass, M, the compactness parameter, $\frac{GM}{c^2 r_b}$, and the first three oscillation frequencies, $f_i = \lambda_i/(2\pi)$. In the second row, the entry with an a indicates the e-folding time in *ms* for the fundamental mode.

$\rho_c[10^{15}\ \mathrm{g/cm^3}]$	r_b [km]	$\frac{M}{M_\odot}$	$\frac{GM}{c^2 r_b}$	f_1 [kHz]	f_2 [kHz]	f_3 [kHz]
3.10	9.692	1.864	0.284	1.066 [a]	18.658	28.835
3.05	9.724	1.865	0.283	0.647	18.678	28.536
2.80	9.891	1.862	0.278	2.366	17.550	27.008
2.50	10.115	1.851	0.270	3.270	16.378	25.093
2.00	10.545	1.801	0.252	3.998	14.266	21.678
1.50	11.058	1.669	0.223	4.147	11.916	17.940
1.00	11.554	1.358	0.174	3.800	9.271	13.809

values of the fundamental constants. Nevertheless, our conclusions regarding the stability are in perfect agreement with those of Ref. [10]. To attest to the consistency of our results, we confirmed that a zero value for the eigenfrequency of the fundamental mode occurs exactly for the value of the central matter density that leads to the maximum gravitational mass. This is the expected result since we have considered that a fluid with the same equation of state permeates both the background and the perturbed spacetimes. The consistency of our approach was further confirmed by comparing the predictions of the new perturbation scheme with those of metric-based perturbation theory, using the systems introduced in Refs. [8,9].

VI. CONCLUSION

We have studied adiabatic, radial perturbations of static, self-gravitating perfect fluids within the theory of general relativity employing the new perturbative formalism introduced in Ref. [11], by considering the point of view of radially static observers. For these observers, the fluid is no longer perceived as being perfect, and momentum transfer terms have to be included in the stress-energy tensor. We discussed in detail the thermodynamic description of the matter fluid in the static frame, showing explicitly that at linear order the perturbations are still adiabatic. Nonetheless, the covariance of the formalism allows us to deduce easily that a radially static observer will measure an apparent entropy flux between the volume elements of the fluid.

This result has to be considered carefully.

In general, in a nonequilibrium state, it is not necessarily possible to describe thermodynamical systems with the state variables considered for systems in equilibrium. To this end, we have considered the local-thermodynamic equilibrium ansatz. Notice, however, that this ansatz is reasonable only if the oscillation evolution timescale is much bigger than a characteristic local relaxation time of the fluid sources. This leads to a limitation in the applicability of thermodynamical arguments for modes associated with eigenfrequencies that are not small enough. Such a caveat should always be considered with care, as nontrivial thermodynamics, in general, will influence the dynamical evolution of perturbations. Beyond this limit, without a complete theory of nonequilibrium thermodynamics, it is impossible to predict the thermodynamic evolution of the perturbed matter fluid. This discussion shows how important it would be to measure the oscillation modes of compact stellar objects directly. These data would offer a window into the behavior of matter in strong relativistic regimes and guide the development of such a theory. If we accept that the static observer will measure a true entropy flow, we can deduce that it will be possible to define an apparent temperature for the fluid. Unfortunately, the analysis of the entropy flow does not bring any information on the higher-order corrections to the temperature. Our inability to determine these corrections ultimately stems from the choice of the state variables. In particular, the fact that the local temperature appears as an integration factor multiplying a first-order variation of a state variable implies that higher-order temperature corrections cannot appear in any perturbative analysis employing the local-thermodynamical equilibrium ansatz. In other words, the very assumption that a fluid out of thermodynamic equilibrium depends locally on the same state variables verifying the same relations as if the system were in a state of thermodynamic equilibrium effectively prevents us from obtaining information on the higher-order corrections to the temperature and even their very meaning.

Although the thermodynamic description of the fluid in the static frame is more complex than the one in the fluids' local rest frame, we have shown that its dynamical description is greatly simplified. Indeed, analytic solutions were developed for a general class of equilibrium spacetimes, leading to a significantly more efficient algorithm to find power series solutions when compared with the results found from the point of view of comoving observers [12]. To illustrate the new system of equations, we have analyzed adiabatic, radial perturbations of selected nontrivial exact solutions of the theory of general relativity for the equilibrium spacetime, and computed the first eigenfrequencies and the corresponding eigenfunctions. We have discussed how the results compare with the predictions of metric-based perturbation formalisms for the same spacetimes, finding complete agreement with the considered accuracy.

Last, we have analyzed adiabatic, radial perturbations of cold neutron stars composed of a perfect fluid characterized by the Bethe-Johnson model I equation of state. In this case, the background spacetime is not real analytic, and the exact power-series solutions are not applicable. Nonetheless, the example illustrates that the system is well-conditioned and suitable for numerical methods. Indeed, we have studied the eigenfrequencies of the perturbed fluid, showing their consistency with those found using metric-based perturbation frameworks in the literature.

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APPENDIX A: THE 1+1+2 DECOMPOSITION

1. Projectors and the Levi-Civita volume form

Assume the existence of an affinely parametrized congruence of timelike curves with normalized tangent vector field u in some open neighborhood of a Lorentzian manifold of dimension 4, $(\mathcal{M}, g)$, where g is a metric tensor with components $g_{\alpha\beta}$. We can use the vector field u to locally foliate the manifold in 3-surfaces, such that any tensor quantity can be pointwise decomposed in its projection along the direction of u and onto the tangent and cotangent space of a 3-surface V. Such decomposition is called $1+3$ formalism and relies on a projector tensor h. Using the metric tensor g and the vector field u, we can naturally define h such that

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \tag{A1}$$

where u_α represent the components of the 1-form associated with u, with the properties

$$\begin{aligned} h_{\alpha\beta} &= h_{\beta\alpha}, \qquad h_{\alpha\beta}h^{\beta\gamma} = h_\alpha{}^\gamma, \\ h_{\alpha\beta}u^\alpha &= 0, \qquad h_\alpha{}^\alpha = 3. \end{aligned} \tag{A2}$$

In addition to the timelike congruence, we assume the existence of another congruence of spacelike curves with normalized tangent vector field e such that, in analogy with the $1+3$ decomposition, any tensor quantity defined in the submanifold V can be decomposed along e and the 2-surfaces W, orthogonal to both u and e. This defines the $1+1+2$ covariant formalism. This formalism relies on the existence of a projector, N, onto the cotangent space of W. Considering the metric tensor and the vector fields u and e, we define

$$N_{\alpha\beta} = h_{\alpha\beta} - e_\alpha e_\beta, \tag{A3}$$

where e_α represents the components of the 1-form associated with e. This operator has the following properties:

$$\begin{aligned} N_{\alpha\beta} &= N_{\beta\alpha}, \qquad N_{\alpha\beta}N^{\beta\gamma} = N_\alpha{}^\gamma, \\ N_{\alpha\beta}u^\alpha &= N_{\alpha\beta}e^\alpha = 0, \qquad N_\alpha{}^\alpha = 2. \end{aligned} \tag{A4}$$

To complete the representation of the geometry of $(\mathcal{M}, g)$ we introduce the totally skew-symmetric tensors

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma} &= \varepsilon_{\alpha\beta\gamma\sigma}u^\sigma, \\ \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta\gamma}e^\gamma, \end{aligned} \tag{A5}$$

where $\varepsilon_{\alpha\beta\gamma\sigma}$ is the covariant Levi-Civita tensor. Last, we will indicate the symmetric and antisymmetric parts of a 2-tensor χ using parentheses and brackets, such that

$$\chi_{(\alpha\beta)} = \frac{1}{2}(\chi_{\alpha\beta} + \chi_{\beta\alpha}), \qquad \chi_{[\alpha\beta]} = \frac{1}{2}(\chi_{\alpha\beta} - \chi_{\beta\alpha}). \tag{A6}$$

2. Covariant derivatives of *u* and *e*

The $1+1+2$ decomposition can be applied to the covariant derivatives of the vector fields u and e and define the kinematical quantities associated with the respective congruences. To lighten the notation, given a tensor quantity χ, we will use

$$\dot{\chi} \coloneqq u^\mu \nabla_\mu \chi, \qquad \hat{\chi} \coloneqq e^\mu \nabla_\mu \chi, \tag{A7}$$

for the directional derivatives along the integral curves of u and the integral curves of e, respectively.

Now, the covariant derivative of the vector field u can be decomposed as

$$\nabla_\alpha u_\beta = -u_\alpha(\mathcal{A}e_\beta + \mathcal{A}_\beta) + \frac{1}{3}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \tag{A8}$$

where

$$\mathcal{A} = -u_\mu u^\nu \nabla_\nu e^\mu, \qquad \mathcal{A}_\alpha = N_{\alpha\mu}\dot{u}^\mu, \tag{A9}$$

and θ represents the expansion scalar, $\sigma_{\alpha\beta}$ the shear tensor, and $\omega_{\alpha\beta}$ the vorticity tensor, such that

$$\begin{aligned} \theta &= h^{\mu\nu}\nabla_\mu u_\nu, \\ \sigma_{\alpha\beta} &= \left(\frac{h_\alpha{}^\mu h_\beta{}^\nu + h_\alpha{}^\nu h_\beta{}^\mu}{2} - \frac{1}{3}h_{\alpha\beta}h^{\mu\nu}\right)\nabla_\mu u_\nu \\ &= \Sigma_{\alpha\beta} + 2\Sigma_{(\alpha}e_{\beta)} + \Sigma\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right), \\ \omega_{\alpha\beta} &= \frac{1}{2}(h_\alpha{}^\mu h_\beta{}^\nu - h_\alpha{}^\nu h_\beta{}^\mu)\nabla_\mu u_\nu = \varepsilon_{\alpha\beta\mu}(\Omega e^\mu + \Omega^\mu), \end{aligned} \tag{A10}$$

with

$$\Sigma_{\alpha\beta} = \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)\sigma_{\mu\nu},$$
$$\Sigma_\alpha = N_\alpha{}^\mu e^\nu \sigma_{\mu\nu}, \qquad \Sigma = e^\mu e^\nu \sigma_{\mu\nu}, \tag{A11}$$

and

$$\Omega^\alpha = \frac{1}{2}N_\gamma{}^\alpha \varepsilon^{\mu\nu\gamma}\nabla_\mu u_\nu, \qquad \Omega = \frac{1}{2}\varepsilon^{\mu\nu}\nabla_\mu u_\nu. \tag{A12}$$

From their definitions, we see that vector and 2-tensor quantities characterize the behavior of the kinematical quantities on the 2-surfaces W, and the scalars characterize their behavior along u or e.

Similarly, the covariant derivative of the vector field e can be written as the following sum:

$$\nabla_\alpha e_\beta = \frac{1}{2}N_{\alpha\beta}\phi + \zeta_{\alpha\beta} + \varepsilon_{\alpha\beta}\xi + e_\alpha a_\beta - u_\alpha \alpha_\beta - \mathcal{A}u_\alpha u_\beta$$
$$+ \left(\frac{1}{3}\theta + \Sigma\right)e_\alpha u_\beta + (\Sigma_\alpha - \varepsilon_{\alpha\mu}\Omega^\mu)u_\beta, \tag{A13}$$

where

$$\phi = N^{\mu\nu}\nabla_\mu e_\nu,$$
$$\zeta_{\alpha\beta} = \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)\nabla_\mu e_\nu,$$
$$\xi = \frac{1}{2}\varepsilon^{\mu\nu}\nabla_\mu e_\nu \tag{A14}$$

are the kinematical quantities of the congruence associated with the vector field e on W; namely, ϕ is the expansion scalar, $\zeta_{\alpha\beta}$ the shear tensor, and the ξ twist, and

$$a_\alpha = e^\mu h_\alpha{}^\nu \nabla_\mu e_\nu, \qquad \alpha_\alpha = u^\mu h_\alpha{}^\nu \nabla_\mu e_\nu. \tag{A15}$$

3. Weyl tensor

In covariant approaches, the conformal structure of the spacetime plays a fundamental role. For this reason, the Weyl tensor, with components $C_{\alpha\beta\gamma\delta}$, which describes the tidal forces and the properties of gravitational waves, needs to be expressed in terms of $1+1+2$ variables. As is well known, the Weyl tensor can be defined as the trace-free part of the Riemann curvature tensor, $R_{\alpha\beta\gamma\delta}$, such that, in four spacetime dimensions, we have

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + R_{\alpha[\gamma}g_{\delta]\beta} - R_{\beta[\gamma}g_{\delta]\alpha} - \frac{1}{3}Rg_{\alpha[\gamma}g_{\delta]\beta}. \tag{A16}$$

Remarkably, we need only two 2-tensors to characterize the Weyl 4-tensor in general relativity completely:

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu, \tag{A17}$$

$$H_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\mu\nu\beta\delta}u^\delta, \tag{A18}$$

where E is called the "electric" part of the Weyl tensor, while H is called the "magnetic" part of the Weyl tensor. They are both symmetric and traceless tensors, such that

$$C_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}u_{\delta]} + 2u_\beta E_{\alpha[\gamma}u_{\delta]}$$
$$- 2\varepsilon_{\alpha\beta\mu}H^\mu{}_{[\gamma}u_{\delta]} - 2\varepsilon_{\mu\gamma\delta}H^\mu{}_{[\alpha}u_{\beta]}. \tag{A19}$$

This represents the famous $1+3$ decomposition of the Weyl tensor. In the $1+1+2$ spacetime decomposition formalism, the electric and magnetic parts of the Weyl tensor can be further decomposed as

$$E_{\alpha\beta} = \mathcal{E}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{E}_\alpha e_\beta + e_\alpha \mathcal{E}_\beta + \mathcal{E}_{\alpha\beta},$$
$$H_{\alpha\beta} = \mathcal{H}\left(e_\alpha e_\beta - \frac{1}{2}N_{\alpha\beta}\right) + \mathcal{H}_\alpha e_\beta + e_\alpha \mathcal{H}_\beta + \mathcal{H}_{\alpha\beta}, \tag{A20}$$

where

$$\mathcal{E} = E_{\mu\nu}e^\mu e^\nu = -N^{\mu\nu}E_{\mu\nu}, \qquad \mathcal{H} = e^\mu e^\nu H_{\mu\nu} = -N^{\mu\nu}H_{\mu\nu},$$
$$\mathcal{E}_\alpha = N_\alpha{}^\mu e^\nu E_{\mu\nu} = e^\mu N_\alpha{}^\nu E_{\mu\nu}, \qquad \mathcal{H}_\alpha = N_\alpha{}^\mu e^\nu H_{\mu\nu} = e^\mu N_\alpha{}^\nu H_{\mu\nu},$$
$$\mathcal{E}_{\alpha\beta} = E_{\{\alpha\beta\}}, \qquad \mathcal{H}_{\alpha\beta} = H_{\{\alpha\beta\}}. \tag{A21}$$

4. Stress-energy tensor

Last, to complete the $1+1+2$ gravitational equations, we also need a decomposition of the stress-energy tensor, with components $T_{\alpha\beta}$. Using the projector operators (A1) and (A3) yields

$$T_{\alpha\beta} = \mu u_\alpha u_\beta + (p+\Pi)e_\alpha e_\beta + \left(p - \frac{1}{2}\Pi\right)N_{\alpha\beta} + 2Qe_{(\alpha}u_{\beta)} + 2Q_{(\alpha}u_{\beta)} + 2\Pi_{(\alpha}e_{\beta)} + \Pi_{\alpha\beta}, \tag{A22}$$

with

$$\begin{aligned}
&\mu = u^\mu u^\nu T_{\mu\nu}, \qquad Q_\alpha = -N_\alpha{}^\mu u^\nu T_{\mu\nu},\\
&p = \frac{1}{3}(e^\mu e^\nu + N^{\mu\nu})T_{\mu\nu}, \qquad \Pi_\alpha = N_\alpha{}^\mu e^\nu T_{\mu\nu},\\
&\Pi = \frac{1}{3}(2e^\mu e^\nu - N^{\mu\nu})T_{\mu\nu}, \qquad \Pi_{\alpha\beta} = \left(\frac{N_\alpha{}^\mu N_\beta{}^\nu + N_\alpha{}^\nu N_\beta{}^\mu}{2} - \frac{1}{2}N_{\alpha\beta}N^{\mu\nu}\right)T_{\mu\nu}.\\
&Q = -e^\mu u^\nu T_{\mu\nu},
\end{aligned} \tag{A23}$$

For an observer with 4-velocity u, μ represents the perceived mass-energy density of the fluid, p is the isotropic pressure, Q characterizes the energy-momentum flow along e, Q_α is the energy-momentum flux in W, and Π, Π_α and $\Pi_{\alpha\beta}$ characterize the anisotropic pressure of the matter fluid.

APPENDIX B: CHANGE IN THE STRESS-ENERGY TENSOR UNDER ISOTROPIC FRAME TRANSFORMATIONS

In this appendix, we will define an isotropic frame transformation associated with two dyads and discuss how this type of transformation changes the stress-energy tensor used to describe the fluid in each frame.

1. Isotropic frame transformations and projectors

Let a local frame be partially defined by a dyad (u, e) composed, respectively, by a timelike and a spacelike vector field in a spacetime, such that $u^\alpha u_\alpha = -1$ and $e^\alpha e_\alpha = +1$. Then, let another frame be partially defined by another dyad $(\bar{u}, \bar{e})$ also formed, respectively, by a timelike and a spacelike vector field, such that $\bar{u}^\alpha \bar{u}_\alpha = -1$ and $\bar{e}^\alpha \bar{e}_\alpha = +1$. As explained in detail in Ref. [11], a general isotropic frame transformation can be represented by the following relations:

$$\begin{aligned}
\bar{u}^\alpha &= u^\alpha \cosh\beta + e^\alpha \sinh\beta,\\
\bar{e}^\alpha &= u^\alpha \sinh\beta + e^\alpha \cosh\beta,
\end{aligned} \tag{B1}$$

or, equivalently,

$$\begin{aligned}
u^\alpha &= \bar{u}^\alpha \cosh\beta - \bar{e}^\alpha \sinh\beta,\\
e^\alpha &= -\bar{u}^\alpha \sinh\beta + \bar{e}^\alpha \cosh\beta,
\end{aligned} \tag{B2}$$

where β is called the tilting angle between the frames.

Using these relations, we can find the transformation of the projector tensor h, defined in Eq. (A1),

$$\begin{aligned}
\bar{h}_{\alpha\beta} &= h_{\alpha\beta} + (u_\alpha u_\beta + e_\alpha e_\beta)\sinh^2\beta\\
&\quad + \frac{1}{2}(u_\alpha e_\beta + e_\alpha u_\beta)\sinh(2\beta),\\
h_{\alpha\beta} &= \bar{h}_{\alpha\beta} + (\bar{u}_\alpha \bar{u}_\beta + \bar{e}_\alpha \bar{e}_\beta)\sinh^2\beta\\
&\quad - \frac{1}{2}(\bar{u}_\alpha \bar{e}_\beta + \bar{e}_\alpha \bar{u}_\beta)\sinh(2\beta),
\end{aligned} \tag{B3}$$

with the following properties:

$$\begin{aligned}
\bar{h}_{\alpha\beta}\bar{u}^\alpha &= 0,\\
\bar{h}_{\alpha\beta}\bar{e}^\alpha &= \bar{e}_\beta.
\end{aligned} \tag{B4}$$

Moreover, we find $\bar{N}_{\alpha\beta} = N_{\alpha\beta}$, such that $\bar{N}_{\alpha\beta}\bar{u}^\beta = 0$ and $\bar{N}_{\alpha\beta}\bar{e}^\beta = 0$.

2. Tilting angle, the stress-energy tensor, and the expansion scalar

We are now interested in relating the tilting angle β of an isotropic frame transformation with the thermodynamic variables of the fluid measured in the resultant frame.

Consider a metric stress-energy tensor of an isotropic fluid decomposed accordingly with Eq. (A22), that is, the vector and tensor components $Q_\alpha = \Pi_\alpha = \Pi_{\alpha\beta} = 0$. Under the transformation (B1) we have

$$\begin{aligned}
\bar{\mu} &= \mu - Q\sinh(2\beta) + (\mu + p + \Pi)\sinh^2\beta,\\
\bar{p} &= p - \frac{1}{3}Q\sinh(2\beta) + \frac{1}{3}(\mu + p + \Pi)\sinh^2\beta,\\
\bar{Q} &= Q\cosh(2\beta) - \frac{1}{2}(\mu + p + \Pi)\sinh(2\beta),\\
\bar{\Pi} &= \Pi\left(1 + \frac{2}{3}\sinh^2\beta\right) - \frac{2}{3}Q\sinh(2\beta) + \frac{2}{3}(\mu + p)\sinh^2\beta.
\end{aligned} \tag{B5}$$

Here, an overline characterizes variables measured in the tilted frame $(\bar{u}, \bar{e})$.

Now, assuming that the stress-energy tensor $T_{\alpha\beta}$, associated with the dyad (u, e), is such that $Q = 0$ and $\Pi = 0$, that is, in the frame associated with that dyad, the fluid can be described by a perfect fluid model, we find that in the resultant barred frame the fluid is characterized by

$$T_{\alpha\beta} = \bar{\mu}\bar{u}_\alpha\bar{u}_\beta + (\bar{p} + \bar{\Pi})\bar{e}_\alpha\bar{e}_\beta + \left(\bar{p} - \frac{1}{2}\bar{\Pi}\right)\bar{N}_{\alpha\beta} + 2\bar{Q}\bar{e}_{(\alpha}\bar{u}_{\beta)}, \tag{B6}$$

where

$$
\begin{aligned}
\bar{\mu} &= \mu + (\mu + p)\sinh^2\beta, \\
\bar{p} &= p + \frac{1}{3}(\mu + p)\sinh^2\beta, \\
\bar{Q} &= -\frac{1}{2}(\mu + p)\sinh(2\beta), \\
\bar{\Pi} &= \frac{2}{3}(\mu + p)\sinh^2\beta.
\end{aligned} \tag{B7}
$$

For the discussion in the main body of the text, it is also useful to find the transformation for the expansion scalars measured by the fiducial curve of each frame: the barred and nonbarred frames. Given the frame transformations (B1)–(B3) we find the following relations:

$$
\bar{\theta} = \theta\cosh\beta + (\mathcal{A} + \phi)\sinh\beta + u^\mu\nabla_\mu(\cosh\beta) + e^\mu\nabla_\mu(\sinh\beta). \tag{B8}
$$

Now, we assume that the fluid that permeates the spacetime can be considered as a perturbation of a perfect fluid with energy density and pressure μ_0 and p_0, respectively. Using Eq. (B7), we can determine a relation between the tilting angle β and $\bar{Q}$, such that, to linear order of perturbation theory, we find (cf. Ref. [11])

$$
\beta = -\frac{\bar{Q}}{\mu_0 + p_0}. \tag{B9}
$$

Moreover, the stress-energy tensor, at linear perturbative order, is given simply by

$$
T_{\alpha\beta} = \bar{\mu}\bar{u}_\alpha\bar{u}_\beta + \bar{p}(\bar{e}_\alpha\bar{e}_\beta + \bar{N}_{\alpha\beta}) + 2\bar{Q}\bar{e}_{(\alpha}\bar{u}_{\beta)}. \tag{B10}
$$

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